Balanced Labeled Trees : Density, Complexity and Mechanicity

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LIG

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Sturmian Words: 3 equivalent definitions

Consider an infinite words:

001010010010100101001 00 . . .

- minimal complexity : $n + 1$ factors of length n.
  example: 4 factors of length 3: 001, 010, 100 and 101.
- balanced : number of 1 only differ by 1 in factors of same length.
  - length 3: 1 or 2.
  - length 4: 1 or 2.
  - . . .
- mechanical:
  - for all i: $w_i = \lfloor \alpha * (i + 1) + \theta \rfloor - \lfloor \alpha * i + \theta \rfloor$
    or for all i: $w_i = \lceil \alpha * (i + 1) + \theta \rceil - \lceil \alpha * i + \theta \rceil$

Problem

Can we extend theses notions to trees?

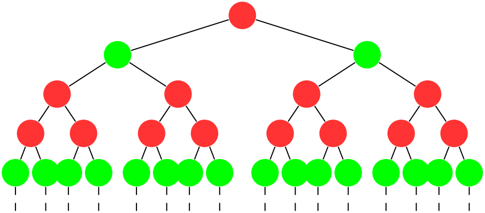
- sturmian
- balanced
- mechanical

Previous Work

Definition (Berstel, Boasson, Carton and Fagnot, 2007)

A Sturmian tree is a tree with $n+1$ subtrees of size n.

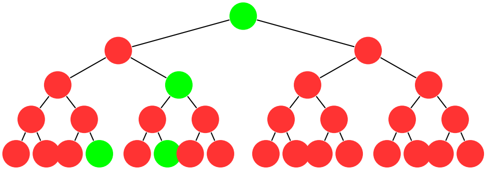
Simple example:



Example: The uniform tree corresponding to 0100101 . . .

Properties

- Link with language theory
- Interesting examples:



Example: Dyck Trees

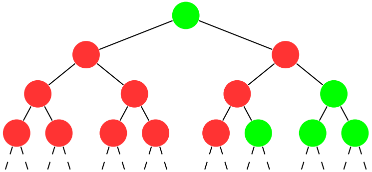
But

- the *balanced* property is lost (important in optimization problems)
- no simple equivalent characterization

Our Words are Infinite Labeled Trees

Our trees are:

- rooted
- labeled by 0 or 1
- infinite
- Non-planar
  (≠ Original definition
  of Sturmian Trees)



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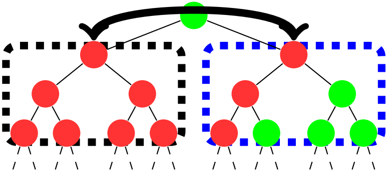

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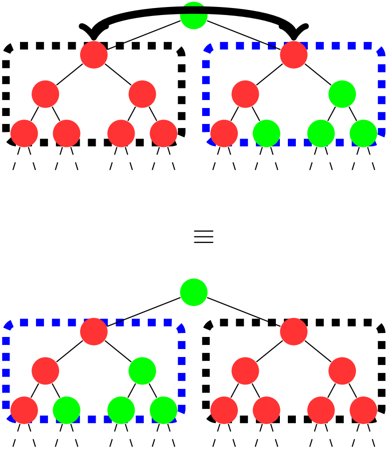
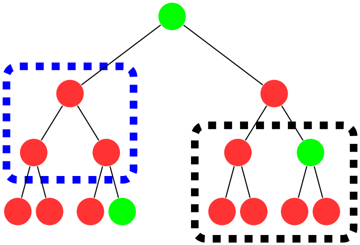


What are Subtrees and Density?

We define:

- Subtree of height n
- Subtree of width k and height n
- Density of a subtree = average number of 1.
- If d_n is the density of the subtree of height n:
  - density $= \lim_n d_n$
  - average density $= \lim_n \frac{1}{n} \sum_{k=1}^{n} d_k$



First simple case

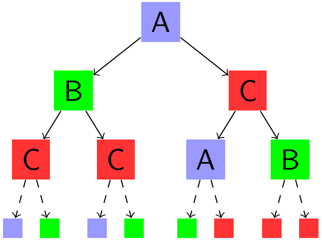
What is a non-planar Rational Tree?

