# Asymptotic Optimality in Restless Bandit

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joint work with **Bruno Gaujal, Dheeraj Narasimha and Chen Yan**

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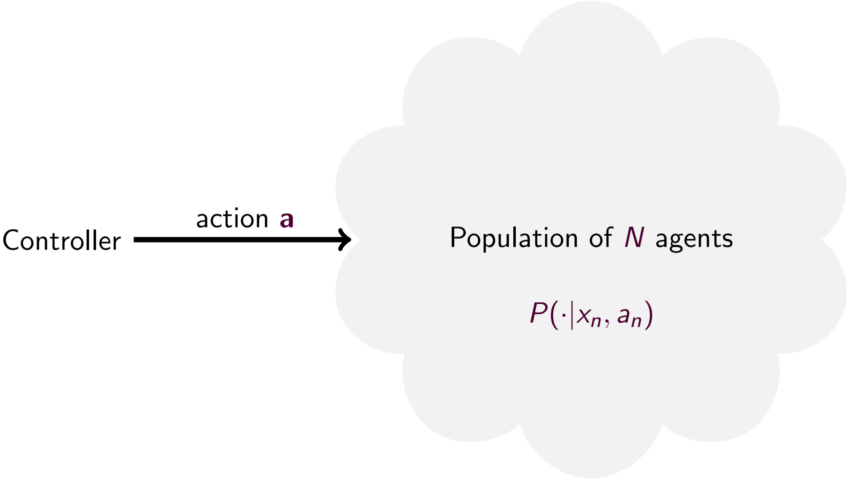
# Mean field control



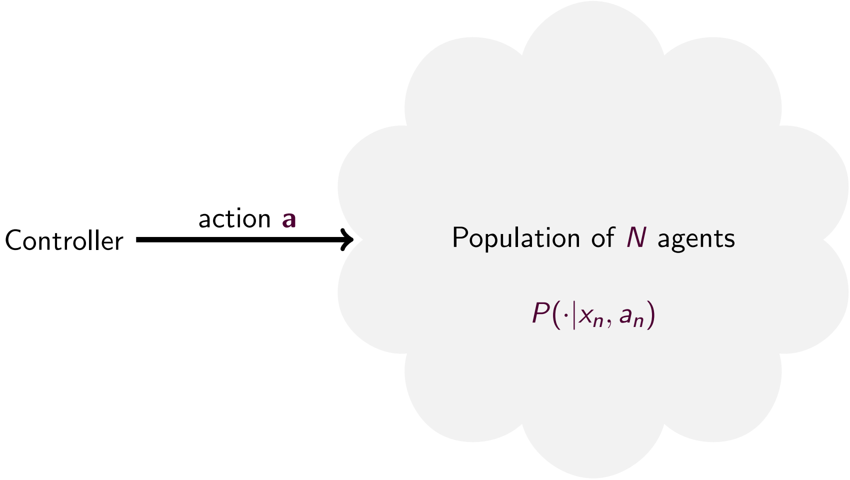
Controller

action $\mathbf{a}$

Population of $N$ agents

$P(\cdot | x_n, a_n)$

# Mean field control



Controller

action **a**

Population of $N$ agents

$P(\cdot|x_n, a_n)$

The computational difficulty increases with $N$ but “$N = \infty$” is easy.

- How to use the $N = +\infty$ solution for finite $N$?
- How efficient is this? (i.e., how fast does it become optimal?)

# This talk will focus on *Markovian bandits*

$N$ statistically identical arms (=agents)

- Discrete time, finite state space.
- $P(\cdot|s_n, a_n)$ and $r(s_n, a_n)$.

Maximize expected reward

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{N} r(s_n(t), a_n(t)).$$

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Hard constraint: $\forall t : \sum_{n=1}^{N} a_n(t) \le \alpha N.$

- If $a_n(t) \in \{0, 1\}$: Markovian bandit (this talk)
- If $a_n(t) \in \{0, 1\}^d$: Weakly coupled MDP.

