Queuing Networks

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Outline

Introduction to Queuing Networks

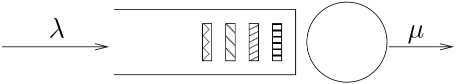
- Single queues have simple results
- They are quite robust to slight model variations
- We may have multiple contention resources to model:
  - servers
  - communication links
  - databases

  with various routing structures.

Queuing networks are direct results for interaction of classical single queues with probabilistic or static routing.

Outline

Refresher: M/M/1

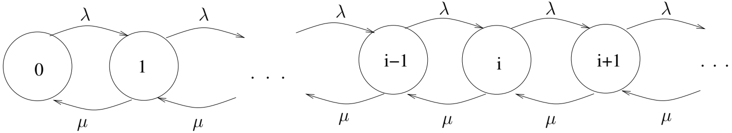


M/M/1 queue

- Infinite capacity
- Poisson(λ) arrivals
- Exp(μ) service times
- FIFO discipline

Definition

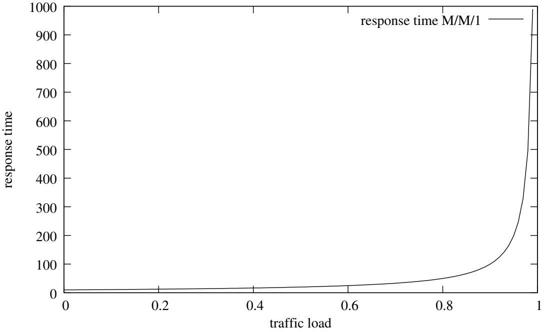
$\rho = \frac{\lambda}{\mu}$ is the traffic intensity of the queueing system.



Number of clients $X(t)$ in the system follows a birth and death process.

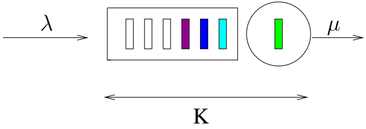
Results for M/M/1 queue

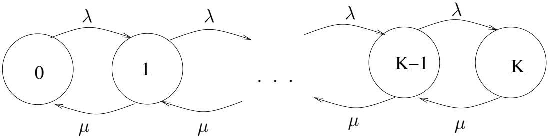
1. Stable if and only if $\rho < 1$
2. Clients follow a geometric distribution $\forall i \in \mathbb{N},\ \pi_i = (1-\rho)\rho^i$
3. Mean number of clients $\mathbb{E}[X] = \frac{\rho}{(1-\rho)}$
4. Average response time $\mathbb{E}[T] = \frac{1}{\mu-\lambda}$



M/M/1/K

In reality, buffers are finite: M/M/1/K is a queueing system with blocking.



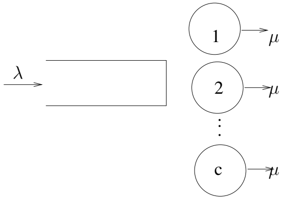


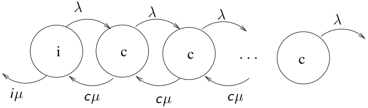
Results for M/M/1/K queue

Geometric distribution with finite state space

$$\pi(i) = \frac{(1-\rho)\rho^i}{1-\rho^{K+1}}$$

M/M/c





Results for M/M/c queue

Stability condition $\lambda < c\mu$.

where $\rho = \frac{\lambda}{\mu}$ and with

$$\pi(i) = \begin{cases} C_{\rho,c}\frac{\rho^i}{i!} & \text{if } i \leq c \\ C_{\rho,c}\frac{1}{c!}\left(\frac{\rho}{c}\right)^i & \text{if } i > c \end{cases}$$

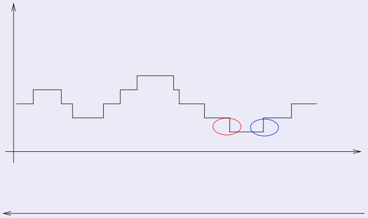
$$C_{\rho,c} = \frac{1}{\sum_{i=0}^{c-1}\frac{\rho^i}{i!} + \frac{\rho^c}{c!}\frac{1}{1-\rho/c}}$$

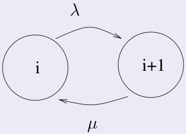
Burke's theorem

Theorem

The output process of an M/M/s queue is a Poisson process that is independent of the number of customers in the queue.

