Restless Bandits, Weakly Coupled MDPs and LP Relaxations

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Inria

Data-driven Queueing Challenges – September, 2022

Motivation problem 1: Applicant screening problem

N applicants, T rounds of interview.
Each round: you can interview up to αN candidates.
Goal: maximize the expected quality of candidates.

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Each candidate has an (unknown) quality p_n.

- Result of an interview: Bernoulli(p_n)

Goal: find the βN highest p_n.

Possible heuristics:

- Greedy (exploitation)?
- Random (exploration)?

Motivation problem 2: What to work on?

Job Scheduling

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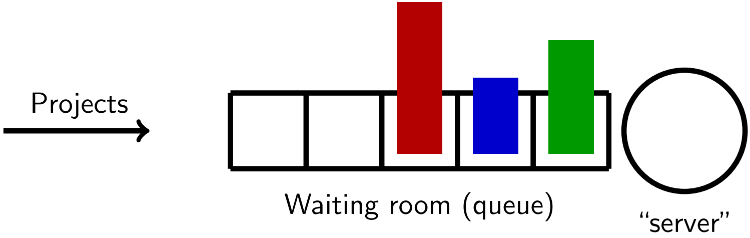

- Examples: research projects, tasks allocations, electric vehicle charging, wireless scheduling,...
- We allow preemption (preempt-resume).

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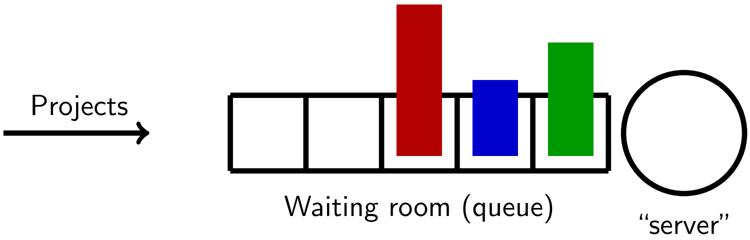

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Possible heuristic: SRPT ((Shortest Remaining Processing Time).)

- "Strongly optimal" [Schrage, 1966] if you know the project durations and you want to minimize the waiting time

Can we do better?

These two problems are restless bandit problems.

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We construct policies by assuming independence.

- Asymptotically optimal policies.
- LP-based (= computationally efficient)

Outline

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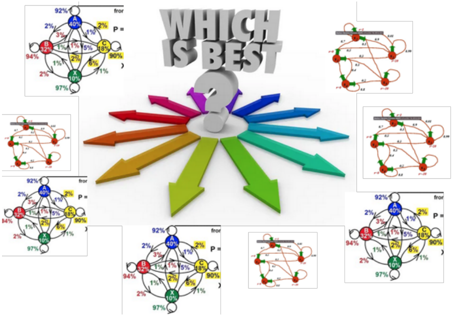
Finite-horizon restless bandits

Subsidy, infinite-horizon and index policies

How to Compute Indices: A Sub-Cubic Algorithm

Conclusion

Reslless Markov bandit problem



A decision maker faces N arms.
Time is discrete.

- Each arm is a 2-action MDP (passive / active)
- Controller can *activate* $\alpha N < N$ arms each time.

Policy : you observe the states. Which ones do you activate?

For simplicity, we consider statistically identical bandits

An arm is a 2-action MDP: if in state s:

- Activation: earn $r(s, active)$, jump to $s' \sim \mathbf{P}\left[s' \mid s, active\right]$.
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The N arms are *statistically identical* and we denote by:

$$X_s^{(N)}(t) = \frac{1}{N}\{\#\text{ arms in state } s \text{ at time } t\}.$$

$$Y_{s,a}^{(N)}(t) = \frac{1}{N}\{\#\text{arms in state } s \text{ for which action } a \text{ is taken at } t\}.$$

Restless bandits are difficult to solve

A admissible policy is a sequence of functions $\pi_t : \mathcal{X} \to \mathcal{Y}$ such that $Y^{(N)}(t) = \pi_t(X^{(N)}(t))$ is feasible with respect to $X^{(N)}(t)$, and

$$\sum_s a Y_{s,a}^{(N)}(t) \leq \alpha \qquad\qquad (\textit{activation constraint})$$

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Theorem (Papadimitriou, Tsitsiklis 1994)

Finding the best admissible policy is PSPACE-complete.