# Example: Resource allocation

Arm/agent can be:

- Tasks (e.g., scheduling)
- Workers (e.g., maintenance problems)
- Electric vehicles (e.g., charging)

# Outline

1. The (relaxed) mean-field control problem
2. Three types of policies
    - Index policies
    - FTVA
    - Model predictive control
3. Performance guarantee
4. Conclusion

# The mean-field control problem (Whittle's relaxation)

Replace "For all $t$, $\sum_{n=1}^{N} a_n(t) \le \alpha N$" by in steady-state: $\sum_{n=1}^{N} \mathbb{E}[a_n] \le \alpha N$"

$\Rightarrow$ This is a constrained MDP and can be solved by an LP (Altman 99).

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$$
\begin{aligned}
V_{rel} := \max_{x\in\Delta, y\geq 0} \quad & \sum_{s,a} r_{s,a} y_{s,a} \\
\text{s.t.} \quad & x_{s'} = \sum_{s} y_{s,a} P(s'|s,a) && \text{Markov transitions} \\
& x_s = \sum_{a} y_{s,a} && \text{action taken} \\
& \sum_{s} y_{s,1} = \alpha && \text{relaxed budget contraint}
\end{aligned}
$$

where $x_s = \mathbf{P}[s_n = s]$ and $y_{s,a} = \mathbf{P}[s_n = s, a_n = a]$.

# How does a solution look like?

```
bandit_lp.BanditRandom(4, seed=1).relaxed_lp_average_reward(alpha=0.4)
```

$$y^* = \begin{array}{c} \begin{array}{cc} \text{Action } 0 & \text{Action } 1 \end{array} \\ \left[\begin{array}{cc} & 0.232 \\ 0.028 & 0.168 \\ 0.210 & \\ 0.171 & \\ 0.191 & \end{array}\right] \end{array}$$

Note: $0.232 + 0.168 = \alpha = 0.4$.

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Note: $0.232 + 0.168 = \alpha = 0.4$.

# Can I apply this to the original (non-relaxed) problem?

$\pi^*$ is optimal for the constrained MDP $\sum_n \mathbb{E}\left[A_n\right] = \alpha N$.

Tentative solution:

$$S(t) = [2, 0, 1, 3, 0, 0, 0, 3, 2, 3, 1, 1, 2, 0, 4]$$

$$\Downarrow \text{ Sample } A_n(t) \sim \pi^*(S_n(t)) \text{ (indep.)}$$

$$\tilde{A}_{\pi^*}(t) = [0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0]$$

Problem: here $8 = \sum_{n=1}^{N} \tilde{A}_n(t) \neq \alpha N = 6$.

# Historical perspective

## and possible solutions

1. Whittle index (88) (Nino-Mora, 90s-2000s) / LP-index (Verloop 15)
    - Works extremely well in practice
    - Often asymptotically optimal (UGAP, Weber and Weiss 91).
    - When they are: exponentially fast. (G, Gaujal, Yan 2023).
2. FTVA – Follow the virtual advice (Hong et al, 2023, 2024)
    - Whittle index can fail (when UGAP fails)
    - Asymptotically optimal in theory, not in practice.
3. Model predictive control (G., Narasimha 2024, G, Gaujal, Yan 2023)
    - Best of both worlds

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# Index policy: LP-index and Whittle index

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Index policy: priority to largest index: $0 > 1 > 4 > 2 > 3$.

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$$S(t) = [0,0,0,0,0,1,1,1,2,2,2,3,3,3,4]$$
$$A_{Idx}(t) = [1,1,1,1,1,1,0,0,0,0,0,0,0,0,0]$$

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$$\left(\tilde{A}_{\pi^*}(t) = [1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0]\right) \qquad \text{from } \pi^*$$

# Where does the LP-index comes from?

The $N = \infty$ is a constraint MDP:

- $P(\cdot|s_n, a_n)$ and $r(s_n, a_n)$ s.t. in steady-state, $\mathbf{P}[a_n] = \alpha$.