Sketch of Proof.

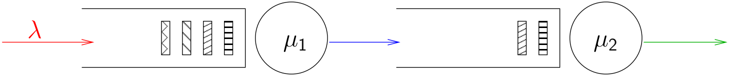




$X(t)$ increases by 1 at rate $\lambda\pi_i$ (Poisson process λ). Reverse process increases by 1 at rate $\mu\pi_{i+1} = \lambda\pi_i$ by reversibility. □

Outline

Tandem queues



Let X_1 and X_2 denote the number of clients in queues 1 and 2 respectively.

Lemma

X_1 and X_2 are independent rv's.

Proof

Arrival process at queue 1 is Poisson(λ) so future arrivals are independent of $X_1(t)$.
By time reversibility $X_1(t)$ is independent of past departures.
Since these departures are the arrival process of queue 2, $X_1(t)$ and $X_2(t)$ are independent. □

Tandem queues

Theorem

The number of clients at server 1 and 2 are independent and

$$P(n_1, n_2) = \left(\frac{\lambda}{\mu_1}\right)^{n_1} \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^{n_2} \left(1 - \frac{\lambda}{\mu_2}\right)$$

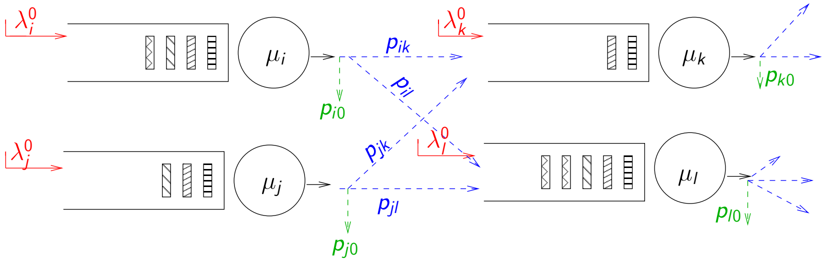
Proof

By independence of X_1 and X_2 the joint probability is the product of M/M/1 distributions. □

This result is called a product-form result for the tandem queue.
This product form also appears in more general networks of queues.

Acyclic networks

Example of a feed-forward network:



- Exponential service times
- output of i is routed to j with probability p_{ij}
- external traffic arrives at i with rate λ_i^0
- packets exiting queue i leave the system with probability p_{i0}.

Routing matrix:

$$R = \begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ p_{ji} & 0 & p_{jk} & p_{jl} \\ p_{ki} & p_{kj} & 0 & p_{jl} \\ p_{li} & p_{lj} & p_{lk} & 0 \end{pmatrix}$$

Decomposition of a Poisson Process

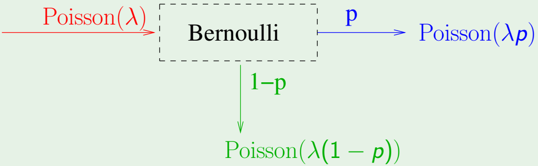
Problem

- $N(t)$ Poisson process with rate λ
- $Z(n)$ sequence of iid rv's $\sim$ Bernoulli(p) independent of N.

Suppose the nth trial is performed at the nth arrival of the Poisson process.

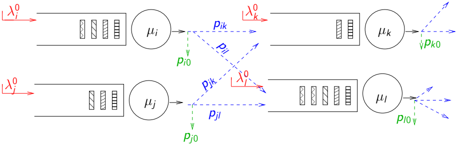
Result

- Resulting process $M(t)$ of successes is Poisson(λp).
- Process of failures $L(t)$ is Poisson$\big(\lambda(1-p)\big)$ and is independent of $M(t)$.



Acyclic networks

Total arrival rate at node i: λ_i $(1 \leq i \leq K)$.