(harder than NP-hard).

The LP-relaxation

Replace the activation constraint $\sum_s a Y_{s,a}^{(N)}(t) \leq \alpha$ by

$$\sum_s a \mathbb{E}\left[Y_{s,a}^{(N)}(t)\right] \leq \alpha.$$

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Lemma

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Lemma

Finding the optimal allocation can be solved by an LP.

Proof. Let $y_{s,a}(t) := \mathbb{E}\left[Y_{s,a}^{(N)}(t)\right]$ and $x_s(t) = \mathbb{E}\left[X_s^{(N)}(t)\right]$.

$$\begin{aligned}
\max \sum_{s,a,t} & y_{s,a}(t) r_{s,a} \\
\text{s.t.} \quad & \sum_a a\, y_{s,a}(t) \leq \alpha N && \forall t \\
& \sum_{s,a} y_{s,a}(t) p(s' \mid s, a) = x_{s'}(t+1) && \forall t, s' \\
& \sum_a y_{s',a}(t) = x_{s'}(t).
\end{aligned}$$

How can we use the LP to build a policy for the original problem?

$\sum_s a Y_{s,a}^{(N)}(t) \leq \alpha$		$\sum_s a \mathbb{E}\left[Y_{s,a}^{(N)}(t) \right] \leq \alpha$
Original problem (Hard)	$\longrightarrow$	LP relaxation (Easy)
V_N^*	$\leq$	V_{rel}^*

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$$V_N^* \leq V_{rel}^*$$

$$V_{rel}^* \downarrow x^*, y^*$$

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Original problem (Hard) $\longrightarrow$ LP relaxation (Easy)

$$V_N^* \le V_{rel}^*$$

$$V_N^* \wr V_N^\pi$$

V_N^π —— Can we build π that is close to optimal? —— x^*, y^* (obtained from V_{rel}^*)

Subject of (G, Gaujal, Yan 2021), (Frazier et al 2020), (Brown and Smith 2019)

LP-relaxation vs Original problem

Relaxed problem: Optimal sequence $x_s^*(t)$, $y_{s,a}^*(t)$.

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Theorem (G., Gaujal, Yan 2021)

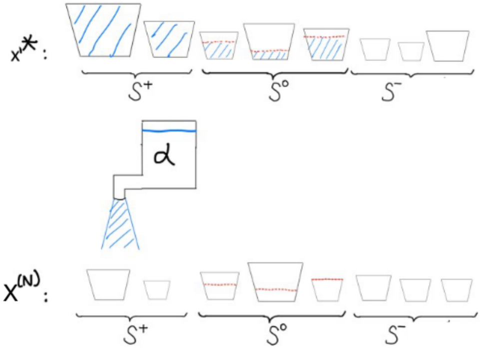
- *A continuous policy is LP-compatible iff* $\lim_{N\to\infty} V_N^\pi = V_{rel}^*$.
- *A locally linear LP-compatible policy satisfies*

$$|V_N^\pi - V_{rel}^*| \leq C_1 e^{-C_2 N}.$$

Proof (sketch). If π is continuous: $\lim_{N\to\infty} X_\pi^{(N)}(t) = x_\pi(t)$ and therefore $\lim_{N\to\infty} V_N^\pi = V_{rel}^\pi$.

How to build an LP-compatible policy: water-filling

LP-compatible: find a function $\pi : \mathcal{X} \to \mathcal{Y}$ such that $\pi(x^*) = y^*$.



where:
- $S^+ = \{s : y_{s,0}(t) = 0\}$
- $S^- = \{s : y_{s,1}(t) = 0\}$
- $S^0 = \{s : y_{s,0}(t) > 0 \wedge y_{s,1}(t) > 0\}$

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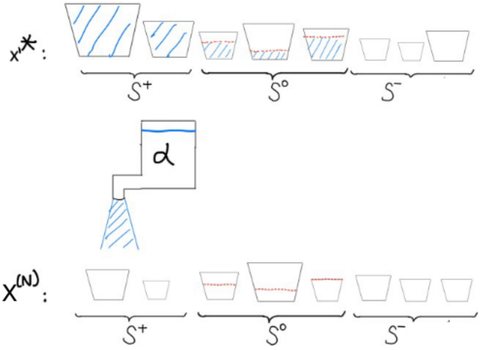