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Idea: use a Lagrangian relaxation:

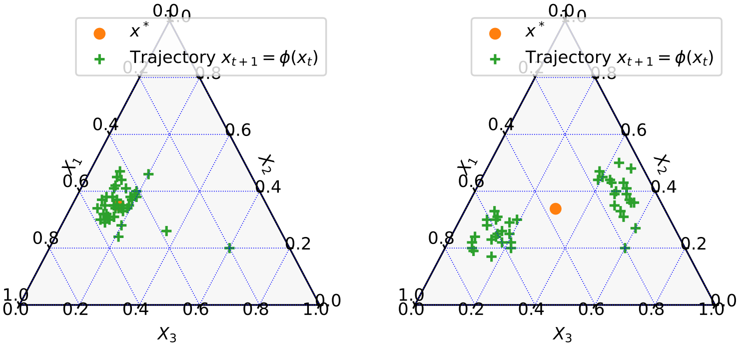
- $P(\cdot|s_n, a_n)$ and $r(s_n, a_n) - \underset{\substack{\Big\downarrow \\ \text{Penalty for activation}}}{\lambda} a_n$.

Index of state $s$: $I_s = Q_\lambda(s, 1) - Q_\lambda(s, 0)$.

# Whittle index / LP-index can fail

## Theorem (Weber-Weiss 90)

*Whittle is asymptotically optimal under UGAP (=stability) but can fail.*



(a) UGAP is satisfied (b) UGAP is not satisfied

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Sample $\tilde{A}_n \sim \pi(\tilde{S}_n)$ and $A \leftarrow \mathrm{cap}(\tilde{A})$

**If** $S_n = \tilde{S}_n$ and $A_n = \tilde{A}$: **then**: couple $S_n(t+1)$ and $\tilde{S}_n(t+1)$

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| A | 0 0 1 1 1 0 1 1 0 0 1 0 0 0 0 | [0 0 1 1 1 0 1 1 0 0 1 0 0 0 0] |
| S | 3 0 2 4 3 3 0 1 1 0 4 3 4 4 2 | [3 4 2 4 3 3 0 1 1 0 4 3 4 4 2] |
| A | 1 1 0 0 0 0 1 1 1 1 0 0 0 0 0 | [0 0 0 0 0 0 1 1 1 1 0 0 0 0 0] |
| S | 4 3 3 3 2 1 1 0 4 3 3 2 0 1 4 | [2 0 3 3 2 1 1 0 4 3 3 2 0 1 4] |
| A | 0 1 0 0 0 1 1 1 0 0 0 0 1 1 0 | [0 1 0 0 0 1 1 1 0 0 0 0 1 1 0] |
| S | 0 4 2 2 1 2 4 4 0 2 4 4 1 0 3 | [4 4 2 2 1 2 4 4 0 2 4 4 1 0 3] |
| A | 1 1 0 0 1 0 0 0 1 0 0 0 1 1 0 | [0 0 0 0 1 0 0 0 1 0 0 0 1 1 0] |

# Model predictive control

## The "LP-update policy"

We define a finite-horizon deterministic problem:

$$V_\tau(\mathbf{S}) := \max_{y\geq 0} \; \sum_{t=0}^{\tau} \sum_{s,a} r_{s,a} y_{s,a}(t)$$

$$\text{s.t.} \quad \sum_a y_{s,a}(t+1) = \sum_s y_{s,a}(t) P(s'|s,a) \qquad \text{Markov transitions}$$

$$\sum_s y_{s,1}(t) = \alpha \qquad \text{relaxed budget contraint}$$

$$\sum_a y_{s,a}(0) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\{S_n(t)=s\}} \qquad \text{initial state}$$

We then apply $y_{s,a}(0)$ to all states.

# Outline

1. The (relaxed) mean-field control problem
2. Three types of policies
   - Index policies
   - FTVA
   - Model predictive control
3. Performance guarantee
4. Conclusion

# Assumptions

We consider the following deterministic dynamical system:

$$\phi(\mathbf{x}) = \mathbb{E}\left[\mathbf{X}(t+1) \mid \mathbf{X}(t) = \mathbf{x} \wedge \ A \ \sim \text{index}\right],$$

and we call $y^*$ the solution of $V_{rel}$, with $x^*_s = \sum_a y^*_{sd,a}$.

We define the following conditions:

UGAP $\lim_{t\to\infty} x_{t+1} = \phi(x_t)$ converges to $x^*$ uniformly for all $x$.

Local stability $\phi$ is locally stable around $x^*$.

Degenerate $y_{s,1} = 0$ or $y_{s,0} = 0$ for all $s$.