No feedback: using Burke theorem, all internal flows are Poisson!
Thus we can consider K independent M/M/1 queues with Poisson arrivals with rate λ_i, where

$$\lambda_i = \lambda_i^0 + \sum_{j=0}^{K} \lambda_j p_{ji} \qquad \text{i.e. } \vec{\Lambda} = \vec{\Lambda}^0 + \vec{\Lambda}\mathbf{R} \quad \text{in matrix notation.}$$

Stability condition

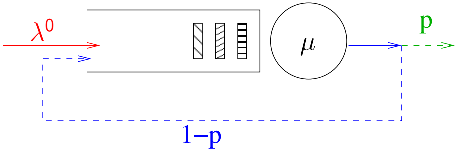
All queues must be stable independently: $\lambda_i < \mu_i, \quad \forall i = 1, 2, \ldots, K.$

Backfeeding

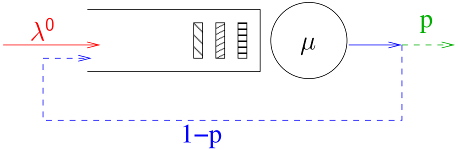
Example

Switch transmitting frames with random errors. A NACK is sent instantaneously if the frame is incorrect.

- Frame success probability is p.
- Arrivals $\sim$ Poisson(λ^0)
- Frame transmission times $\sim$ Exp (μ)



Backfeeding



Remark

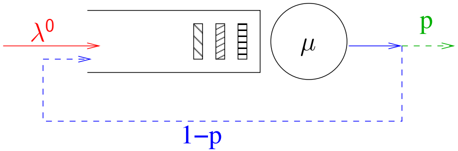
Arrivals are not Poisson anymore!

Result

The departure process is still Poisson with rate λp.

Proof in [Walrand, An Introduction to Queueing Networks, 1988].

Backfeeding: steady-state



Balance equations:

$$\begin{aligned} \pi(0)\lambda^0 &= \mu p \pi(1) \\ \pi(n)(\lambda^0 + p\mu) &= \lambda^0 \pi(n-1) + \mu p \pi(n+1), \quad n > 0 \end{aligned}$$

Actual arrival rate $\lambda = \lambda^0 + (1-p)\lambda,$ so $\lambda^0 = \lambda p$ which gives

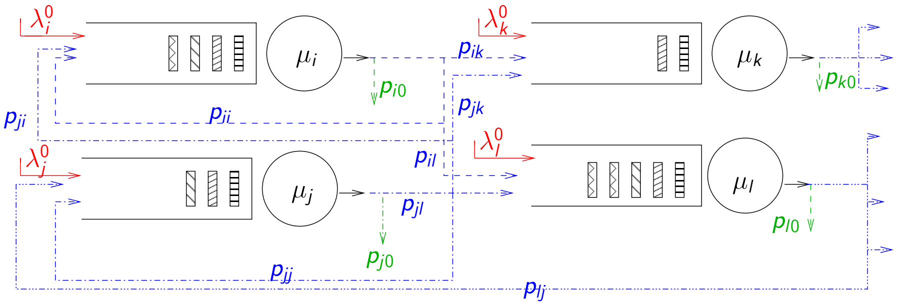
$$\begin{aligned} \pi(0)\lambda &= \mu \pi(1) \\ \pi(n)(\lambda + \mu) &= \lambda \pi(n-1) + \mu \pi(n+1), \quad n > 0 \end{aligned}$$

M/M/1!

The unique solution is:

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda^0}{p\mu}\right)\left(\frac{\lambda^0}{p\mu}\right)^n$$

Jackson networks: example



Backfeeding allowed.

System state (CTMC): $X(t) = (n_1(t), n_2(t), \ldots, n_K(t))$ where K is the number of queues and $n_i(t)$ the number of clients at queue i.

Jackson networks

Theorem (Jackson, 1957)

If $\lambda_i < \mu_i$ (stability condition), $\forall i = 1, 2, \ldots K$ then

$$\pi(\vec{n}) = \prod_{i=1}^{K} \left(1 - \frac{\lambda_i}{\mu_i}\right) \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \quad \forall \vec{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K.$$

where $\lambda_1, \ldots, \lambda_K$ are the unique solution of the system

$$\lambda_i = \lambda_i^0 + \sum_{j=0}^{K} \lambda_j p_{ji}$$

Product form even with backfeeding!