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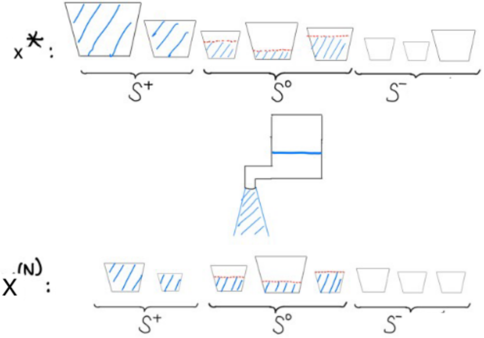

- Priority to S^+
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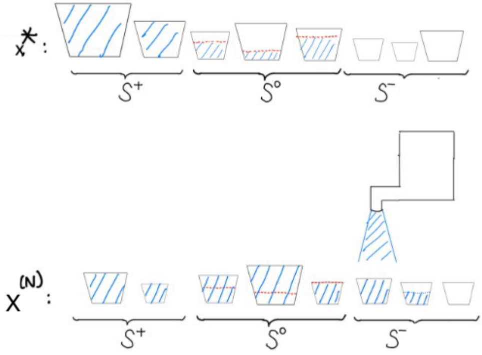

- Priority to S^+
- Second priority to S^0 up to $y^*_{s,1}$.
- Fill with the rest.

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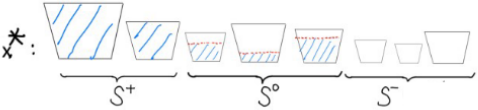
Existence of an LP-compatible policy

Non-degenerate = for all t: $|S^0(t)| \geq 1$.
Rankable = for all t: $S^0(t) \leq 1$.



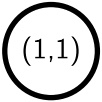
Theorem

- *For any problem, there exists an LP-compatible policy.*
- *If the problem is non-degenerate, then there exists a locally linear LP-compatible policy.*
- *If the problem is rankable, there exists a strict priority policy that is LP-compatible.*

Illustration: Applicant screening problem

Figure from (Brown Smith 2020)

Candidates with prior quality Beta(1,1), Interview budget $\alpha = 0.25$



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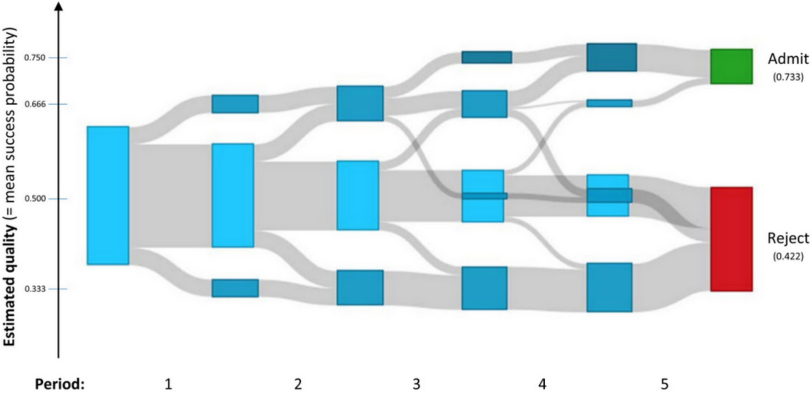
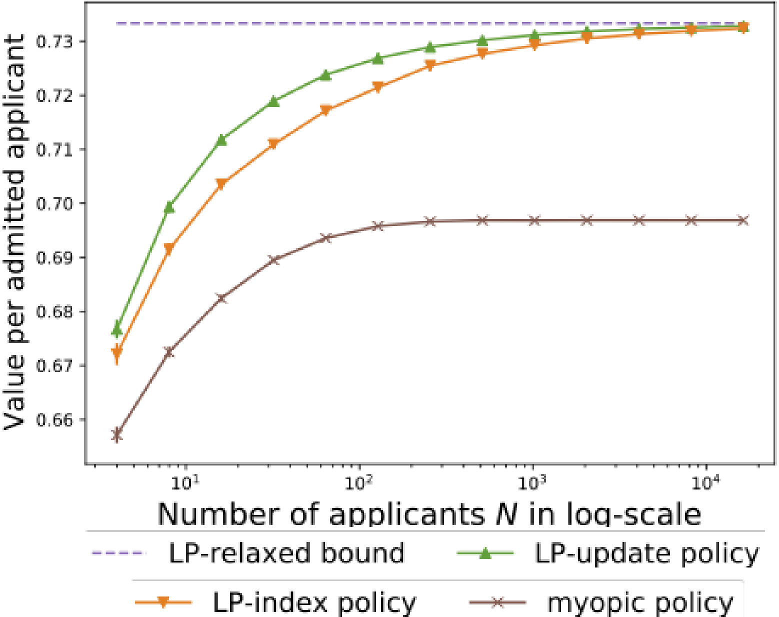