# Theoretical guarantees

### Theorem (Weber-Weiss, G,G,Y23)

*Under UGAP and non-degenerate:* $V_{index} \geq V_{rel} - e^{-\Omega(N)}$.

### Theorem (Hong et al. 23)

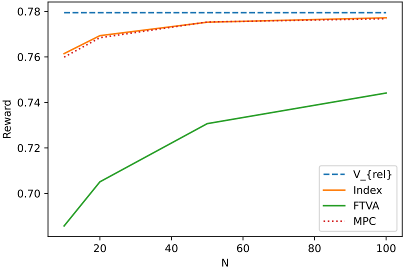
*If* $P$ *is ergodic, then:* $V_{FTVA} \geq V_{rel} - O(1/\sqrt{N})$.

### Theorem (G,N 24)

1. *If* $P$ *is ergodic:* $V_{MPC} \geq V_{rel} - O(1/\sqrt{N})$.
2. *Under non-degenerate and local stability:* $V_{MPC} \geq V_{rel} - e^{-\Omega(N)}$.

# Illustration

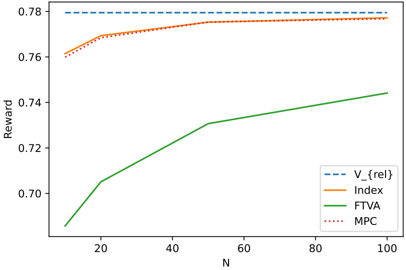
The random example.



UGAP + non-degenerate.

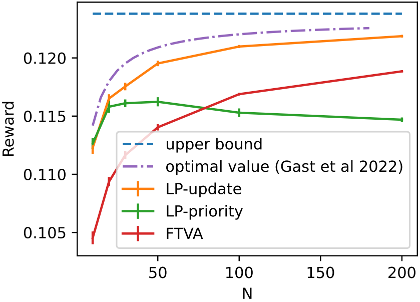
# Illustration

The random example.



UGAP + non-degenerate.

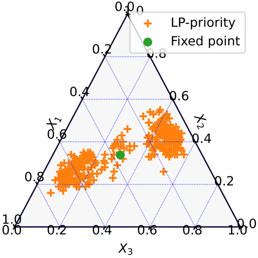
Example from Yan 2023.



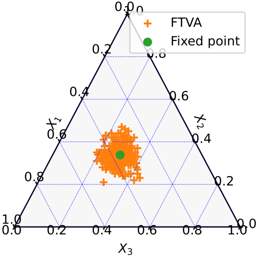
No UGAP nor local stability.

# Illustration of non-UGAP

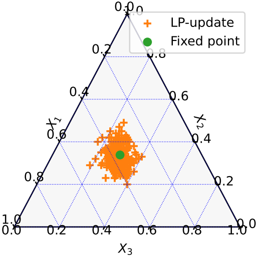
## Example from Yan 2023 (3D example)



(a) Index



(b) FTVA



(c) MPC

# Outline

1. The (relaxed) mean-field control problem

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3. Performance guarantee

4. **Conclusion**

# Conclusion

For Markovian bandits, mean-field control can be solved by an LP.

- Can be generalized to weakly coupled MDPs.

Simple policies (priority rule) are not always optimal.

- When they are, they become optimal exponentially fast.

- This talk: comparison of various approaches.

# Conclusion

For Markovian bandits, mean-field control can be solved by an LP.

- Can be generalized to weakly coupled MDPs.

Simple policies (priority rule) are not always optimal.

- When they are, they become optimal exponentially fast.

- This talk: comparison of various approaches.
- Open questions: learning, continuous state-spaces.

`http://polaris.imag.fr/nicolas.gast/`

- *LP-based policies for restless bandits: necessary and sufficient conditions for (exponentially fast) asymptotic optimality*. G. Gaujal Yan. MMOR 2023. `https://arxiv.org/abs/2106.10067`
- *Restless Bandits with Average Reward: Breaking the Uniform Global Attractor Assumption*. Hong, Xie, Chen, and Wang. NeurIPS 2023.
- *Model Predictive Control is Almost Optimal for Restless Bandit*. G, Narasimha. 2024. Under review.