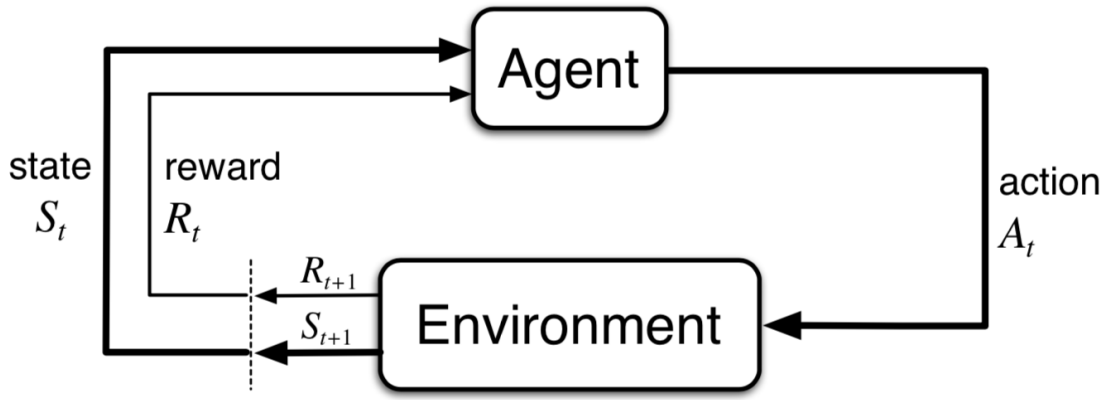
Outline

Reminder: states, actions and policy



$\mathcal{S}$, $\mathcal{A}$ = state/action spaces.

A (determinisitic) policy is a function
$\pi : \mathcal{S} \to \mathcal{A}$

.

Gain and value function

The gain is:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots \\ &= R_{t+1} + \gamma G_{t+1}, \end{aligned}$$

where $\gamma \in (0, 1)$ is the discount factor.

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where $\gamma \in (0, 1)$ is the discount factor.

The value function V and action-value function Q are:

$$\begin{aligned} V_\pi(s) &= \mathbb{E}\left[G_{t+1} \mid S_t = s, \pi\right] \\ Q_\pi(s, a) &= \mathbb{E}\left[G_{t+1} \mid S_t = s, A_t = a, \pi\right] \end{aligned}$$

Two problems

- Policy evaluation

For a given policy π, find
$V^\pi(x)$ and $Q^\pi(x,a)$.

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- Policy evaluation

For a given policy π, find
$V^\pi(x)$ and $Q^\pi(x,a)$.

- Control problem / optimization

Find / use π^* such that
$V^{\pi^*} = \max_\pi V^\pi(x)$.

Bellman's equation

$$V^*(s) =$$

$$Q^*(s,a) =$$

Bellman's equation

$$V^*(s) = \max_{a \in \mathcal{A}} Q^*(s,a)$$

$$Q^*(s,a) = r(s,a) + \gamma \sum_{s'} V^*(s') p(s' \mid s,a)$$

Two problems:

- Requires the knowledge of systems dynamics and rewards.
- $|\mathcal{S}|$ can be large

Bellman's equation

$$r(s,a) = \sum_{s'} s' \mathbb{P}(s' \mid s, a)$$

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~~Two problems:~~

- ~~Requires the knowledge of systems dynamics and rewards.~~
  - We assume to have access to a simulator.
- ~~$|\mathcal{S}|$ can be large~~
  - We assume $|\mathcal{S}|$ to be small for now.

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Monte Carlo methods

Class of algorithms where we replace a deterministic computation by an estimation of $\mathbb{E}[X]$. We then sample many values of X and compute the average (law of large numbers: $\frac{1}{n}\sum_{i=1}^{n} X_i \approx \mathbb{E}[X]$).

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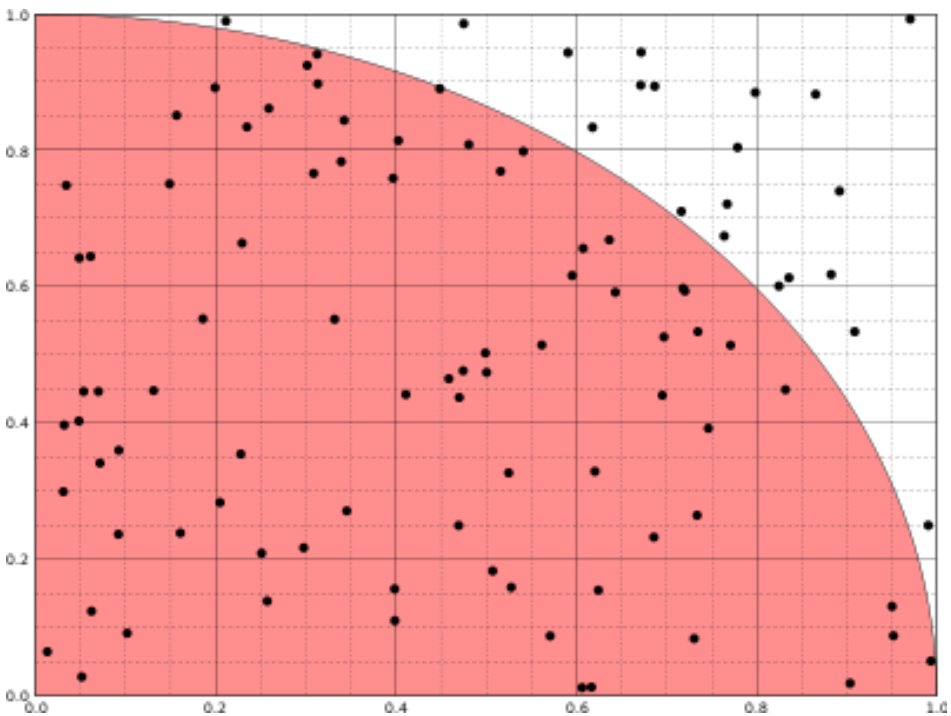
Example:



Source: wikipedia

- Area is $\pi/4$. A point (x, y) is in the red zone if $x^2 + y^2 \leq 1$.

Monte Carlo for policy Evaluation

$$V^{\pi}(S_t) = \mathbb{E}\left[G_t \mid S_t = s, \pi\right].$$

Monte-Carlo = sample G_t by using rollout.

Monte Carlo for policy Evaluation

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Monte-Carlo = sample G_t by using rollout.

Recipe:

- Play many episodes with π
- Record the return from the first visit to each state
- Return the average as an approximation of $V^\pi(s)$.

Note: every-visit also works but the samples are not independent.

Monte Carlo learning algorithm

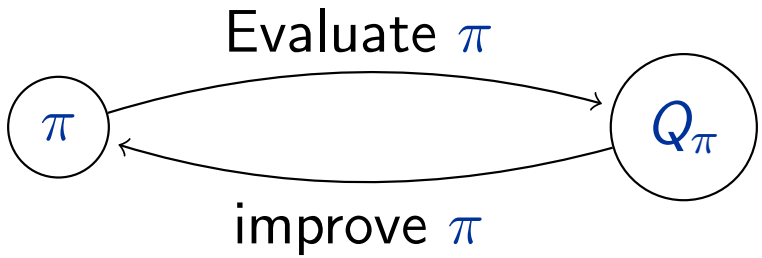
First-visit Monte-Carlo

For all s: R(s):={}
while True **do**
  Simulate an episode from 0 to T using π
  Set $G_T := 0$
  for $t = T$ to 0 (backward) **do**
    $G_t = R_{t+1} + \gamma G_{t+1}$.
    If S_t does not appear in $S_0 \dots S_{t-1}$, $R(S_t).append(G_t)$.
  end for
end while
$V(s) = mean(R(s))$.

If a state has been seen n times, the error is $O(1/\sqrt{n})$.

Monte-Carlo optimization

Monte-Carlo can be used to evaluate the state-action function $Q(s, a)$.

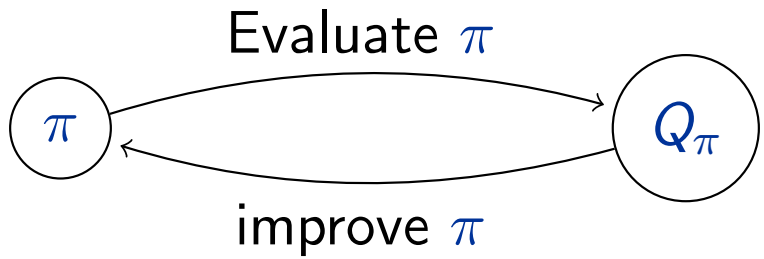


Recall: improve can be done by using greedy:

$$\pi(s) = \arg\max_{a \in \mathcal{A}} Q(s, a).$$

Monte-Carlo optimization

Monte-Carlo can be used to evaluate the state-action function $Q(s, a)$.



Recall: improve can be done by using greedy:

$$\pi(s) = \arg\max_{a \in \mathcal{A}} Q(s, a).$$

Possible problems:

- One may need many samples for all actions.
- Some action-pair might not be visited.

Solutions: exploration/exploitation tradeoff (course 4), importance sampling.

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The temporal difference (TD) error

Bellman's equation states:

$$\begin{aligned} V(S_t) &= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \dots\right] \\ &= \mathbb{E}\left[R_{t+1} + \gamma V(S_{t+1})\right]. \end{aligned}$$

$$V_\pi(s) = r(s, \pi(s)) + \gamma \sum_{s'} V_\pi(s') P(s' | s, \pi(s))$$

The temporal difference (TD) error

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α_t "small"

This is equivalent to

$$0 = \mathbb{E}\left[\underbrace{R_{t+1} + \gamma V(S_{t+1}) - V(S_t)}_{\text{TD error}}\right]$$

The TD learning algorithm uses the updates:

$$V(S_t) := V(S_t) + \alpha_t(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))),$$

where α is a learning rate.

Goal: Estimate $\mathbb{E}[X]$

$$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \lim_{n \to \infty} S_n = \mathbb{E}[X] \quad \text{(a.s.)}$$

$$S_{n+1} = S_n + \frac{1}{n+1}\left(X_{n+1} - S_n\right)$$

$$\sum_{i=1}^{n+1} X_i = (n+1) S_{n+1} = n S_n + X_{n+1} = (n+1) S_n + X_{n+1} - S_n$$

TD learning algorithm

TD(0) for evaluating V^π

1: Initialize $V(s)$ arbitrarily.
2: **while** True **do**
3: Initialize S
4: **for** While S' is not a terminal state **do**
5: Sample $A \sim \pi(S)$ and simulate a transition $S', R \sim p(\cdot \mid S, A)$.
6: $V(S) := V(S) + \alpha_t(R + \gamma V(S') - V(S))$.
7: $S := S'$
8: **end for**
9: **end while**

TD-learning: proof of convergence

TD-update:

$$V(S_t) := V(S_t) + \alpha_t(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))).$$

Theorem

Fix a policy π that visits all states and let $\gamma < 1$.
Assume that we use the TD-update with α_t be decreasing and such that:

- $\sum_t \alpha_t = +\infty$ *and* $\sum_t \alpha_t^2 < +\infty$.

Then the TD-learning converges to V^π almost surely.

Proof

Let $\beta_t(s)$ be such that

$$\beta_t(s) = \begin{cases} 0 & \text{if } s = S_t \\ \alpha_t & \text{otherwise} \end{cases}$$

Let V_t be the V-table at time t. The definition of β_t implies that for all s:

$$V_{t+1}(s) := V_t(s) + \beta_t(s) \left(\underbrace{R_{t+1} + \gamma V_t(S_{t+1})}_{= T^\pi V_t + \text{noise}} - V_t(s) \right).$$

with $\sum_t \beta_t(s) = \infty$ and $\sum_t \beta_t^2(s) < \infty$.

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with $\sum_t \beta_t(s) = \infty$ and $\sum_t \beta_t^2(s) < \infty$.

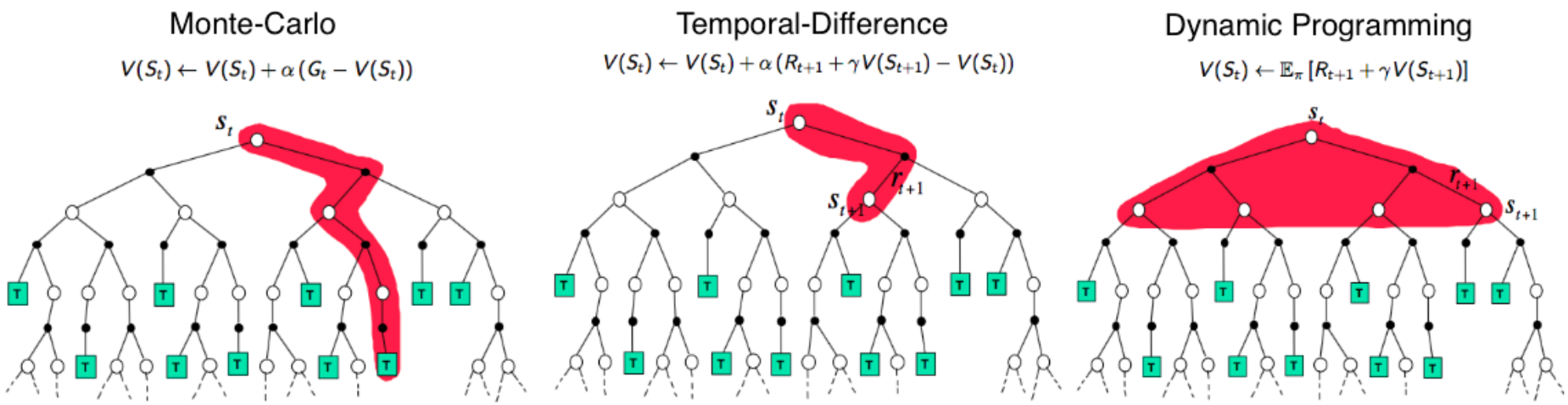
As T^π is contracting, Theorem 1 of On the convergence of stochastic iterative dynamic programming algorithms., Jaakkola, Jordan, Singh, NeurIPS 93 shows that this implies $\lim_{t\to\infty} V_t = V^\pi$ almost surely.