Example: optimal relaxed solution: after two interviews:

- $x^*_{(1,1)} = 2x^*_{(2,1)} = 2x^*_{(1,2)} = 0.5$.
- $y^*_{(2,1),interview} = y^*_{(1,1),interview} = 0.125$.

Optimality

LP-index = policy from (Brown Smith 2020), LP-update = update LP solution.



Outline

Subsidy and indexability

Main idea: use a Lagrangian decomposition and replace the constraint $\sum \mathbb{E}[aY_{s,a}] = \alpha$ by a subsidy $a\mu$ to action a.

An arm is a 2-action MDP: if in state s:

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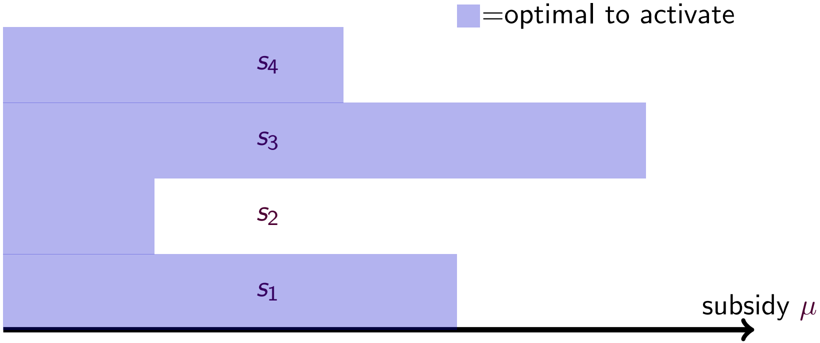
In general, the "optimal" subsidy depends on α.

What is the subsidy and who should we activate?



[1] Verloop. Asymptotically optimal priority policies for indexable and nonindexable restless bandits. (2016) Annals of Applied Probability.

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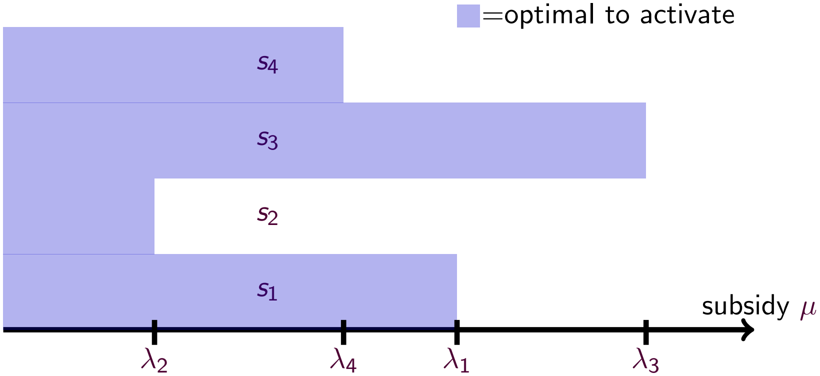

If the subsidy for which "activation" is optimal for s is $(-\infty, \lambda(s)]$, then the state is indexable and $\lambda(s)$ is its Whittle index.

- Activate arms by decreasing order of Whittle index.

Whittle policy is[1] LP-compatible for infinite horizon.