Jackson networks: sketch of proof

Derive balance equations:

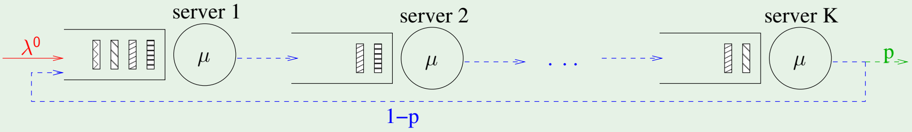
$$\pi(\vec{n})\left(\sum_{i=1}^{K}\lambda_i^0+\sum_{i=1}^{K}\mathbb{1}_{[n_i>0]}\mu_i\right) = \sum_{i=1}^{K}\mathbb{1}_{[n_i>0]}\lambda_i^0\pi(\vec{n}-\vec{e_i}) + \sum_{i=1}^{K}p_{i0}\mu_i\pi(\vec{n}+\vec{e_i}) + \sum_{i=1}^{K}\sum_{j=1}^{K}\mathbb{1}_{[n_j>0]}p_{ij}\mu_i\pi(\vec{n}+\vec{e_i}-\vec{e_j})$$

Then check that $\pi(\vec{n})=\prod_{i=1}^{K}\left(1-\frac{\lambda_i}{\mu_i}\right)\left(\frac{\lambda_i}{\mu_i}\right)^{n_i}$ satisfies the balance equations with $\lambda_i=\lambda_i^0+\sum_{j=0}^{K}\lambda_j p_{ji}$. □

Jackson networks: example

Example

Switches transmitting frames with random errors.



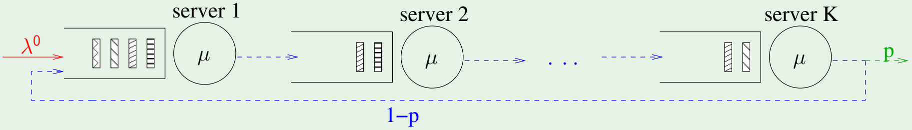
Traffic equations give $\lambda_i = \lambda_{i-1}$ for $i \geq 2$ and $\lambda_1 = \lambda^0 + (1-p)\lambda_K$. The unique solution is clearly $\lambda_i = \frac{\lambda^0}{p}$ for $1 \leq i \leq K$. Apply Jackson's theorem:

$$\pi(\vec{n}) = \left(1 - \frac{\lambda^0}{p\mu}\right)^K \left(\frac{\lambda^0}{p\mu}\right)^{n_1+\ldots+n_K} \qquad \forall \vec{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K.$$

Jackson networks: example (Continued)

Example

Switches transmitting frames with random errors.



Using M/M/1 results for each queue we get the mean number of frames at each queue $\mathbb{E}[X_i] = \frac{\lambda^0}{p\mu - \lambda^0}$
The expected transmission time of a frame is therefore (Little)

$$\mathbb{E}[T] = \frac{1}{\lambda^0}\mathbb{E}[X] = \frac{1}{\lambda^0}\sum_{i=1}^{K}\mathbb{E}[X_i] = \frac{K}{p\mu - \lambda^0}$$

Networks of M/M/c queues

Theorem

Consider an open network of K $M/M/c_i$ queues. Let $\mu_i(n) = \mu_i \min(n, c_i)$ and $\rho_i = \frac{\mu_i}{\lambda_i}$.
Then if $\rho_i < c_i$ for all $1 \leq i \leq K$ then

$$\pi(\vec{n}) = \prod_{i=1}^{K} C_i \left(\frac{\lambda_i^{n_i}}{\prod_{m=1}^{n_i} \mu_i(m)} \right) \quad \forall \vec{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K$$

where $(\lambda_1, \ldots, \lambda_K)$ is the unique positive solution of the traffic equations

$$\lambda_i = \lambda_i^0 + \sum_{j=0}^{K} \lambda_j p_{ji}, \quad \text{and where } C_i = \left(\sum_{m=1}^{c_i - 1} \frac{\rho_i}{i!} + \frac{\rho_i^{c_i}}{c_i!(1 - \rho_i / c_i)} \right)^{-1}$$

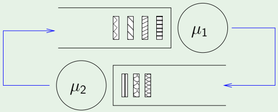
Outline

Closed Queueing Networks

Definition

A closed system is a system in which the number of clients is a constant variable.