Relation between MC, TD and DP

$$V(S_t) = \mathbb{E}[G_t] \qquad MC$$

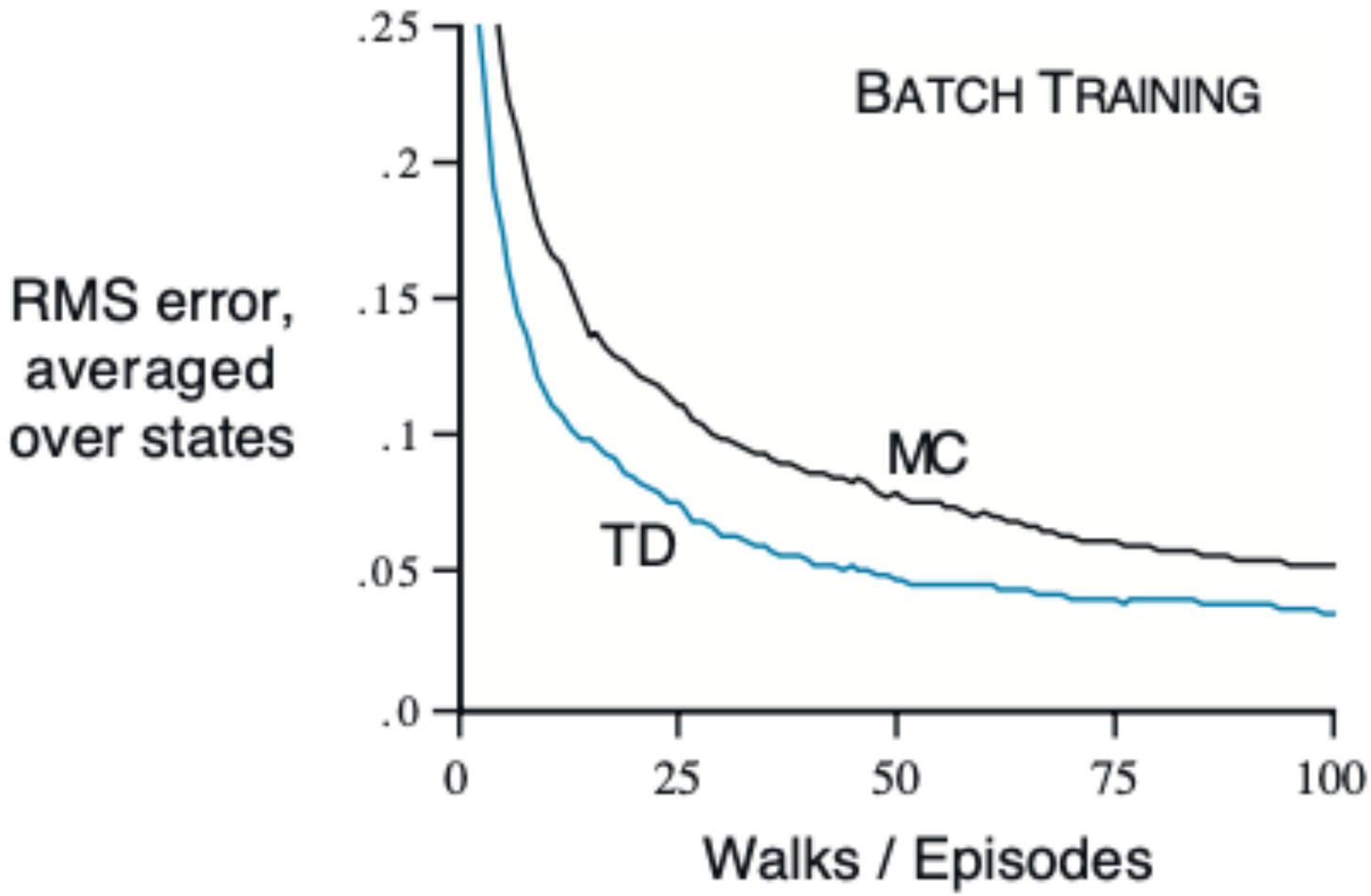
$$V(S_t) = \mathbb{E}[R_{t+1} + \gamma V(S_{t+1})] \qquad TD$$

$$V(S_t) = \mathbb{E}[R_{t+1}] + \gamma \sum_{s'} V(S_{t+1}) \mathbb{P}(S_{t+1} = s') \qquad DP$$



- MC simulates a full trajectory
- TD samples one-step and uses a previous estimation of V.
- DP needs all possible values of $V(s')$.

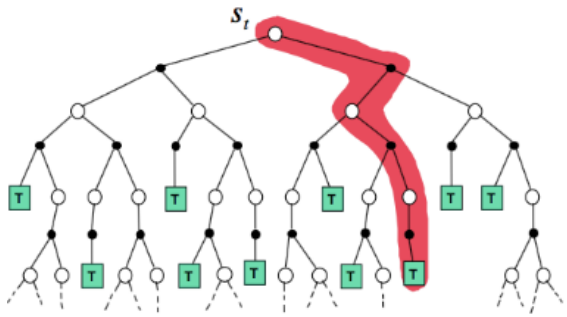
TD vs MC comparison: general case



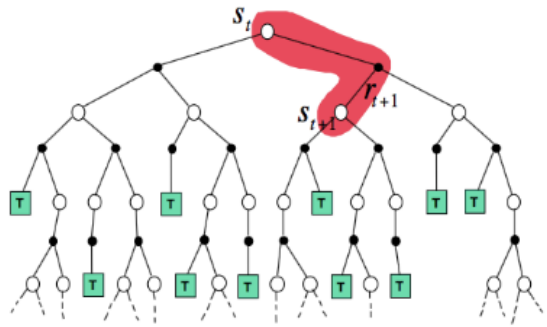
source: Sutton, Barto 2018. For a random-walk example.

Warning: this might very well depend on the choice of learning parameter α_t!

TD v.s. MC and tradeoffs

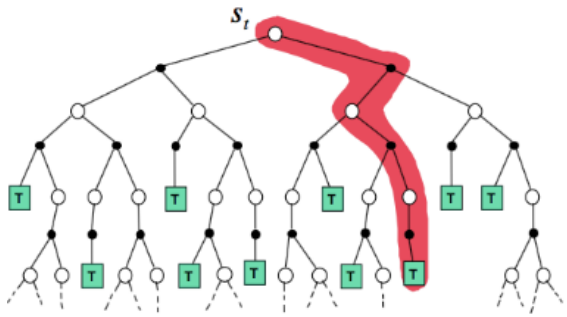


One full trajectory for update



Updates take time to propagate

TD v.s. MC and tradeoffs



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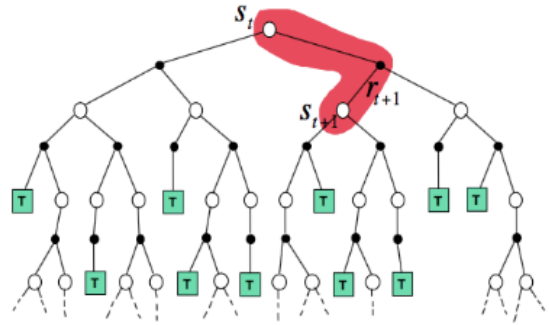


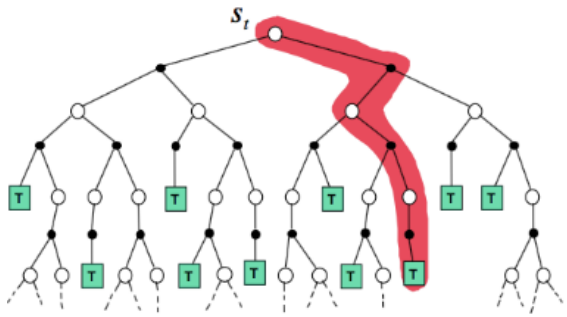
Updates take time to propagate

Tradeoff:

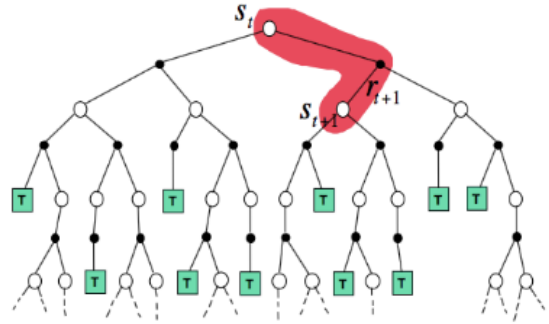
- Use n-step returns (see Sutton-Barto, chapter 7).

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^{t+n} V(S_{t+n}).$$

TD v.s. MC and tradeoffs



One full trajectory for update



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- $TD(\lambda)$ (see Sutton-Barto, chapter 12 or Szepesvári, Section 2.1.3).

$$G_t(\lambda) = (1-\lambda) \sum_{n=1}^{T} \lambda^{n-1} G_{t:t+n} + \lambda^T G_t.$$

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TD learning = policy evaluation. What about optimization?

Bellman's equations are:

$$V^\pi(S_t) = \mathbb{E}^\pi \left[R_{t+1} + \gamma V^\pi(S_{t+1}) \right]$$ to evaluate π

$$R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

TD learning = policy evaluation. What about optimization?

Bellman's equations are:

$$V^\pi(S_t) = \mathbb{E}^\pi \left[R_{t+1} + \gamma V^\pi(S_{t+1}) \right] \qquad \text{to evaluate } \pi$$

$$Q^*(S_t, A_t) = \mathbb{E}\left[R_{t+1} + \gamma \underbrace{\max_a Q^*(S_{t+1}, a)}_{V^*(S_{t+1})} \right] \qquad \text{to find the best policy}$$

TD difference for Q:

$$R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)$$

$S_t \to A_t \to R_{t+1}, S_{t+1} \to A_{t+1} \to \cdots$

TD learning = policy evaluation. What about optimization?

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This leads to two variant of:

- Q-learning = off-policy learning.
  - Choose $A_t \sim \pi$.
  - Apply TD-learning replacing $V(s)$ by $\max_a Q(s,a)$.
- SARSA = on-policy learning:
  - Choose $A_{t+1} \sim \arg\max_{a \in \mathcal{A}} Q(S_{t+1}, a)$.
  - Apply TD-learning replacing $V(s)$ by $Q(s, A_{t+1})$.

Q-learning and convergence guarantee

$$A_t \sim \pi$$

$$Q(S_t, A_t) := Q(S_t, A_t) + \alpha_t \left(R_{t+1} + \gamma \max_{a \in \mathcal{A}} Q(S_{t+1}, a) - Q(S_t, A_t) \right).$$

Q-learning and convergence guarantee

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Theorem

Assume that $\gamma < 1$ and that:

- *Any station-action pair (a, s) is visited infinitely often.*
- $\sum_t \alpha_t = \infty$ *and* $\sum_t \alpha_t^2 < \infty$.

Then: Q converges almost surely to the optimal Q^*-table as t goes to infinity.

Proof: Identical to the proof of TD-learning.

ex: $\alpha_t = \frac{1}{t}$

$$\sum_{i=1}^{t} \alpha_i \simeq \log t$$

$$\sum_{i=1}^{t} (\alpha_i)^2 = \sum_i \frac{1}{i^2} < \frac{\pi^2}{6}$$

$$\alpha_t = t^{-\beta}, \quad \beta \in \left(\frac{1}{2}; 1\right)$$

$$\alpha_t = \frac{1000}{t}$$

Q-Learning and SARSA

Q-learning, (one of the most popular RL algorithm):

$$A_t \sim \pi$$

$$Q(S_t, A_t) := Q(S_t, A_t) + \alpha_t \left(R_{t+1} + \gamma \max_{a \in \mathcal{A}} Q(S_{t+1}, a) - Q(S_t, A_t) \right).$$

Q-Learning and SARSA

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SARSA (name comes from $S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}$)

$$A_{t+1} \sim \arg\max Q(S_t, A_t) \text{ (or } \varepsilon\text{-greedy)}$$
$$Q(S_t, A_t) := Q(S_t, A_t) + \alpha_t \left(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t) \right).$$

Q-learning pseudo-code

The Q learning algorithm

1: Initialize $Q(s, a)$ arbitrarily.
2: **while** True **do**
3: Initialize S
4: **while** S' is not a terminal state **do**
5: $\pi =$ policy derived from Q (e.g. ε-greedy).
6: Sample $A \sim \pi(S)$ and simulate a transition $S', R \sim p(\cdot \mid S, A)$.
7: $Q(S, A) := Q(S, A) + \alpha_t(R + \gamma \max_a Q(S', a) - Q(S, A))$.
8: $S := S'$
9: **end while**
10: **end while**

(in orange, the difference with TD-learning).

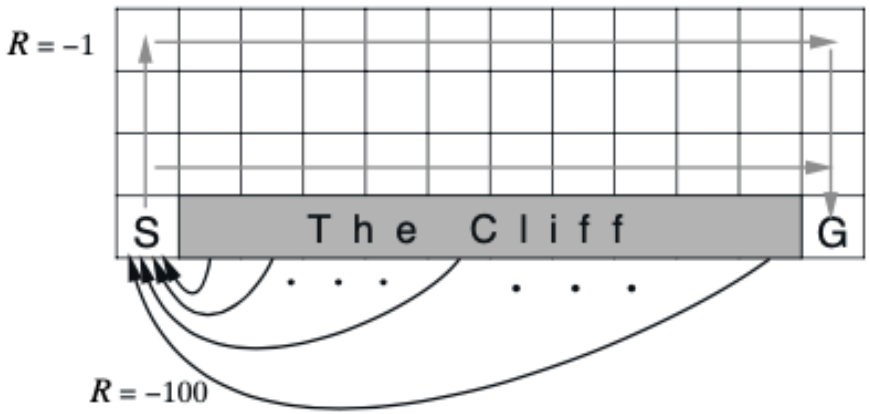
SARSA

SARSA algorithm

Initialize $Q(s, a)$ arbitrarily.
while True **do**
  Initialize S and A
  while S' is not a terminal state **do**
    π = policy derived from Q (e.g. ε-greedy).
    Simulate $S', R \sim p(\cdot \mid S, A)$ and $A' := \pi(S')$.
    $Q(S, A) := Q(S, A) + \alpha_t(R + \gamma Q(S', A') - Q(S, a))$.
    $S := S', A := A'$
  end while
end while

(in orange, the difference with Q-learning).