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Asymptotic optimality of Whittle index

Theorem (Weber Weiss 90s, Verloop 2016)

For infinite horizon, an LP-compatible policy is asymptotically optimal in under the "global attractor condition".

(G. Gaujal Yan 2021) *Holds with exponential rate for non-degenerate problems.*

Illustration with a stochastic scheduling problem

Example: two jobs of sizes X and Y with:

- $X = 10 - \varepsilon$
- $Y = \begin{cases} 2 & \text{proba } 1/2 \\ 18 & \text{proba } 1/2 \end{cases}$

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For this model: Gittins (=Whittle) index policy is optimal:

- serve job with smallest index first.

Running a job costs 1€/sec and you can stop anytime. If you finish the job, you earn x. Gittins index = smallest x so that you running or stopping is equivalent.

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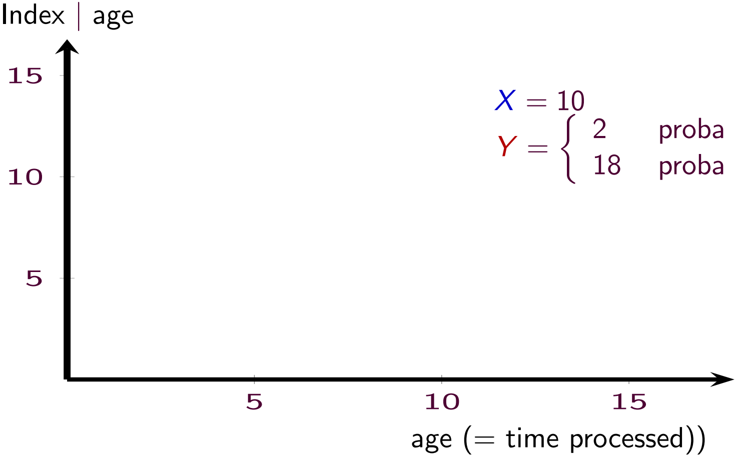

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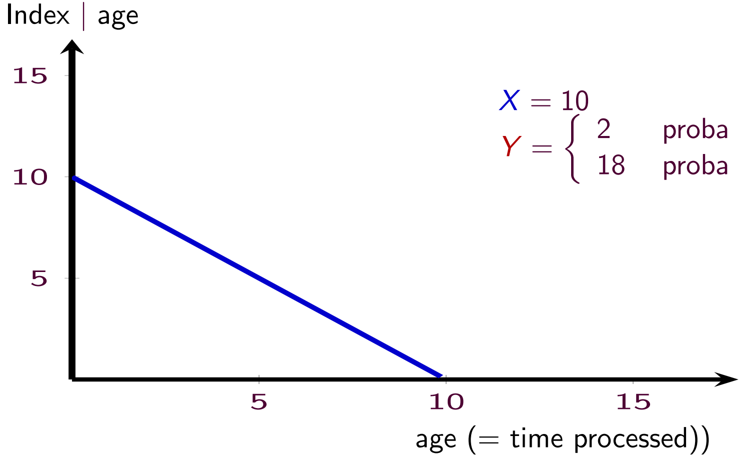

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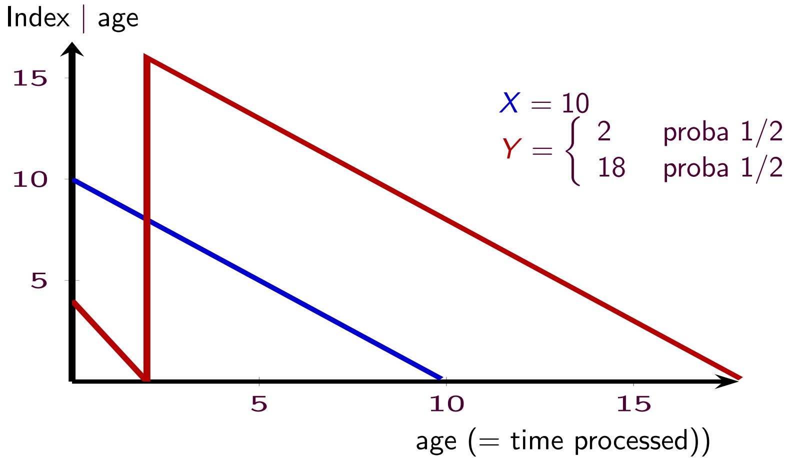