Example



The traffic equations are linearly dependent!

$$\lambda_i = \sum_{j=0}^{K} \lambda_j p_{ji}, \quad 1 \leq i \leq K$$

Therefore the previous Jackson theorem cannot be applied and does not yield the correct result.

Jackson theorem for closed networks

Consider a closed queueing system with K queues and N clients.
Define by $S(N,K)$ the set of vectors $\vec{n} = (n_1, \dots n_k) \in \mathbb{N}^K$ such that $n_1 + \dots + n_K = N$.

Theorem (Closed Jackson networks)

Let $(\lambda_1, \dots, \lambda_K)$ *be an arbitrary non-zero solution of the traffic equation* $\lambda_i = \sum_{j=0}^{K} \lambda_j p_{ji}$, $1 \le i \le K$. *Then for all* $\vec{n} \in S(N,K)$,

$$\pi(\vec{n}) = \frac{1}{C(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}, \quad C(N,K) = \sum_{\vec{n} \in S(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}$$

Not a product-form!

Performance indexes

A typical performance metric is the expected queue length $\mathbb{E}[X_i]$ at node i.

Expected queue length

$$\mathbb{E}[X_i] = \sum_{k=1}^{N} \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{C(N-k,K)}{C(N,K)}$$

Sketch of Proof:

For a $\mathbb{N}-$valued r.v we have: $\mathbb{E}[X_i] = \sum_{k\geq 1} P(X_i \geq k)$

$$\begin{aligned} P(X_i \geq k) &= \sum_{\vec{n}\in S(N,K), n_i\geq k} \pi(\vec{n}) = \sum_{\vec{n}\in S(N,K), n_i\geq k} \dots \\ &= \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{C(N-k,K)}{C(N,K)} \end{aligned} \quad (1)$$

Performance indexes

One may also be interested in the utilization at node i, i.e. the probability that node i is non-empty.

Utilization

$$U_i = 1 - P(X_i = 0) = \frac{\lambda_i}{\mu_i} \frac{C(N-K, K)}{C(N, K)}$$

Proof

Note that $P(X_i = k) = P(X_i \geq k) - P(X_i \geq k-1)$ and apply (1) with $k = 0$.

Convolution algorithm

Computing the normalization factor $C(N,K)$ is a heavy task!

$$C(n,k) = \sum_{\vec{n}\in S(n,k)} \prod_{i=1}^{k} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} = \sum_{m=0}^{n} \sum_{\substack{\vec{n} \in S(n,k) \\ n_k = m}} \prod_{i=1}^{k} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}$$

$$= \sum_{m=0}^{n} \left(\frac{\lambda_k}{\mu_k}\right)^{m} \sum_{\vec{n}\in S(n-m,k-1)} \prod_{i=1}^{k-1} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i}$$

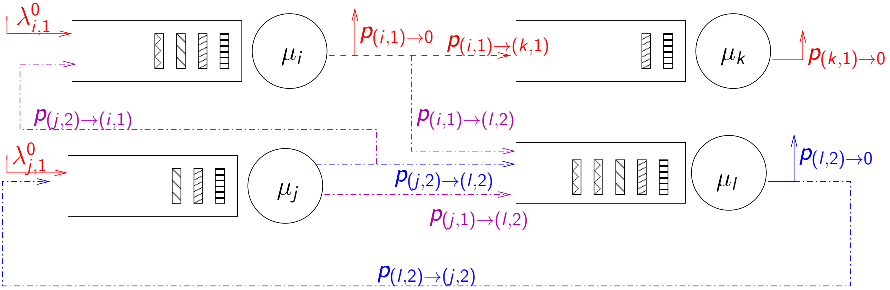
Convolution algorithm (Buzen,1973)

$$C(n,k) = \sum_{m=0}^{n} \left(\frac{\lambda_k}{\mu_k}\right)^{m} C(k-m,k-1) \text{ and } \begin{cases} C(n,1) & = & \left(\frac{\lambda_1}{\mu_1}\right)^{n} \\ C(0,k) & = & 1, \forall 1 \leq i \leq K \end{cases}$$

Outline

Multiclass Networks

Definition



Multiclass Networks

Definition

- $K < \infty$ nodes and $R < \infty$ classes
- Customer at node i in class r will go to node j with class s with probability $p_{(i,r);(j,s)}$
- (i, r) and (j, s) belong to the same subchain if $p_{(i,r);(j,s)} > 0$
- FIFO discipline and exponential service times

Definition

A subchain is open iff there exist one pair (i, r) for which $\lambda^0_{(i,r)} > 0$.