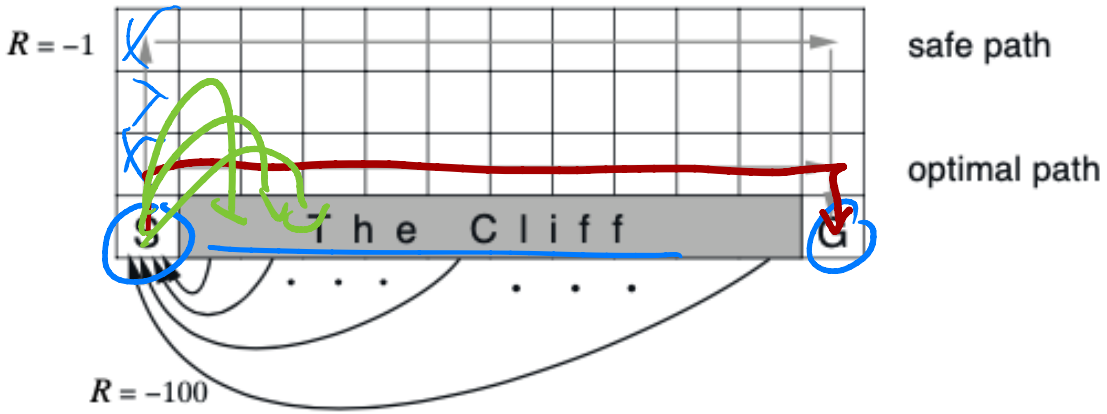
SARSA vs Q-learning



- Model is deterministic.
- Exploration policy (π) is ε-greedy.

SARSA or Q-learning: what will be the difference?

SARSA vs Q-learning



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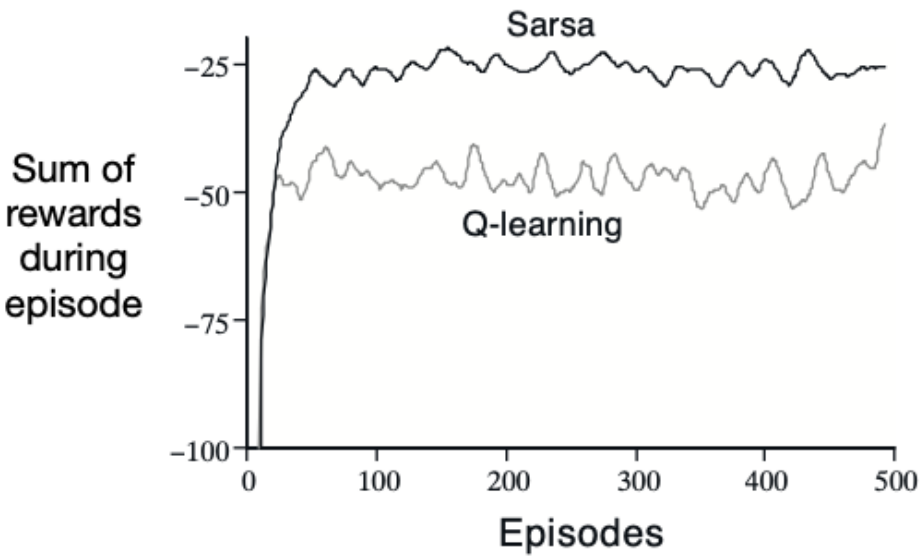

- For large ε, SARSA will avoid the optimal shortest path.
- Q-learning will learn the shortest path but will often fall.

How to choose the learning rate and guarantee exploration?

Recall: for Q learning, you are given an exploration policy π and apply:

$$A_{t+1} \sim \pi$$

$$Q(S_t, A_t) := Q(S_t, A_t) + \alpha_t \left(R_{t+1} + \gamma \max_{a \in \mathcal{A}} Q(S_{t+1}, a) - Q(S_t, A_t) \right).$$

Questions:

- How to choose π?
- How to choose α_t?

Solution: exploration/exploitation tradeoff (course 3), and Q-learning with UCB Exploration is Sample Efficient for Infinite-Horizon MDP by Dong et al 2019.

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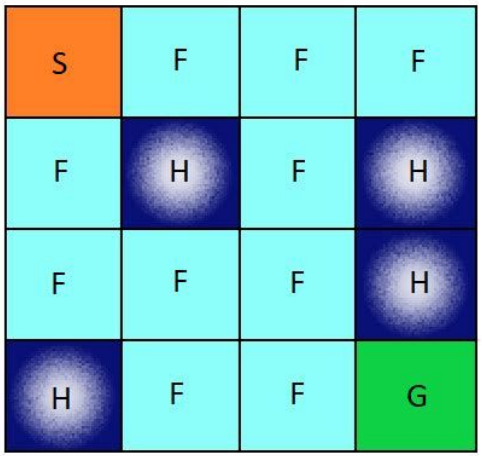
Important notions

(your job here)

TD and Q-learning are tabular method

They can be proven to converge.



S	$V(S)$
(0,0)	
(0,1)	
(0,2)	
(0,3)	
(1,0)	
(1,1)	
(1,2)	
(1,3)	
⋮	

S \ A	N	S	E	W
(0,0)				
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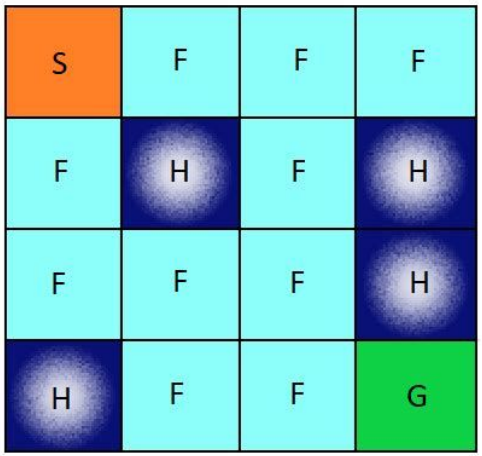

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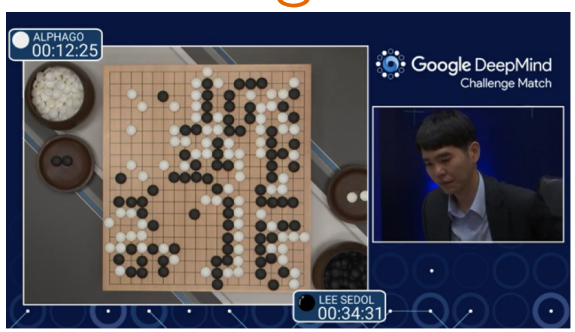
⋮	

S \ A	N	S	E	W
(0,0)				
(0,1)				
(0,2)				
(0,3)				
(1,0)				
(1,1)				
(1,2)				
⋮				

What about large state spaces?



Outline

Reminder: Tabular MDP

We want to find $Q(s,a) \approx Q^*(s,a)$.

$$\pi(s) = \arg\max_{a \in \mathcal{A}} Q(s,a).$$

Two types of methods:

- MC methods:

$$Q^\pi(s,a) = \frac{1}{K} \sum_{k=1}^{K} G^{(k)}$$

- TD methods (SARSA / Q-learning)

Reminder: Tabular MDP

We want to find $Q(s,a) \approx Q^*(s,a)$.

$$\pi(s) = \arg\max_{a \in \mathcal{A}} Q(s,a).$$

Does it scale?

The complexity is $\Omega(|\mathcal{S}||\mathcal{A}|)$.

$Q(s,a)$	a_1	a_2	a_3	...
s_1				
s_2				
s_3				
s_4				
$\vdots$				

Two types of methods:

- MC methods:

$$Q^\pi(s,a) = \frac{1}{K} \sum_{k=1}^{K} G^{(k)}$$

- TD methods (SARSA / Q-learning)

What are typical state space sizes?

The curse of dimensionality

Managing a portfolio of 10 types of product, with 100 product each max.

- $|\mathcal{S}| = 100^{10} = 10^{20}$.
- $\mathcal{A}$ = possible orders (=10×100?)

$\vec{s} = (\#\text{prod } 1, \#\text{prod } 2 \ldots \#\text{prod } 10)$

$\vec{s} = (s^1, s^2 \ldots s^{10})$

What are typical state space sizes?

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- $|\mathcal{S}| = 100^{10} = 10^{20}$.
- $\mathcal{A}$ = possible orders ($=10 \times 100$?)

Game of go

- $|\mathcal{S}| = 3^{19 \times 19}$ (19×19 board game).
- $|\mathcal{A}| = 19 \times 19$.

There are $\approx 10^{170}$ Q-values.

What are typical state space sizes?

The curse of dimensionality

Breakout (1976) ▸ Atari games

- $|\mathcal{S}| = 8^{84 \times 84}$ (84×84 screen, 8 colors).
- $|\mathcal{A}| = 2$ (left, right).

There are $\approx 10^{2000}$ Q-values.

What are typical state space sizes?

The curse of dimensionality

Breakout (1976) ▸ Atari games

- $|\mathcal{S}| = 8^{84\times 84}$ (84×84 screen, 8 colors).
- $|\mathcal{A}| = 2$ (left, right).

There are $\approx 10^{2000}$ Q-values.

Starcraft ▸ alphastar

- $|\mathcal{S}| \gg |\mathcal{A}| \approx +\infty$??

We need approximations.

Table of contents

TD-learning and function approximation

The tabular TD-learning or Q-learning algorithm is:

$$V(S_t) := V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$

$$Q(S_t, A_t) := Q(S_t, A_t) + \alpha \left(R_{t+1} + \gamma \max_{a \in \mathcal{A}} Q(S_{t+1}, a) - Q(S_t, A_t) \right).$$

This does not scale if $|\mathcal{S}|$ (or $|\mathcal{A}|$) are large.

Function approximation

We replace the exact Q-table (or value function V) by an approximation:

$$Q(S, A) \approx q_{\mathsf{w}}(S, A),$$

where w is a vector parameter to be found.

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- (classic): Use a linear approximation. For instance:

  $$Q(S, A) = \mathsf{w}^T \phi(s, a),$$

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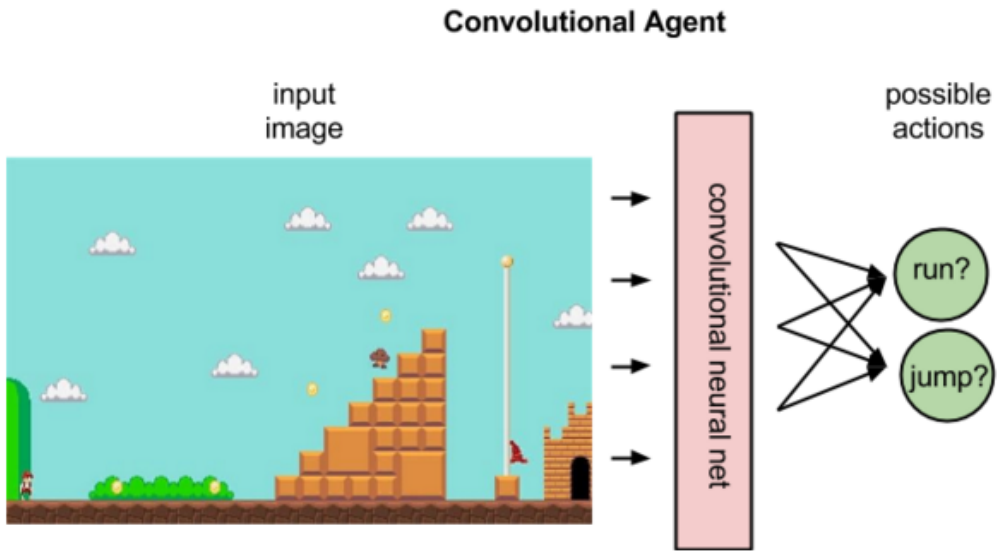
  where $\phi(s, a)$ is a feature vector.
- ("modern"): q_{w} is a deep neural network.



From Q-learning to deep Q-learning

The original Q-learning uses that:

$$Q(S_t, A_t) = \mathbb{E}\left[R_{t+1} + \max_{a \in \mathcal{A}} Q(S_{t+1}, a)\right].$$

We want to find w such that $\underbrace{q_{\mathsf{w}}(S_t, A_t)}_{\text{predictor}} \approx \underbrace{\mathbb{E}\left[R_{t+1} + \gamma \max_{a \in \mathcal{A}} q_{\mathsf{w}}(S_{t+1}, a)\right]}_{\text{target}}$.

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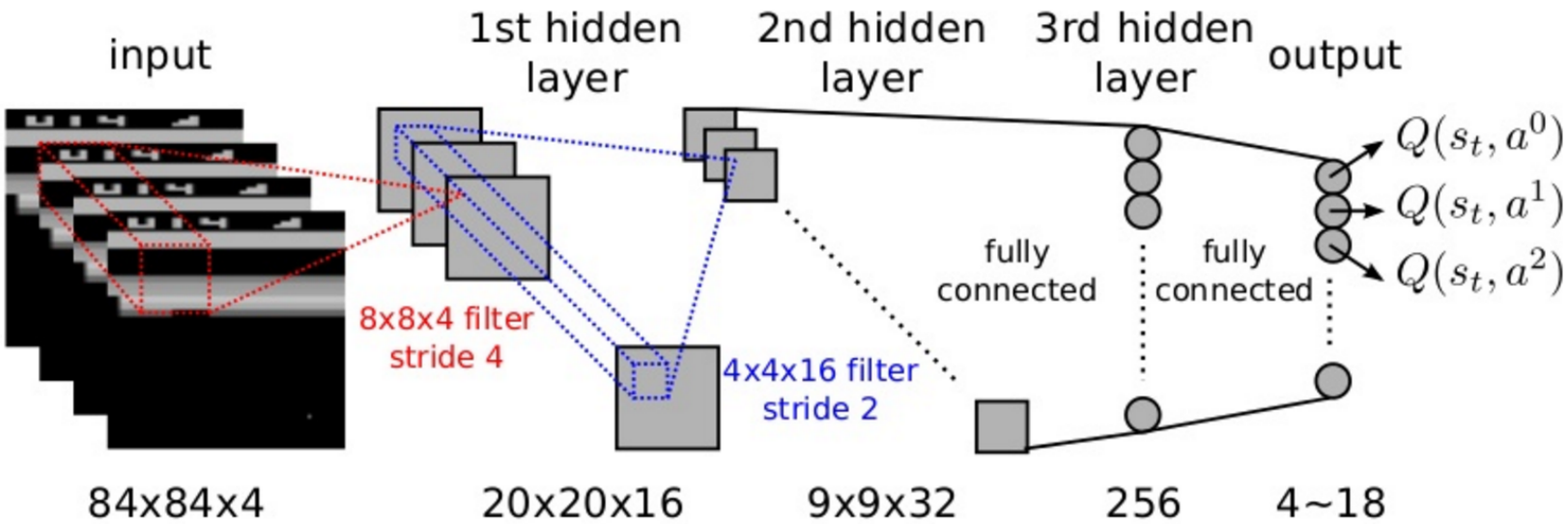
Deep Q-learning minimizes the L_2 norm and use gradient descent:

$$\mathsf{w} := \mathsf{w} + \alpha \left(R_{t+1} + \gamma \max_{a \in \mathcal{A}} q_{\mathsf{w}}(S_t, a) - q_{\mathsf{w}}(S_t, A_t)\right) \nabla_{\mathsf{w}}(q_{\mathsf{w}}(S_t, A_t)).$$

Example of breakout



Why is vanilla unstable?