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(Recently) closed questions:

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- At which speed do Whittle index become optimal? (exponentially fast in most cases [G,Gaujal,Yan 2021])
- Can we define index for finite-horizon problems? [Hu and Frazier 2019-20, Brown Smith 2020, Gaujal, Yan 2022].
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- Can we leverage indexable to problem to obtain better learning algorithms? (No regret learning.[G,G,Khun, 2021])
- Are Whittle index hard to compute? [G, G. Khun, 2022]

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General idea: compute the index by increasing order

- A policy is $\pi \subset \{1 \ldots S\}$
- $g_\pi(x) =$ value for subsidy x.
- $\pi^*(x) = \arg\max_{\pi} g_\pi(x)$.

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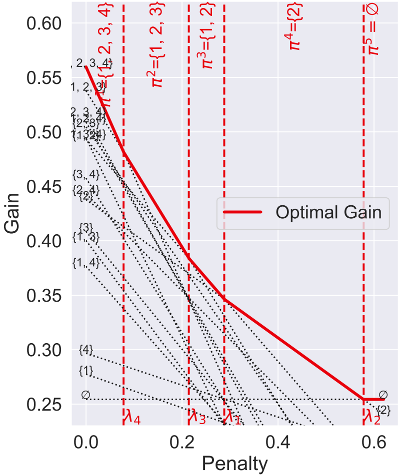
We want to find the inflection points of the red curve.



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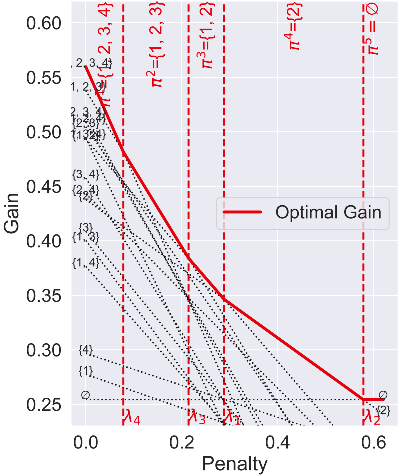

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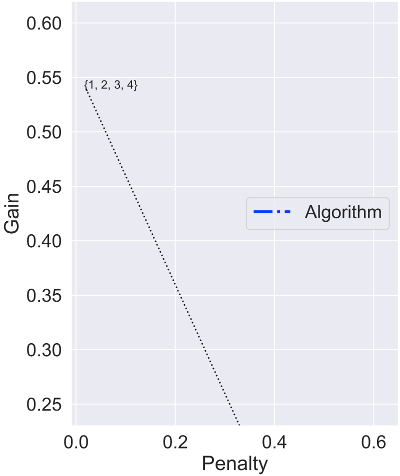

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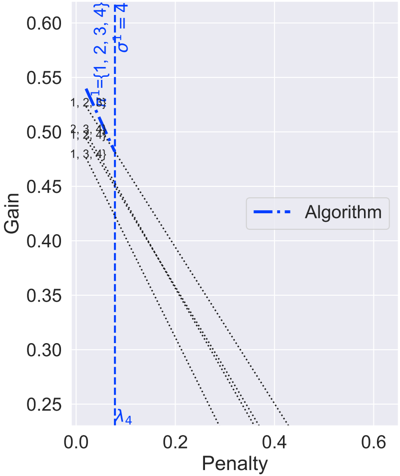

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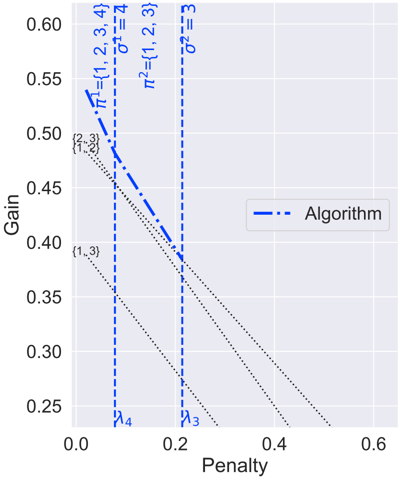