Definition

A mixed system contains at least one open subchain and one closed subchain.

Multiclass Networks

Definition

The state of a multiclass network may be characterized by the number of customers of each class at each node

$$\vec{Q}(t) = (\vec{Q}_1(t), \vec{Q}_2(t), \dots, \vec{Q}_K(t)) \text{ with } \vec{Q}_i(t) = (Q_{i1}(t)), \dots, Q_{iR(t)})$$

Problem

$\vec{Q}(t)$ is not a CMTC!

To see why, consider the FIFO discipline: how do you know the class of the next customer?

Multiclass Networks

Definition

Define $\vec{X}_i(t) = (I_{i1}(t), \ldots, I_{iQ_i(t)}(t))$ with $I_{ij}(t)$ the class of the jth customer at node i.

Proposition

$\vec{X}(t)$ is a CMTC!

Solving the balance equations for X gives a product-form solution. The steady-state distribution of $\vec{X}(t)$ also gives the distribution of $\vec{Q}(t)$ by aggregation of states.

Outline

Other queueing networks

Limitations of Jackson networks

Jackson networks imply

- FIFO discipline
- probabilistic routing

These assumptions can be relaxed using BCMP and Kelly networks.

BCMP networks

[Baskett, Chandy, Muntz and Palacios 1975]

Definition

BCMP networks are multiclass networks with exponential service times and c_i servers at node i.

Service disciplines may be:

- FCFS
- Processor Sharing
- Infinite Server
- LCFS

BCMP networks also have product-form solution!

BCMP networks

Definitions

Consider an open/closed/mixed BCMP network with K nodes and R classes in which each node is either FIFO,PS,LIFO or IS. Define

- $\rho_{ir} = \frac{\lambda_{ir}}{\mu_{ir}}$ for LIFO, IS and PS nodes
- $\rho_{ir} = \frac{\lambda_{ir}}{\mu_i}$ for FIFO nodes
- $\lambda_{ir} = \lambda_{ir}^0 + \sum_{(j,s)\in E_k} \lambda_{js} p_{(i,r);(j,s)}$ for any (i,r) of each open subchain E_k
- $\lambda_{ir} = \sum_{(j,s)\in E_m} \lambda_{js} p_{(i,r);(j,s)}$ for any (i,r) of each closed subchain E_m

BCMP networks

Main result

Theorem

The steady-state distribution is given by: for all $\vec{n}$ in state space $\mathcal{S}$,

$$\pi(\vec{n}) = \frac{1}{G}\prod_{i=1}^{K} f_i(\vec{n}_i) \quad \text{with } G = \sum_{\vec{n}\in\mathcal{S}}\prod_{i=1}^{K} f_i(\vec{n}_i)$$

with $\vec{n} = (\vec{n}_1, \ldots, \vec{n}_K) \in \mathcal{S}$ and $\vec{n}_i = (n_{i1}, \ldots, n_{iR})$, if and only if (stability condition for open subchains) $\sum_{r:(i,r)\in \text{ any open } E_k} \rho_{ir} < 1, \quad \forall 1 \leq i \leq K.$

Moreover, $f_i(\vec{n}_i)$ has an explicit expression for each service discipline.

BCMP networks

Main result

FIFO $f_i(\vec{n}_i) = |n_i|! \prod_{j=1}^{|n_i|} \frac{1}{\alpha_i(j)} \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$ with $\alpha_j(j) = min(c_i, j)$.

PS or LIFO $f_i(\vec{n}_i) = |n_i|! \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$

IS $f_i(\vec{n}_i) = \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}$

Extensions

the BCMP product form result may be extended to the following cases:

- state-dependent routing probabilities
- arrivals depending on the number of customers in the corresponding subchain