We want to find w such that $\underbrace{q_{\mathsf{w}}(S_t, A_t)}_{\text{predictor}} \approx \underbrace{\mathbb{E}\left[R_{t+1} + \gamma \max_{a \in \mathcal{A}} q_{\mathsf{w}}(S_{t+1}, a)\right]}_{\text{target}}$.

For that, we do:

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Problems:

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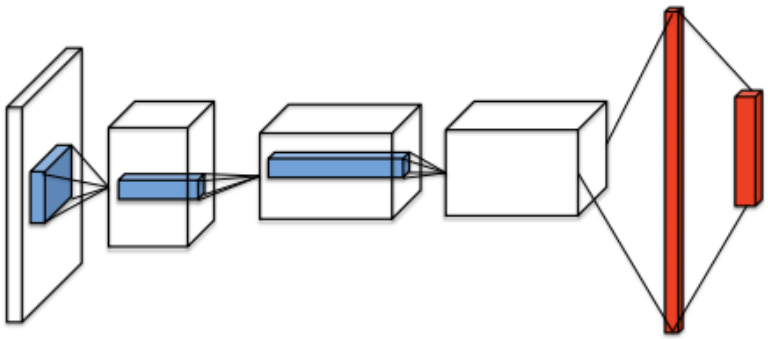
Problems:

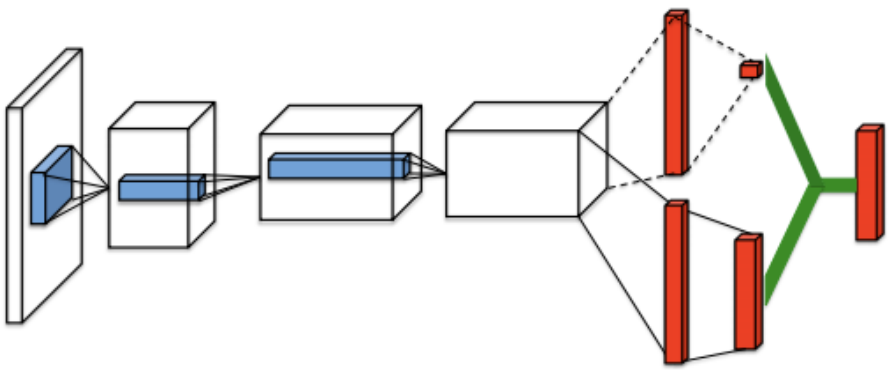
- Target and sources are highly correlated
- Target changes as we learn.
- Exploration is not guaranteed.

Learning algorithm can be unstable.

Possible solution: replay buffer or separate target network



Vanilla Q-learning uses a single network



DDQN uses a slow learning target network and a fast learning q-network.

Applications of Deep RL

- Resource management (energy)
- Computer vision and robotics
- Finance
- ...

Fundamental idea is simple but making the system stable and fast is an issue. Also, delayed actions or sparse rewards is difficult.

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Policy search

We are given a family of policies π_w parametrized by $w \in \mathcal{W}$. Typically:

$$\pi_w(a \mid s) \propto \exp(w^T \phi(s, a)),$$

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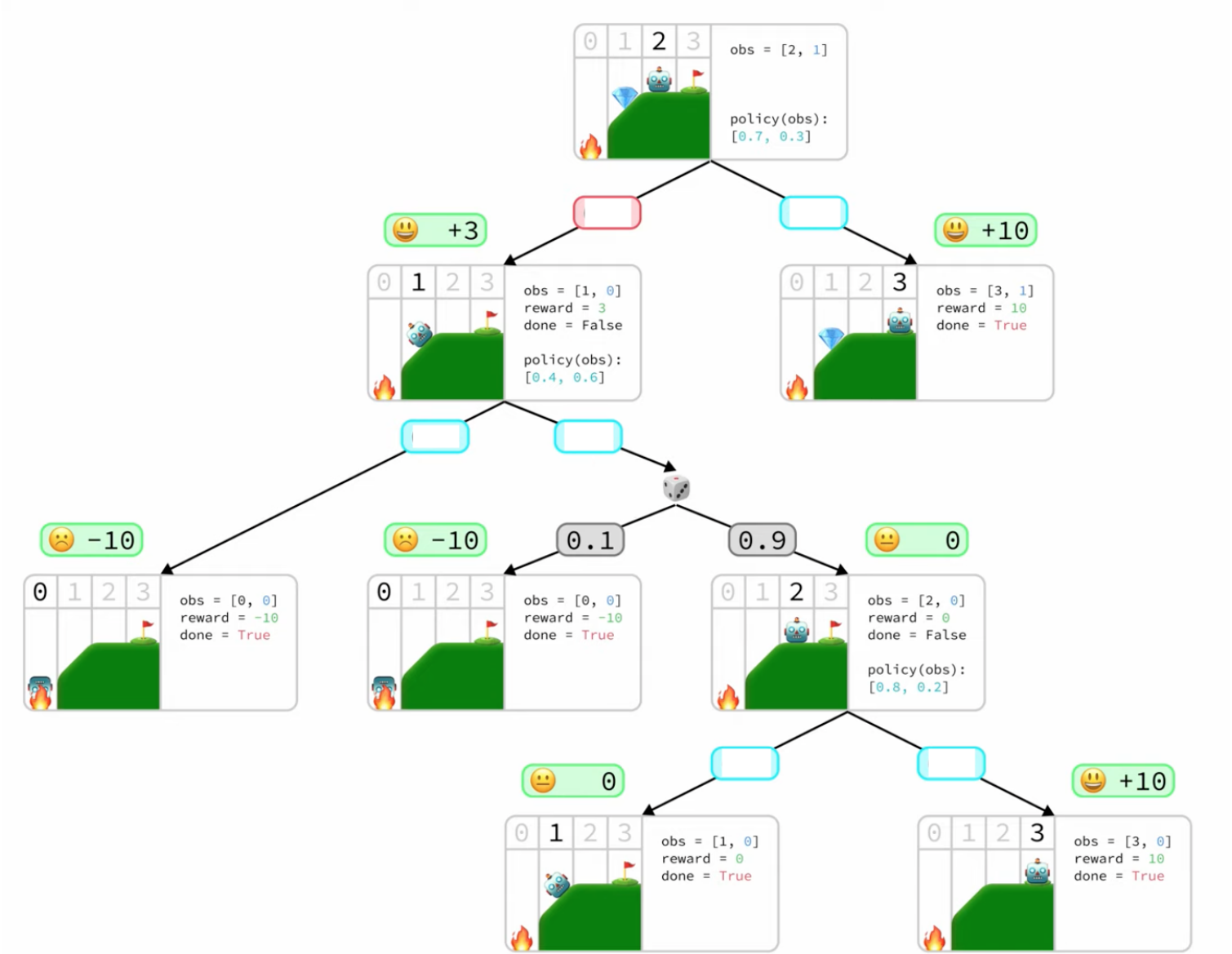
Let $J(w) := V^{\pi_w}(s_0)$ be its performance. We want to find w that maximizes $J(w)$.

- Sometimes, this works well with direct methods (brute-force)
- We can also use policy gradients:

$$w := w + \alpha \nabla_w J(w).$$

On an example https://www.youtube.com/watch?v=cQfOQcpYRzE



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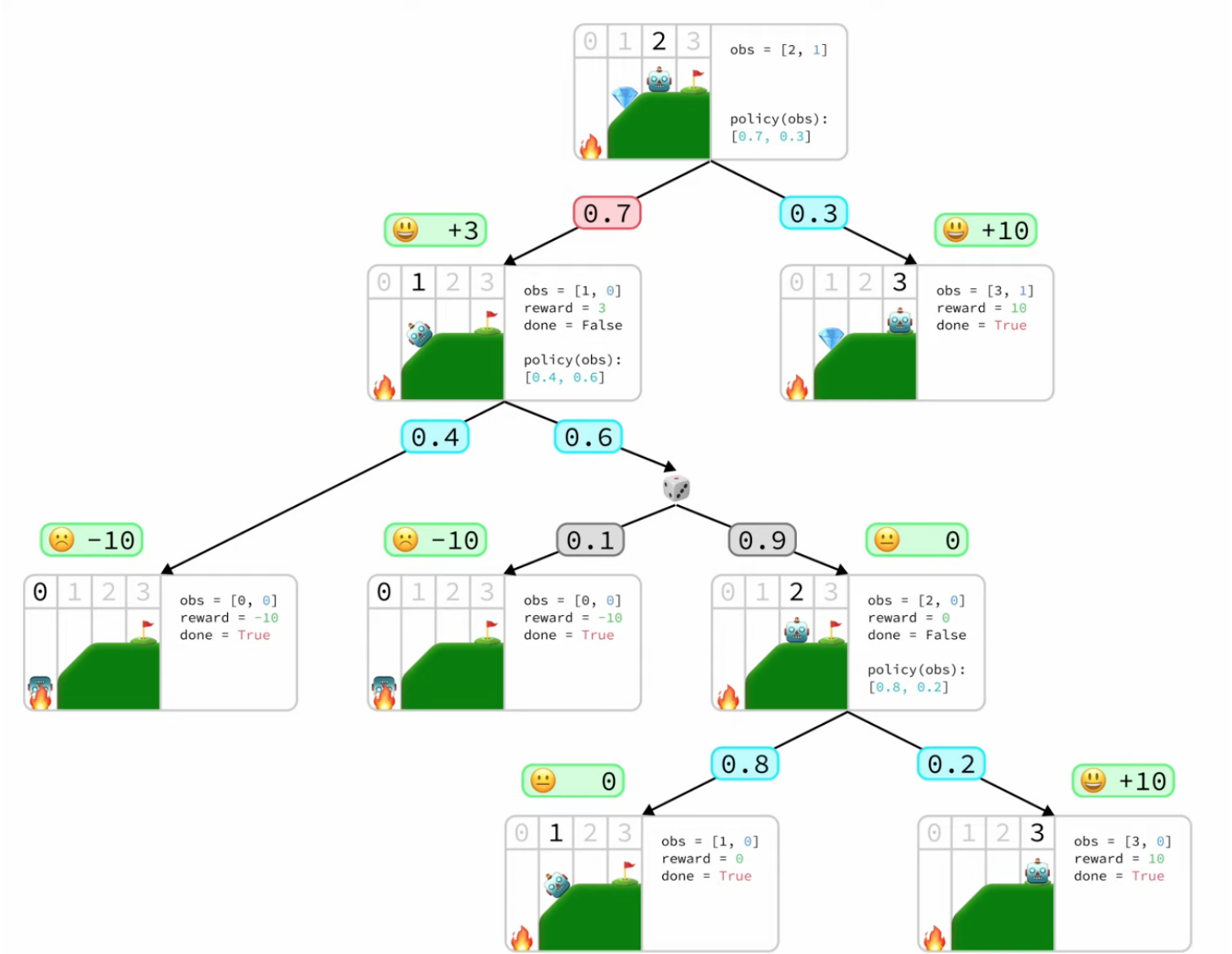

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```
Expected Return (G) =

    (0.7) * (3) +

    (0.3) * (10) +

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    (0.7 * 0.6 * 0.1) * (-10) +