Rational Trees: Definition

We call $P(n)$ = number of subtrees of size n.

Rational Trees: 3 equivalent definitions:

- $P(n)$ bounded.
- $\exists n / P(n) = P(n+1)$
- $\exists n / P(n) \leq n$.



Rational Tree: average Density

Theorem

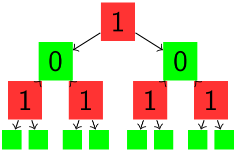
- *A rational Tree has an average density α which is rational.*

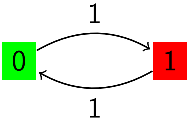
  α is not necessary a density but:

- *If the Markov chain associated is aperiodic then there exists a density.*

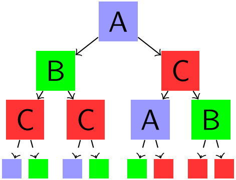
Example of density

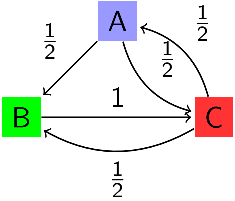
- Periodic = average density $d_{\text{average}} = \frac{1}{2}$





- Aperiodic : density $d = \frac{2}{9}\ell_A + \frac{1}{3}\ell_B + \frac{4}{9}\ell_C$





Second case

Balanced and Mechanical Trees

Balanced Trees and Strongly Balanced Trees

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- Strongly balanced tree: same property with subtrees of height n and width k.

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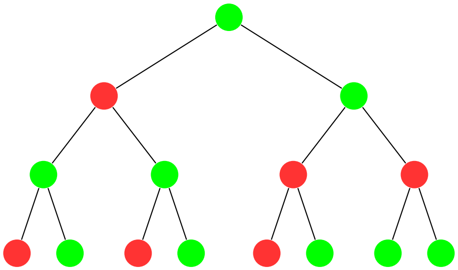

Example: Balanced tree not strongly balanced

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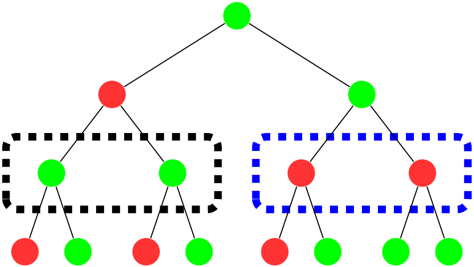

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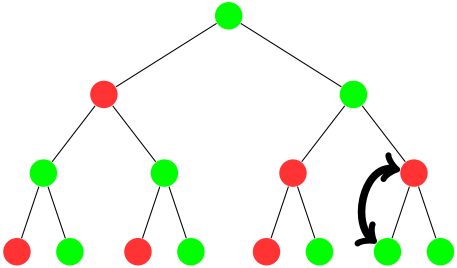

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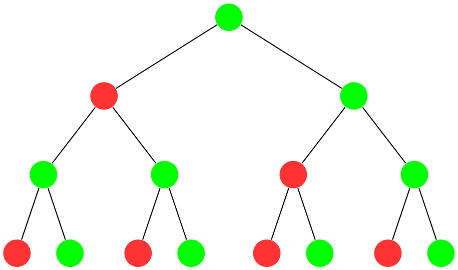


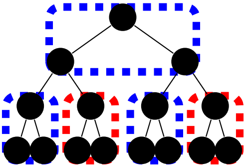
Example: Strongly balanced tree

Density of a Balanced Tree

Theorem

- *A balanced tree has a density.*

Sketch of the proof.

1. A tree of size n has a density α_n or $\alpha_n + \frac{1}{2^n - 1}$
2. 

   If blue has a density α_2 and red $\alpha_2 + \frac{1}{3}$ then $\alpha_2 \leq \alpha_4 \leq \alpha_2 + \frac{1}{3}$
3. Take limit

□

Mechanical Trees

- Subtree of size n has $2^n - 1$ nodes.
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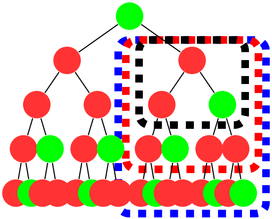
Mechanical tree of density α:

- For all node i, there is a phase $\phi_i \in [0;1)$ such that the number of 1 in a subtree of height n and root i is $\lfloor (2^n - 1)\alpha + \phi_i \rfloor$ (resp. for all i: $\lceil (2^n - 1)\alpha + \phi_i \rceil$)



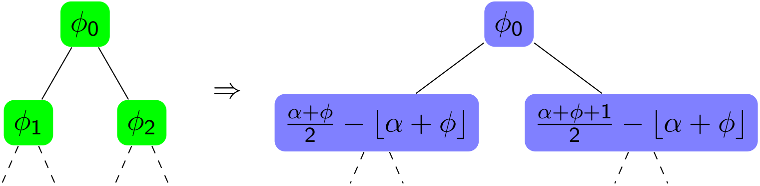
$\alpha = 0.3$, $\phi = 0.55$.

n	$(2^n - 1)\alpha + \phi$
1	0.85
2	1.45
3	2.65
4	5.05

Uniqueness of a mechanical Tree

Theorem

- *There exists a unique mechanical tree if* (α, ϕ_0) *is fixed.*



- *The phase of the root* ϕ_0 *is unique for almost all* α.

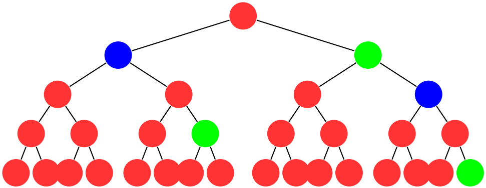
Equivalences?

What are the equivalences between definitions?

Equivalences between Definitions

Theorem (Mechanical $\sim$ strongly balanced)

- *A mechanical tree is strongly balanced*
- *A strongly balanced tree with irrational density is mechanical*
- *A strongly balanced tree with rational density is ultimately mechanical.*



Example: Ultimately mechanical tree

Sketch of Proof

Mechanical implicates strongly balanced.

The number of 1 in a subtree of size n and width k is bounded by $\lfloor (2^n - 2^k)\alpha \rfloor$ and $\lfloor (2^n - 2^k)\alpha \rfloor + 1$ □

Strongly Balanced implicates mechanical.

$\forall \tau \in [0; 1)$, if h_n is the number of 1 in the subtree of size n, at least one of these properties is true:

1. for all n: $h_n \leq \lfloor (2^n - 1)\alpha + \tau \rfloor$,
2. for all n: $h_n \geq \lfloor (2^n - 1)\alpha + \tau \rfloor$.

We choose ϕ the maximal τ such that 1 is true. □

Theorem

- *An irrational mechanical tree is a sturmian tree: it has $n + 1$ subtrees of height n.*

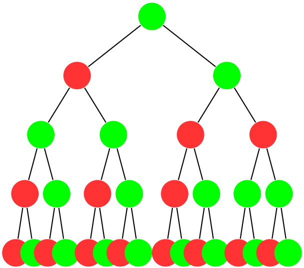
Proof.

- A subtree of size n depends only on its phase
- In fact, it depends on $((2^1 - 1)\alpha + \phi, \ldots, (2^n - 1)\alpha + \phi)$ which takes $n + 1$ values when $\phi \in [0; 1)$.

□

Limit of the Equivalences

- Balanced $\not\Rightarrow$ strongly balanced (no matter the density is rational or not).
- Sturmian $\not\Rightarrow$ balanced.
- Irrational Balanced tree $\not\Rightarrow$ sturmian.



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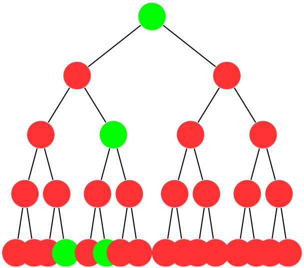

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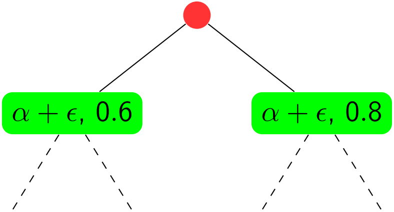

Example: Balanced tree non sturmian

Conclusion

- Non-planar definition better?
- Constructive definition
- Strong inclusions
- Good characterization

but:

- what are exactly balanced trees?
- how many trees of size n?