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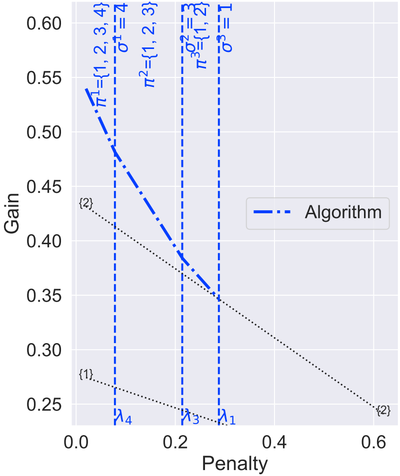

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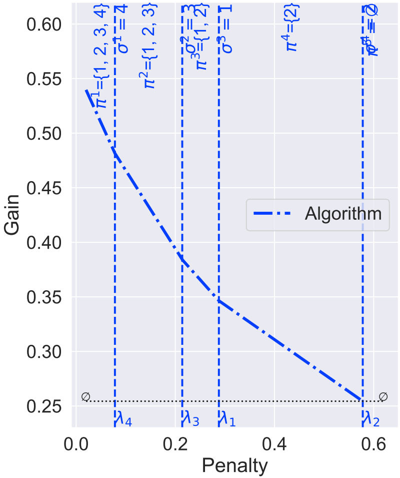

Why are the facts true?

- $g_\pi(x) = (A^\pi)^{-1}(r + x\pi)$

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- $g_\pi(x) = (A^\pi)^{-1}(r + x\pi)$
- If subsidy is $-\infty$ for all states, then we should not activate.

Why are the facts true?

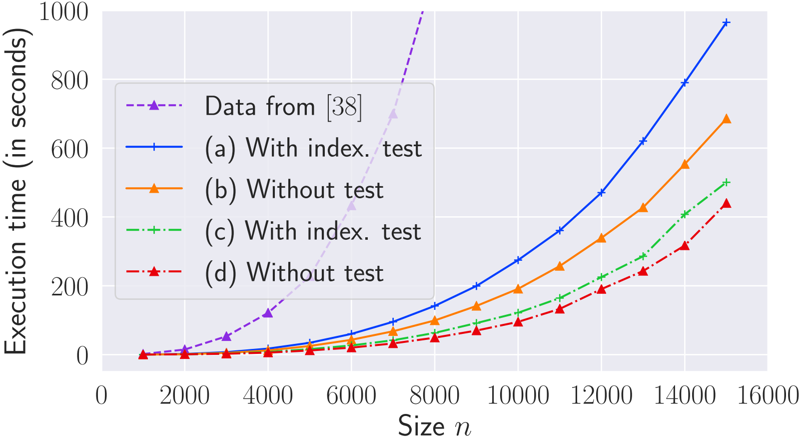
- $g_\pi(x) = (A^\pi)^{-1}(r + x\pi)$
- If subsidy is $-\infty$ for all states, then we should not activate.
- Sherman-Morisson formula: Let A be an invertible matrix, u and v vectors $1D$ such that $1 + v^T A^{-1} u \neq 0$. Then:

$$\left(A + uv^T\right)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

By using fast matrix multiplication, we can compute Whittle indices in $O(S^{2.53})$ operations (conjectured to be at least n^3 in a 2016 paper).

Simulation result

https://pypi.org/project/markovianbandit-pkg/



Outline

Finite-horizon restless bandits

Subsidy, infinite-horizon and index policies

How to Compute Indices: A Sub-Cubic Algorithm

Conclusion

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Index policies / LP relaxation are efficient ways to share resources among tasks.

- Idea: it allows you to compute a "right price" for each resource. You can then activate the cheapest up to your budget.

This scales and performs very well in practice.

- This talk: Optimality of index and computation of index.
- Open questions: learning, continuous state-spaces.

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http://polaris.imag.fr/nicolas.gast/

- *Computing Whittle (and Gittins) Index in Subcubic Time*, G. Gaujal, Khun https://arxiv.org/abs/2203.05207
- *LP-based policies for restless bandits: necessary and sufficient conditions for (exponentially fast) asymptotic optimality*. G. Gaujal Yan. https://arxiv.org/abs/2106.10067