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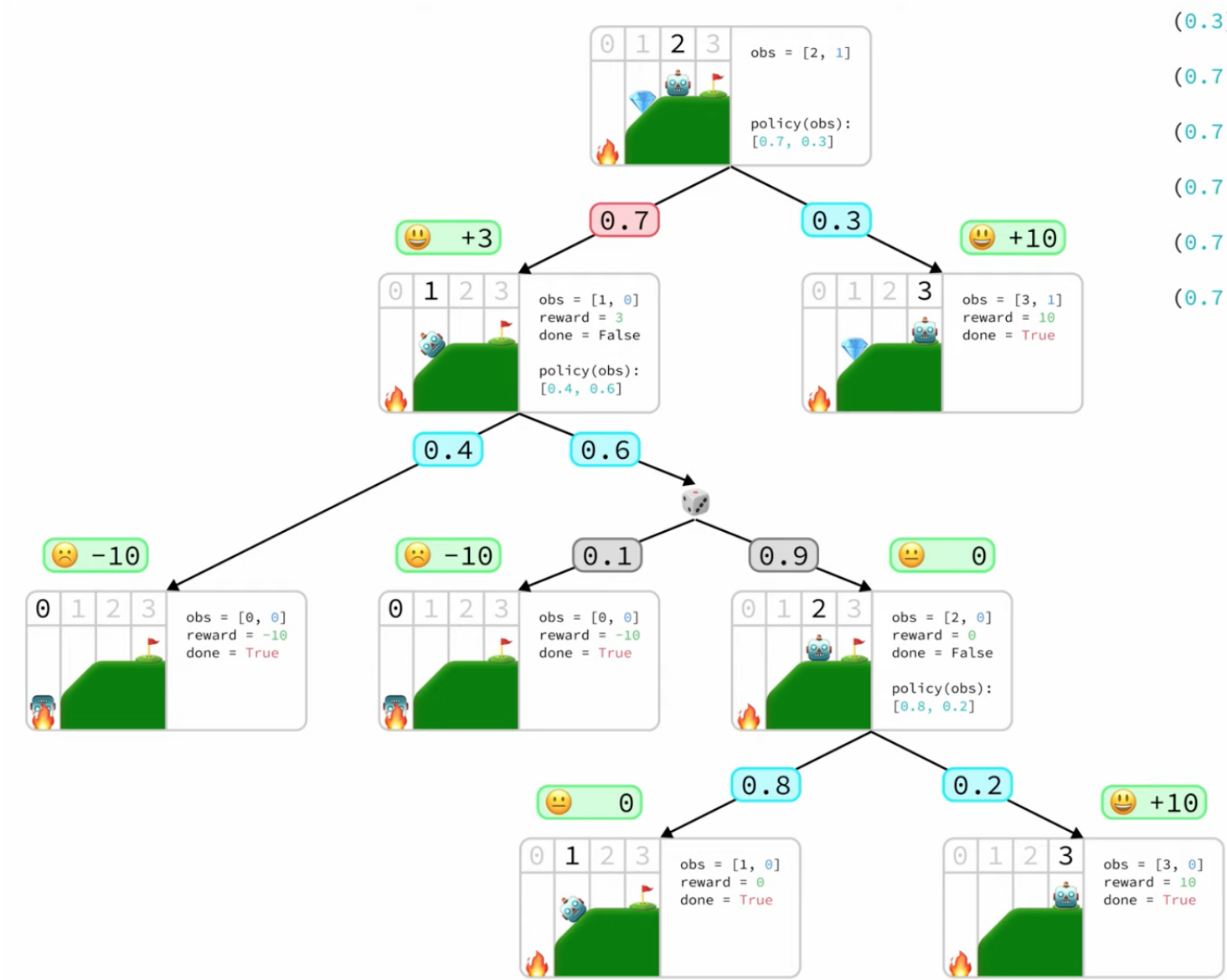

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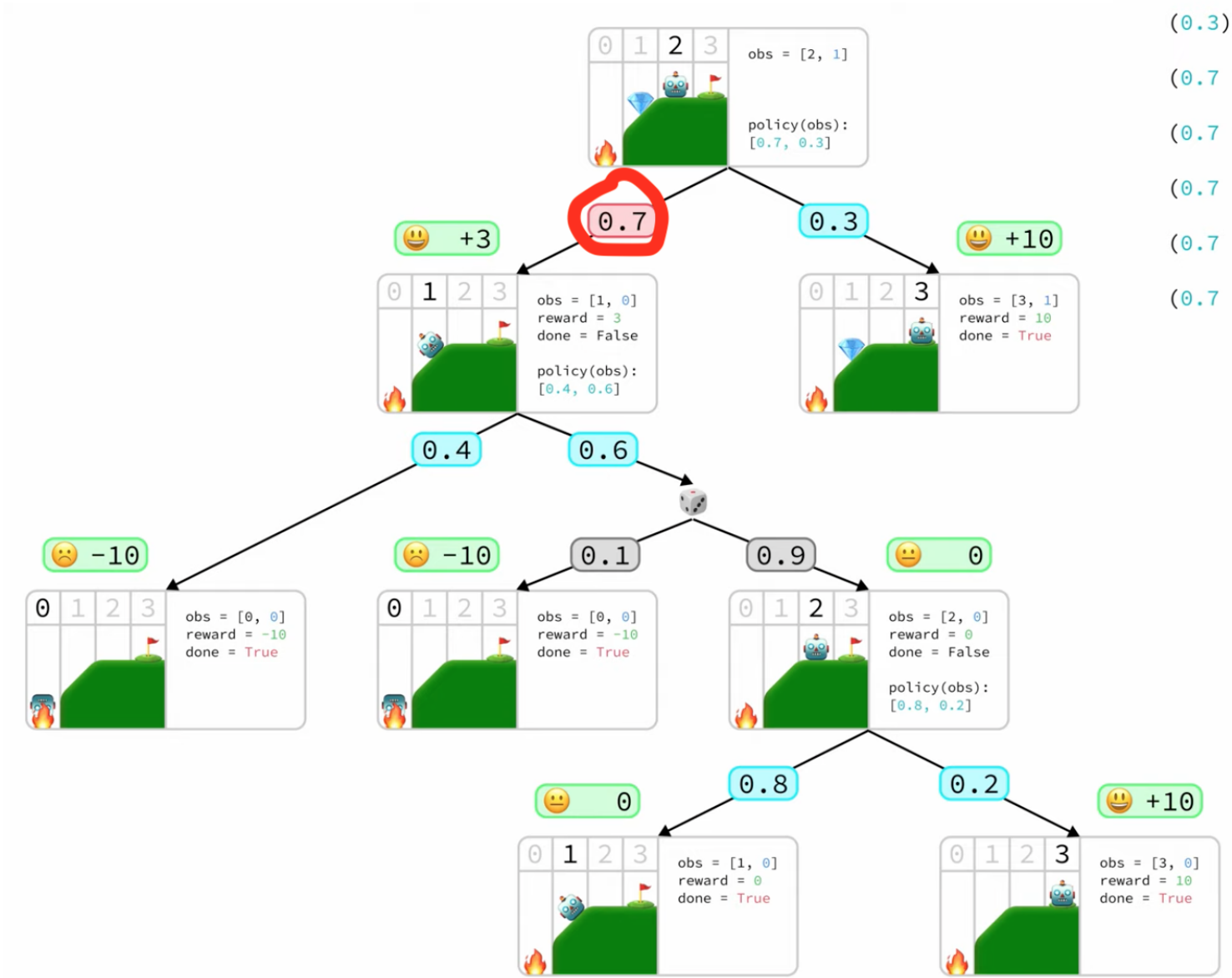

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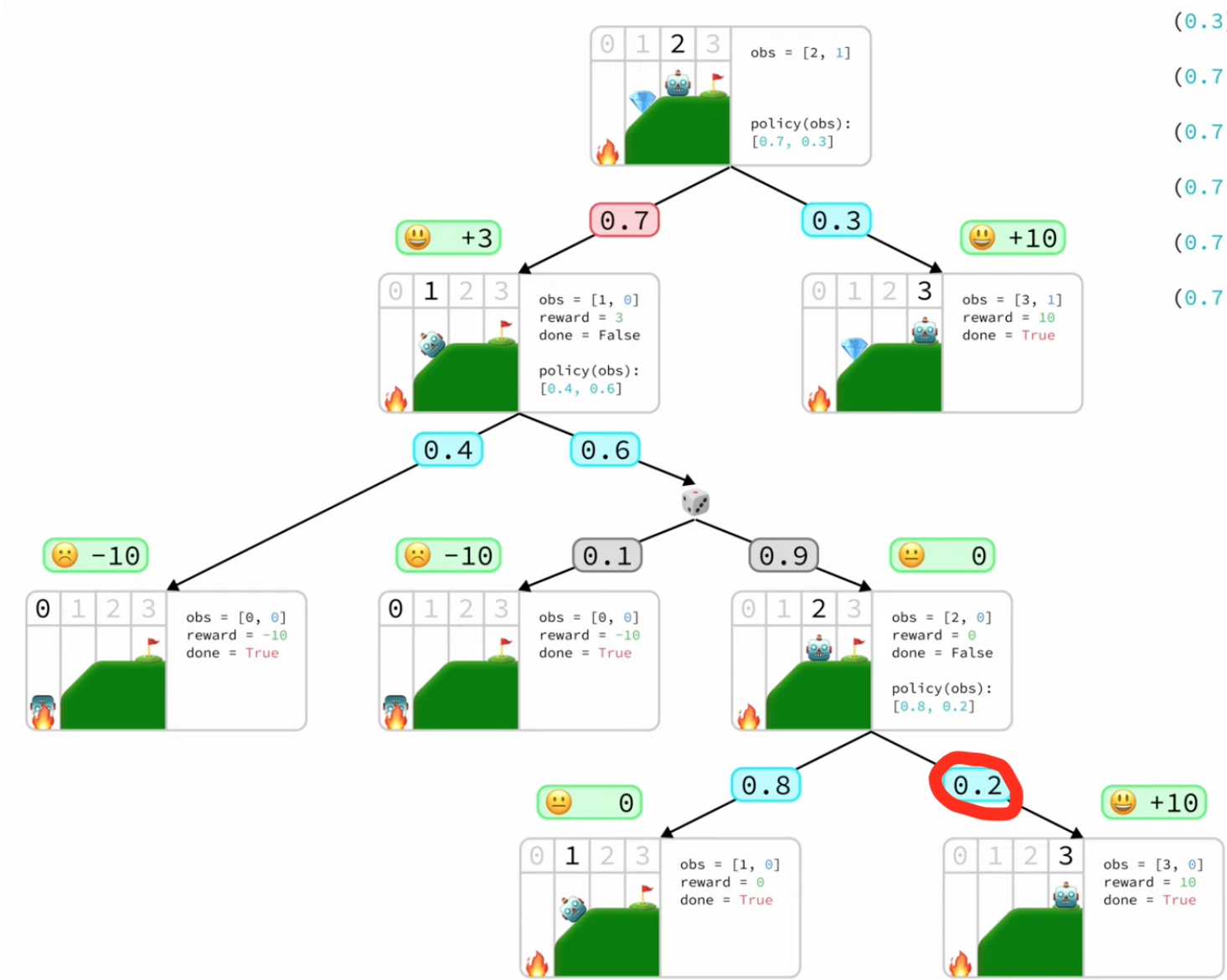

How to estimate the gradient with trajectories?

Assume for simplicity that each state is visited only once.
The probability of choosing a in state s is $\pi(a|s)$.

$$\begin{aligned}\nabla_{\pi(a|s)}\mathbb{E}\left[G_0\right] &= \mathbb{P}(\text{attaining } s)Q(s,a)\\ &= \frac{1}{\pi(a|s)}\mathbb{P}(\text{observing } (s,a))Q(s,a)\end{aligned}$$

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Algorithm: We want to compute $\text{gradient}(S,A) = \nabla_{\pi(a|s)}\mathbb{E}\left[G_0\right]$.

- Run a trajectory and observe S_t, A_t.
- For each t:

$$\widehat{gradient}(S_t,A_t) = \frac{1}{\pi(A_t|S_t)}G_t.$$

Theorem. For all s,a: $\mathbb{E}\left[\widehat{gradient}(s,a)\right] = \nabla_{\pi(a|s)}\mathbb{E}\left[G\right]$.

The policy gradient theorem

Assume that $\pi(a|s) = f_w(s, a)$. We have:

$$\nabla_w \mathbb{E}[G_0] = \sum_{s,a} \nabla_w \pi(a|s) \nabla_{\pi(a|s)} \mathbb{E}[G_0]$$

The policy gradient theorem

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$$\nabla_w \mathbb{E}[G_0] = \sum_{s,a} \nabla_w \pi(a|s) \nabla_{\pi(a|s)} \mathbb{E}[G_0]$$

Hence, an unbiased estimate of the gradient $\nabla_w \mathbb{E}[G_0]$ is

$$\sum_t \frac{(\nabla_w \pi(A_t|S_t))}{\pi(A_t|S_t)} G_t.$$

By using that $\nabla log(y) = \nabla(y)/y$, we get:

An unbiased estimate of the gradient is:

$$\nabla_w \mathbb{E}[G_0] = \mathbb{E}\left[\sum_t (\nabla_w \log \pi(A_t|S_t)) G_t\right].$$

Why is $\nabla \log \pi(a|s)$ easy to compute?

Reminder: if $p_i = e^{u_i} / \sum e^{u_j}$, then

$$\frac{\partial}{\partial u_j} \log p_i = 1_{\{i=j\}} - p_j.$$

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Reminder: if $p_i = e^{u_i} / \sum e^{u_j}$, then

$$\frac{\partial}{\partial u_j} \log p_i = 1_{\{i=j\}} - p_j.$$

If $\pi(a|s) \propto \exp(w^T \phi(s,a))$, then it means that $\pi(a|s) = \frac{\exp(w^T \phi(s,a))}{\sum_{a'} \exp(w^T \phi(s,a'))}$.

As a consequence:

$$\nabla_w \pi_w(a|s) = \phi(a,s) - \sum_{a'} \phi(a'|s) \pi_w(a'|s).$$

The REINFORCE algorithm

REINFORCE

1: Initialize w.
2: **while** True **do**
3: Simulate a trajectory (from $t = 1$ to T)
4: **for** $t = T$ to $t = 1$ **do**
5: $G_t := \sum_{t'=t}^{T} R_{t'}$.
6: $\nabla J := G_t \nabla \log \pi(A_t|S_t)$.
7: $\mathsf{w} := \mathsf{w} + \alpha \nabla J$.
8: **end for**
9: **end while**

Recall that $\nabla \log \pi(a|s)$ is easy to compute when $\pi(a|s) \propto w^T \phi(s, a)$.

Variance reduction

Problem: Monte-Carlo sampling can have a large variance.
Ex: if $Q(s, a_1) = 8 \pm 1$ and $Q(s, a_2) = 8.5 \pm 1$, is a_2 better than a_1?

Variance reduction

Problem: Monte-Carlo sampling can have a large variance.
Ex: if $Q(s, a_1) = 8 \pm 1$ and $Q(s, a_2) = 8.5 \pm 1$, is a_2 better than a_1?

Solution: add a baseline $h : \mathcal{S} \to \mathbb{R}$. Indeed, using the same log-trick:

$$\begin{aligned}\mathbb{E}\left[h(s_t)\nabla \log \pi(a_t|s_t)\right] &= \mathbb{E}\left[\sum_{a\in\mathcal{A}} h(s_t)\nabla\pi(a|s_t)\right] \\ &= 0\end{aligned}$$

This shows that for any function h, one has:

$$\nabla_w J(s_0) \propto \sum_t \mathbb{E}\left[(G_t - h(s_t))\nabla \log \pi(a_t|s_t)\right]\}.$$

Choosing a h close to G_t reduces the variance of the estimator.

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Classes of learning algorithms

We have seen two classes of RL methods:

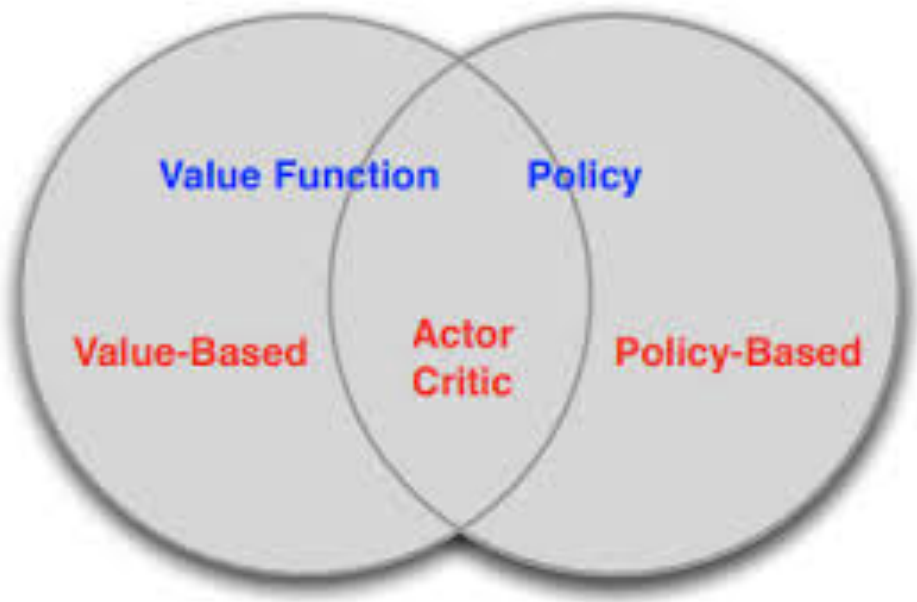
- Value-based (SARSA, Q-learning, Deep QL)
- Policy-based (Policy gradient, REINFORCE)

- Value-based learning can be unstable but uses samples efficiently.
- Policy-based tend to be more robust.

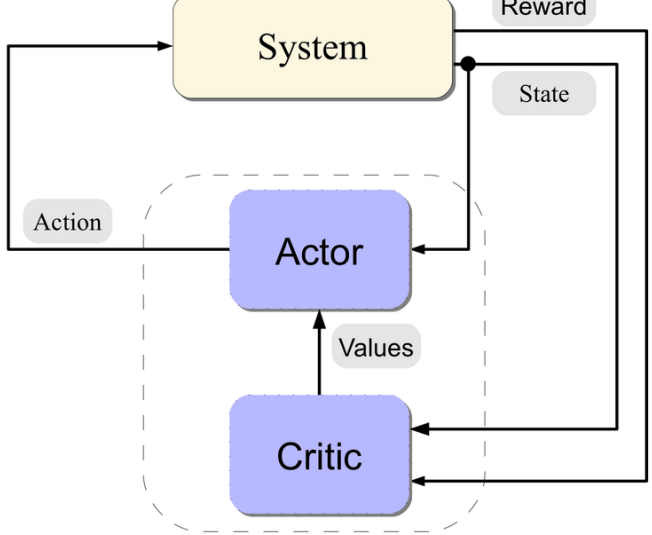
Classes of learning algorithms

We have seen two classes of RL methods:

- Value-based (SARSA, Q-learning, Deep QL) =Critic
- Policy-based (Policy gradient, REINFORCE) =Actor
- Value-based learning can be unstable but uses samples efficiently.
- Policy-based tend to be more robust.



Actor Critic method



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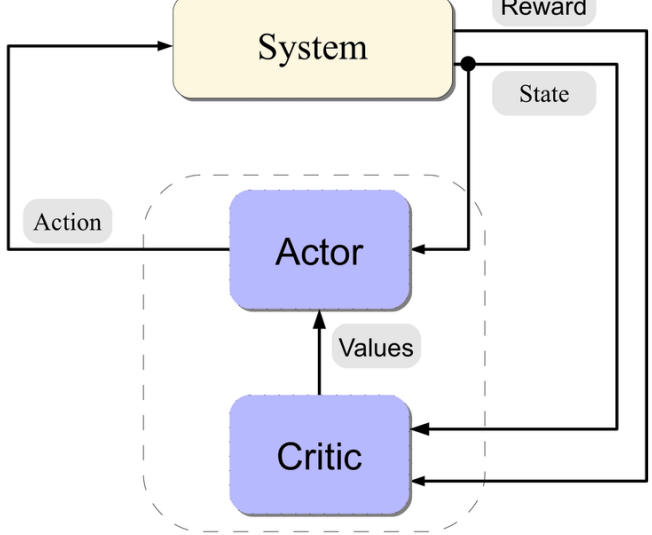


Basic Actor Critic

Initialize parameters $w^{(a)}$ (Actor) and $w^{(c)}$ (Critic)
while True **do**
  Initialize S
  for $t = 1$ to $t = T$ **do**
    $A_t \sim \pi_w(S)$ and simulate R, S'
    $w^{(c)} := w^{(c)} + \alpha^{(c)}(R + \gamma v_{w^{(c)}}(S') - v_{w^{(c)}}(S))$ *# TD-update*
    $w^{(a)} := w^{(a)} + \alpha^{(a)} v_{w^{(c)}}(S) \nabla \log \pi(a_t|s_t)$ *# Policy-gradient*
    S:=S'.
  end for
end while

Going further

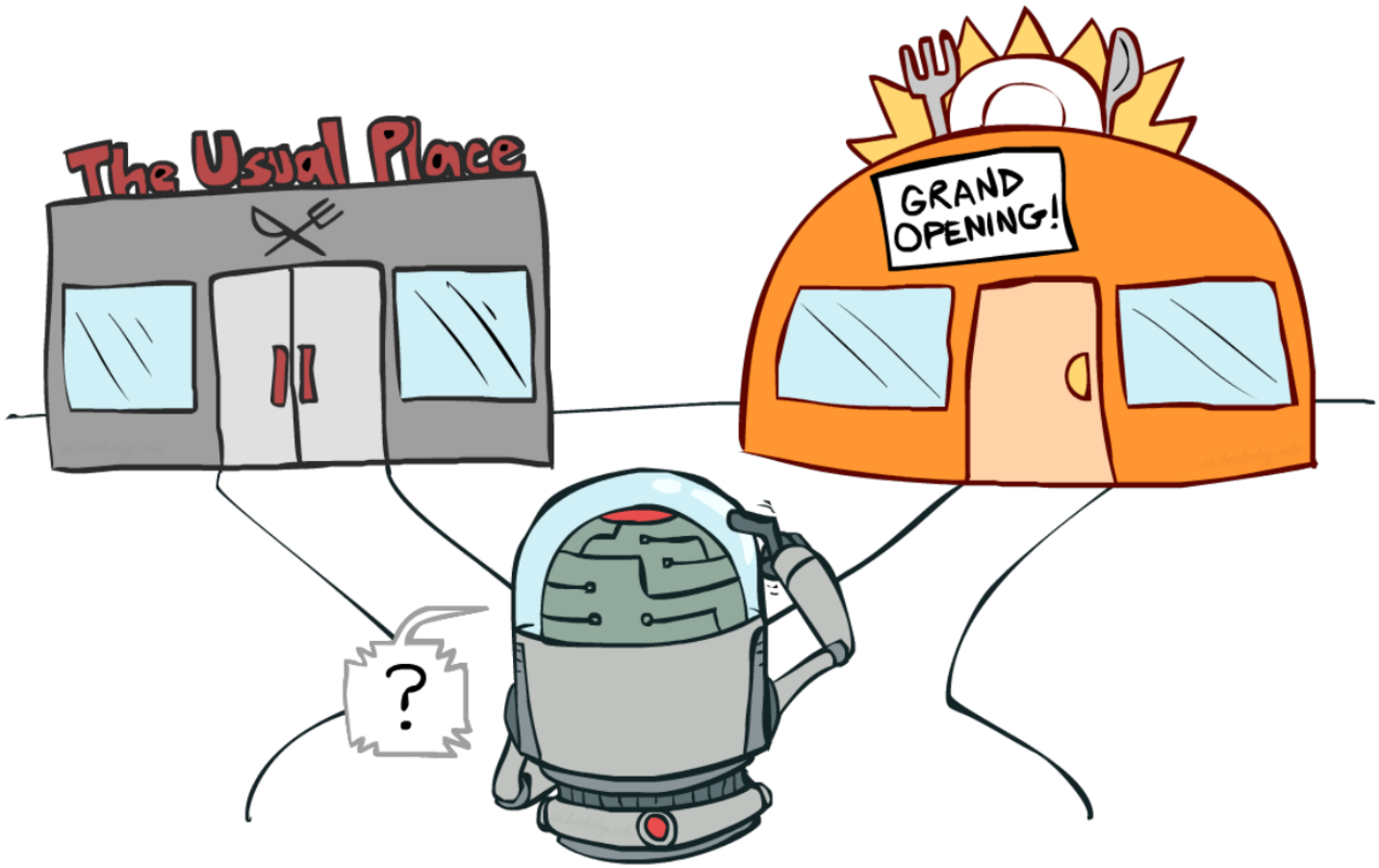
Extra-reading:

- Introduction to Reinforcement Learning (Sutton-Barto, 2018 last ed.)
- Algorithms for Reinforcement Learning (Szepesvari, 2010)
- Deep Reinforcement learning: hands on (Maxim Lapan, 2020)

Next course: some thoughts on exploration / exploitation.

Outline

Reminder: exploration-exploitation dilemma and bandits



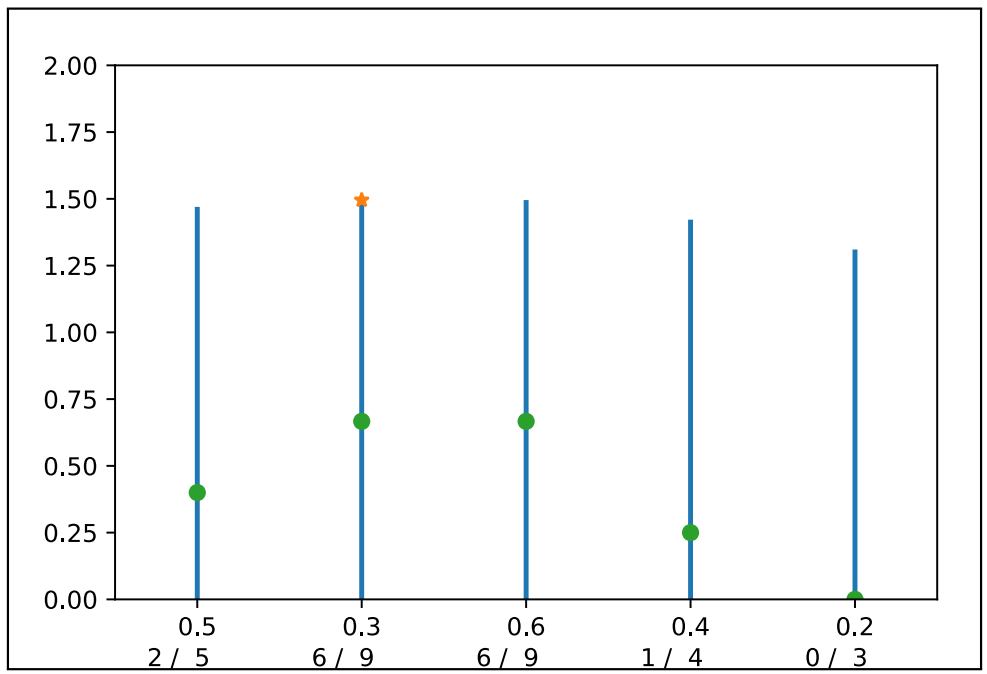
- How useful is this for RL?

Reminder: UCB algorithm

UCB computes a confidence bound $UCB_a(t)$ such that $\mu_a(t) \leq UCB_a(t)$ with high probability. Example : $UCB1$ [Auer et al. 02] uses

$$UCB_a(t) = \hat{\mu}_a(t) + \sqrt{\frac{\alpha \log t}{2N_a(t)}}.$$

- Choose $A_{t+1} \in \arg\max_{a \in \{1 \dots n\}} UCB_a(t)$ (optimism principle).



Can we use optimism for MDPs?

Observe the empirical means $\hat{R}(s,a)$ and $\hat{P}(s' \mid s,a)$.

What bonus should one use?

Can we use optimism for MDPs?

Observe the empirical means $\hat{R}(s,a)$ and $\hat{P}(s' \mid s,a)$.

What bonus should one use?

- UCRL2 (Jaksch 2010) or variant: use bonus on R and P. Let $\delta(s,a) = C\sqrt{t/N_t(s,a)}$ where $N_t(s,a)$ is the number of time that you took action a in state s before time t.

$$\mathcal{R} = \{\text{vector } r \text{ such that for all } s,a\text{: } |r(s,a) - \hat{r}(s,a)| \leq \delta(s,a)\}$$

$$\mathcal{P} = \{\text{trans. matrix } P \text{ s.t. for all } s,a,a' \left| P(s,a,a') - \hat{P}(s,a,a') \right| \leq \delta(s,a$$

Optimism:

  - Apply π that maximizes $V^{\pi}_{r,P\in\mathcal{R},\mathcal{P}}$ (by using extended value iteration) and re-update the policy periodically.

Tree search

For turn-based two players zero sum games

From a given position, takes the best decision.

- Generate a tree of possibilities.
- Explore this tree.

What if the tree is too big?

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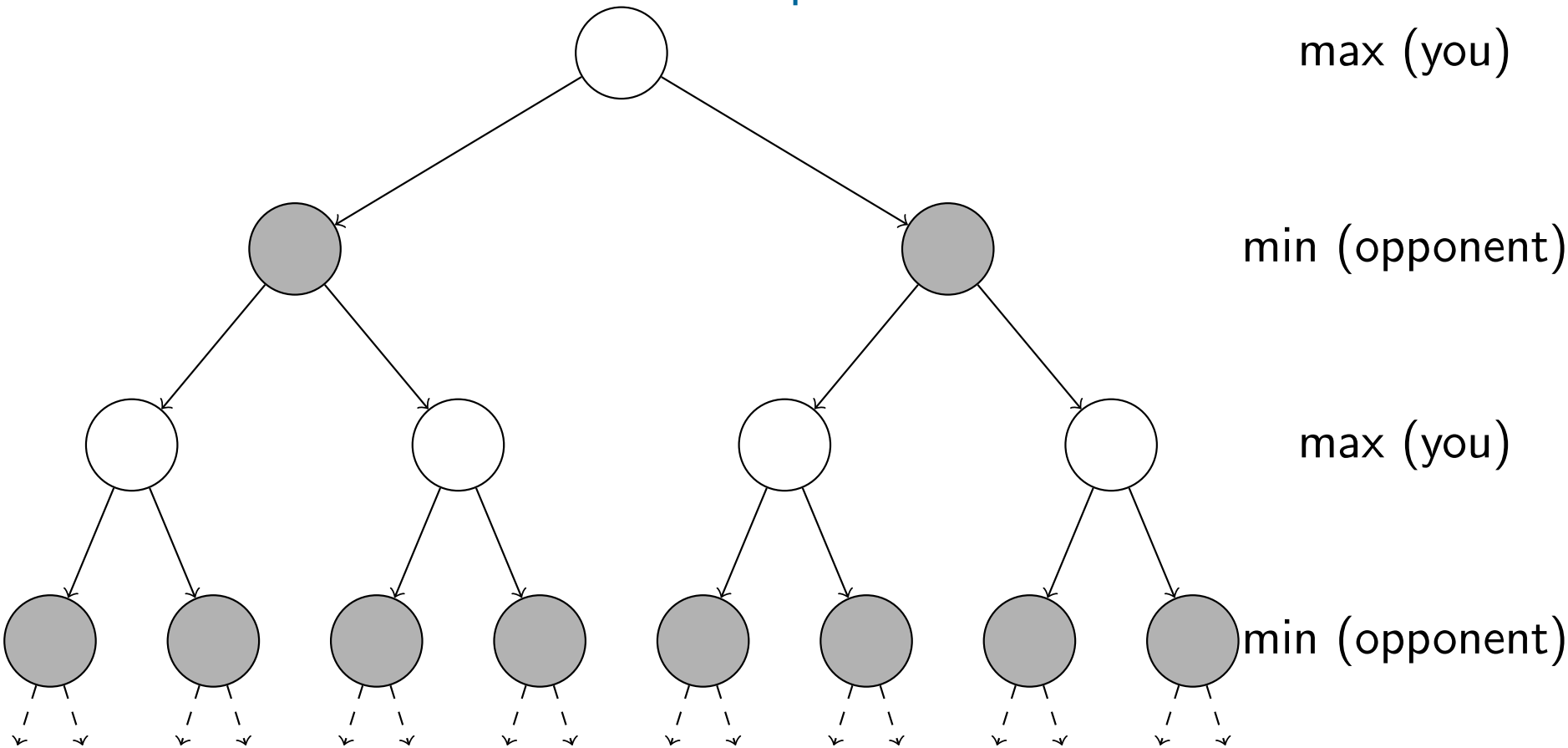
You can construct the tree of possibilities



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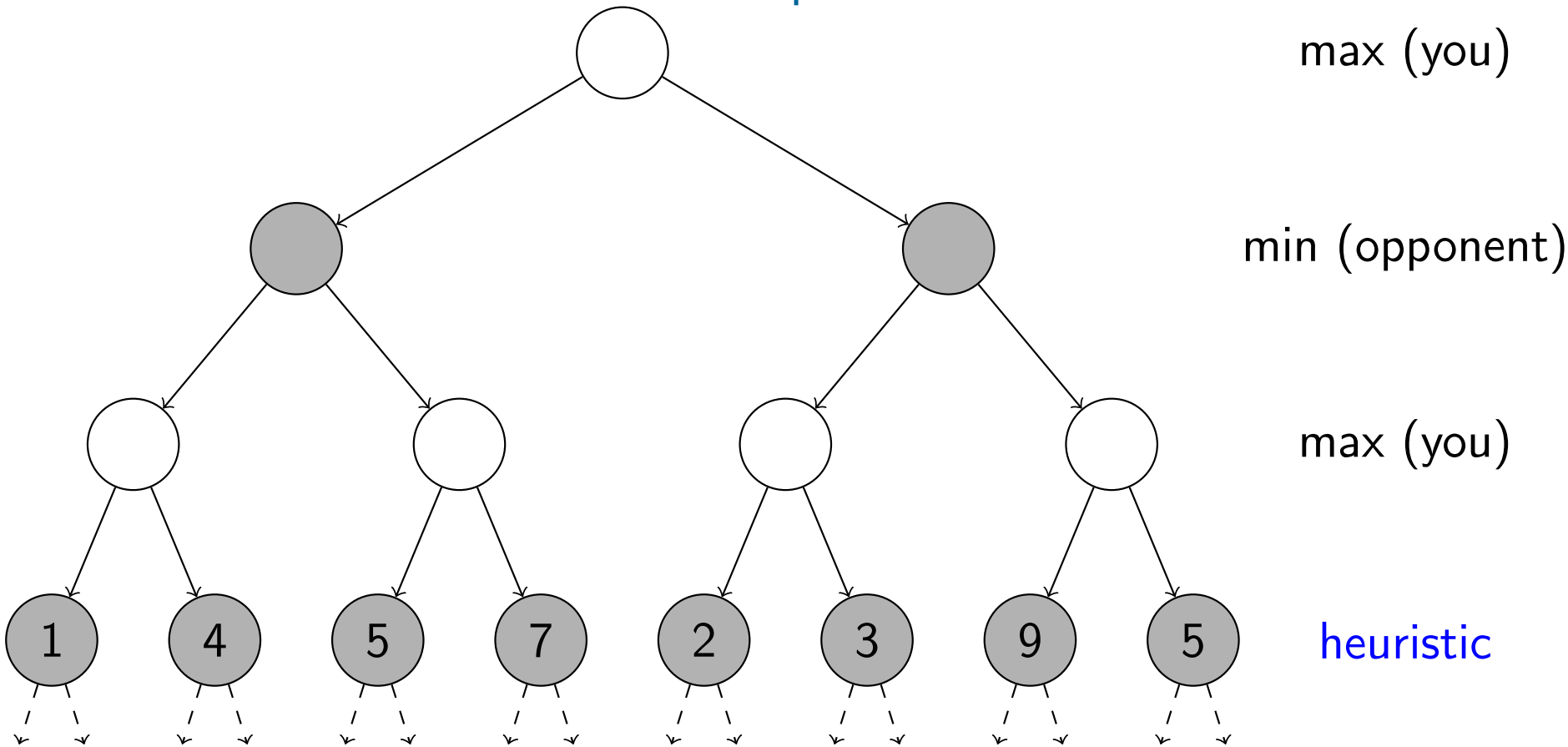
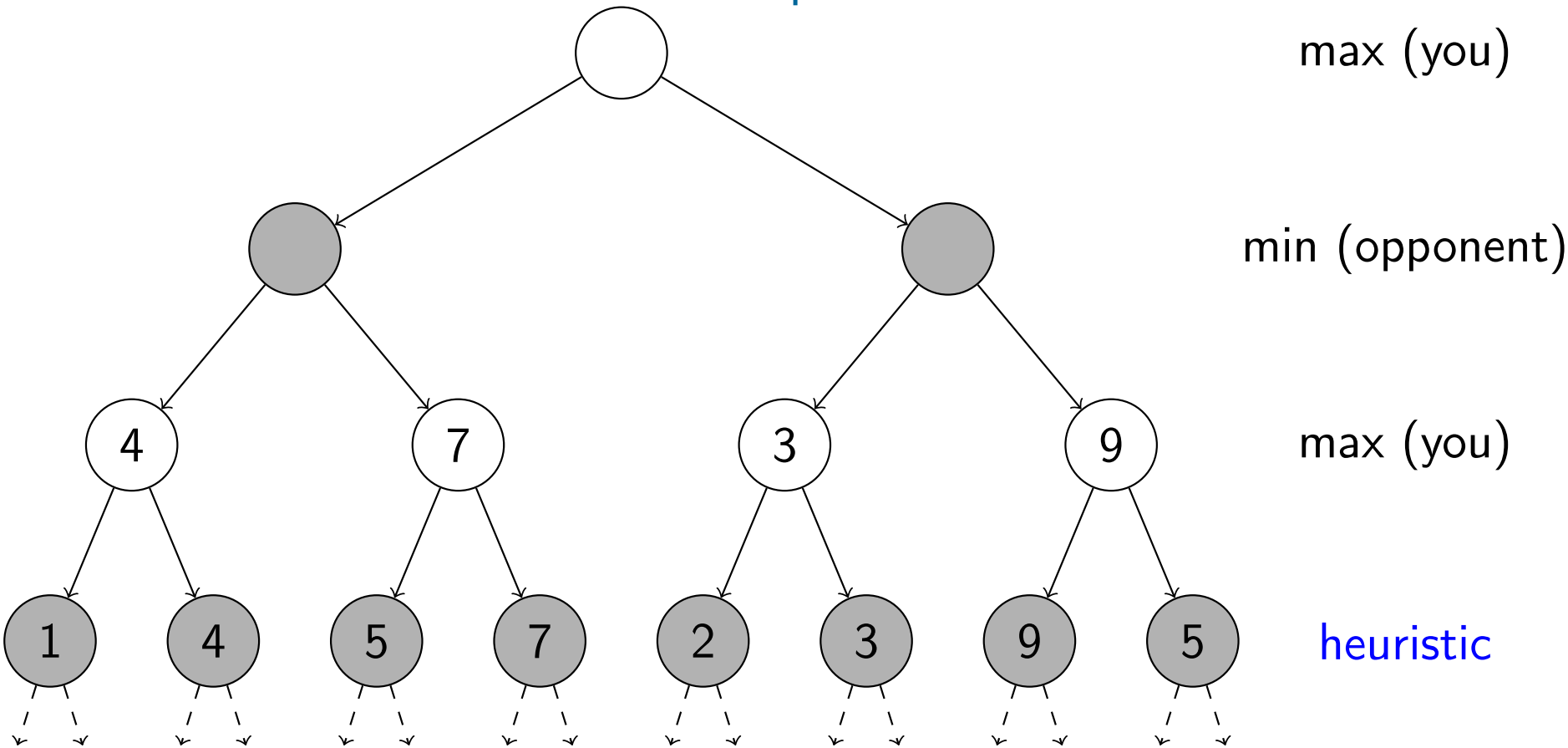


If the tree is two big, you stop at depth D and use a heuristic.

- You can backtrack with the min-max algorithm.

You can construct the tree of possibilities



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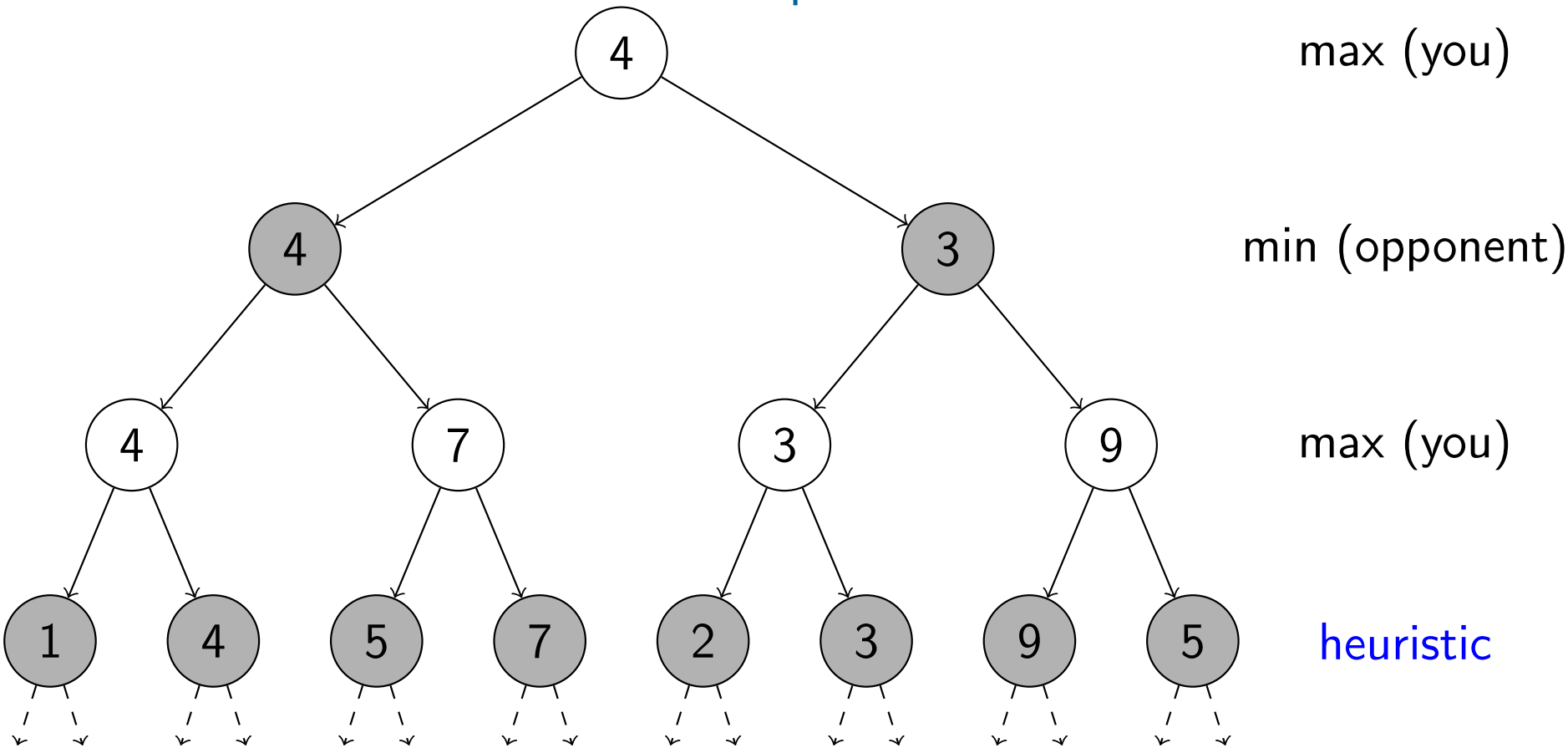

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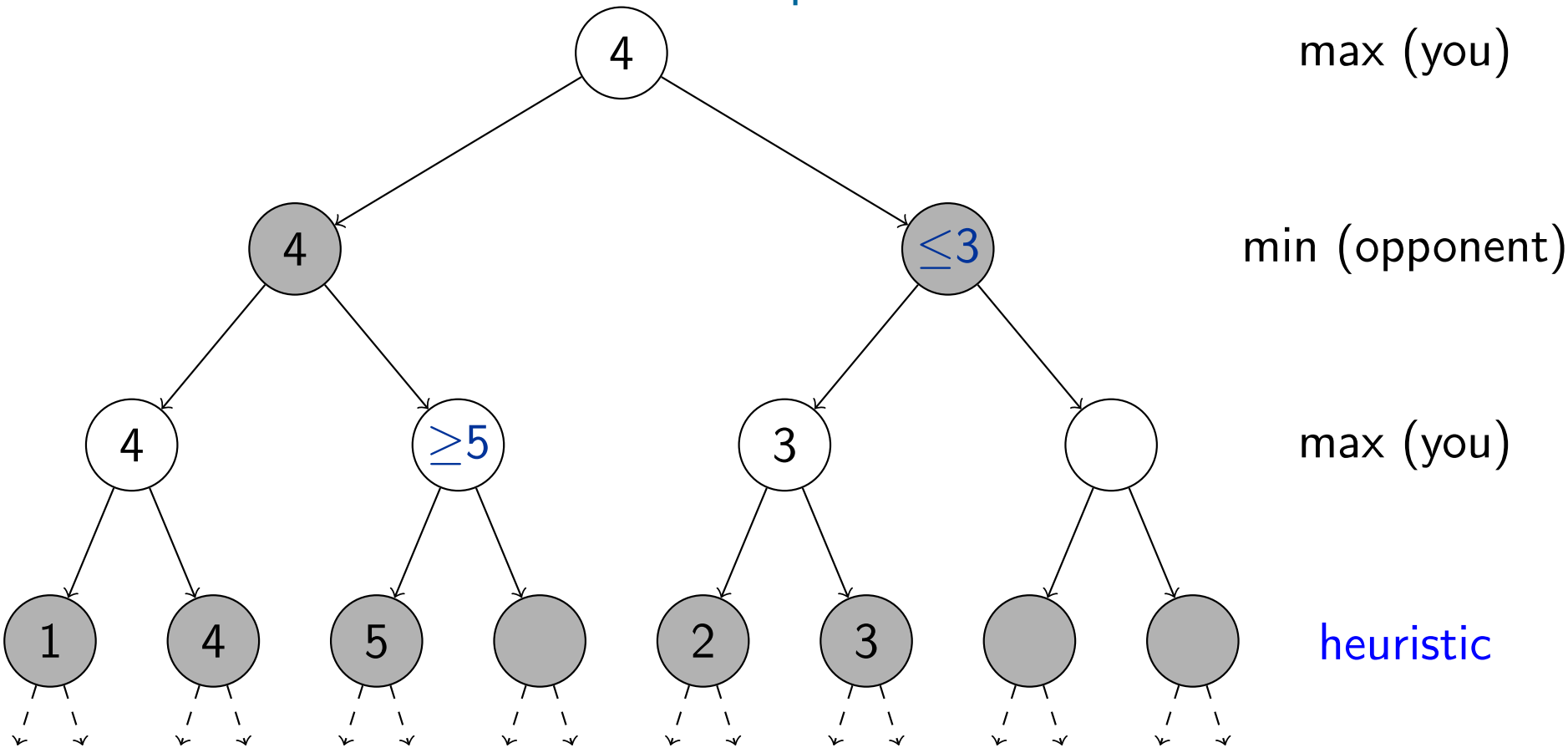

If the tree is two big, you stop at depth D and use a heuristic.

- You can backtrack with the min-max algorithm.
- For optimization, you can use alpha-beta pruning.

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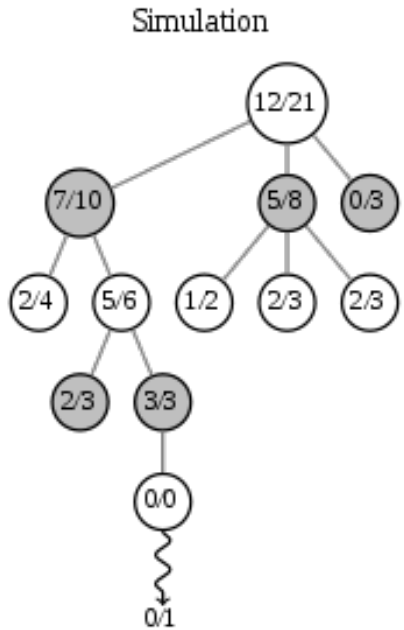
Min-max and alpha-beta perform well (ex: Chess)...

...but can be limited (ex: go)

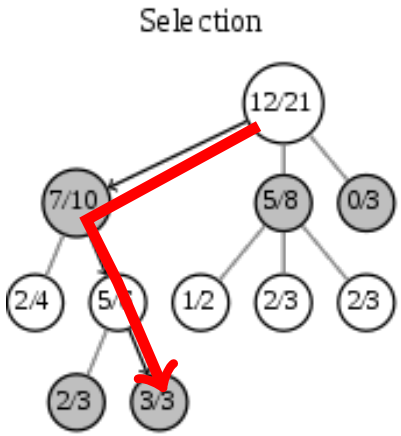
- Tree can still be very big (A^D)
- You need a good heuristic.
  - Result is only available at the end
- You might want to avoid the exploration of not promising parts.
  - For that you need a good heuristic.

MCTS (Monte Carlo Tree Search) uses simulation to conduct the tree search



- Simulate many games and compute how many were won.
- Explore carefully which actions were best.

MCTS (Monte Carlo Tree Search) uses simulation to conduct the tree search

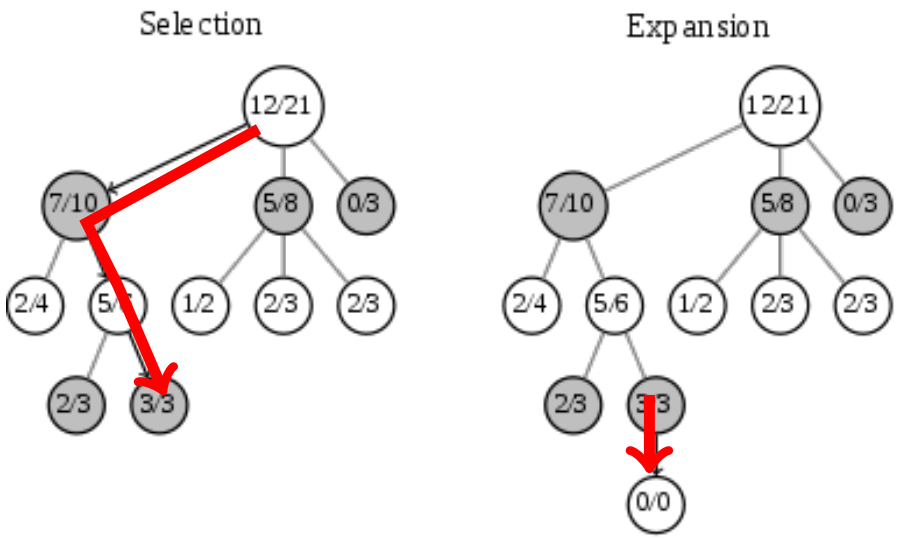


For each child, let $S(c)$ be the number of success and $N(c)$ be the number of time you played c, and $t = \sum_{c'} N(c')$.

- Explore $\arg\max_c \frac{S(c)}{N(c)} + 2\sqrt{\frac{\log t}{N(c)}}$.

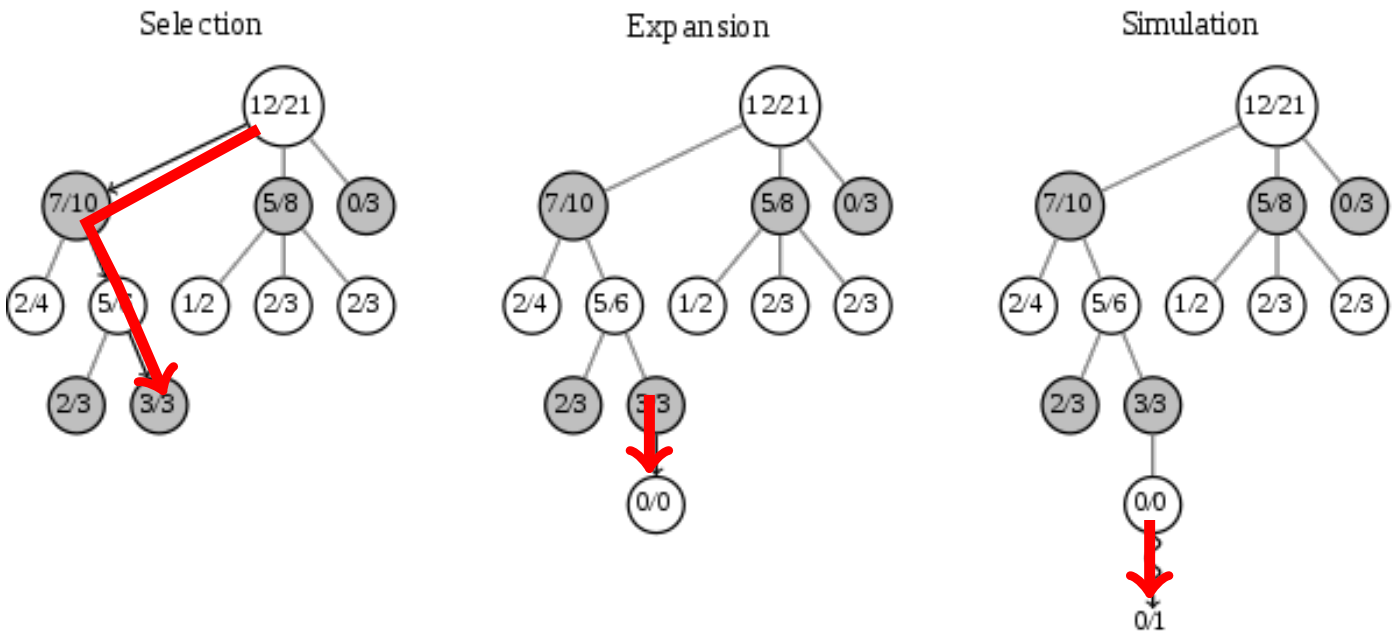
Open question: no guarantee with $\sqrt{\log t / N(c)}$. Is $\sqrt{t}/N(c)$ better?

MCTS (Monte Carlo Tree Search) uses simulation to conduct the tree search



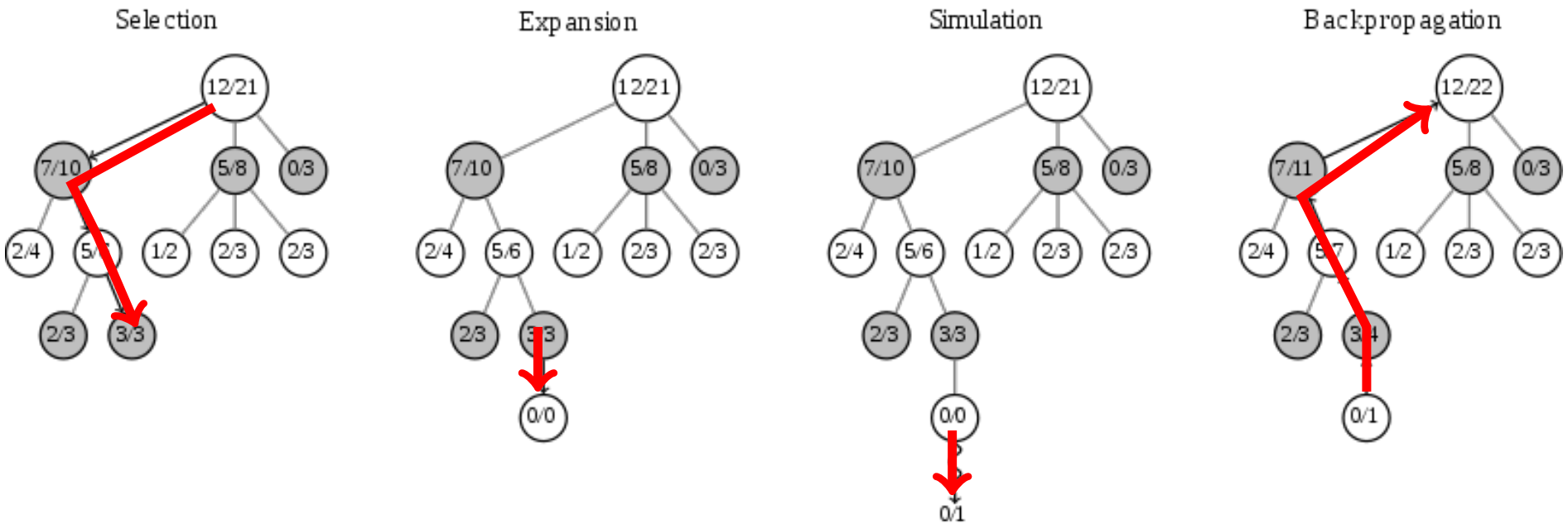
- Create one or multiple children of the leaf.

MCTS (Monte Carlo Tree Search) uses simulation to conduct the tree search



- Obtain a value of the node (e.g. rollout)

MCTS (Monte Carlo Tree Search) uses simulation to conduct the tree search



Rollout policy
(ex: random)

- Backpropagate to the root

MCTS algorithm

MCTS

1: **while** Some time is left **do**
2: Select a leaf node *#UCB-like*
3: Expand a leaf
4: Use rollout (or equivalent) to estimate the leaf *#random sampling*
5: Backpropagate to the root
6: **end while**
7: Return $\arg\max_{c \in \text{children(root)}} N(c)$ *#or* $S(c)/N(c)$.

Demo / exercice

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Conclusion

Exploration v.s. exploitation is central in RL

- Bandits and regret help formalizing this idea.
- One important notion is the use of optimism to force exploration.
  - Bayesian sampling can also be used
- Theoretical tools guide practical implementations.