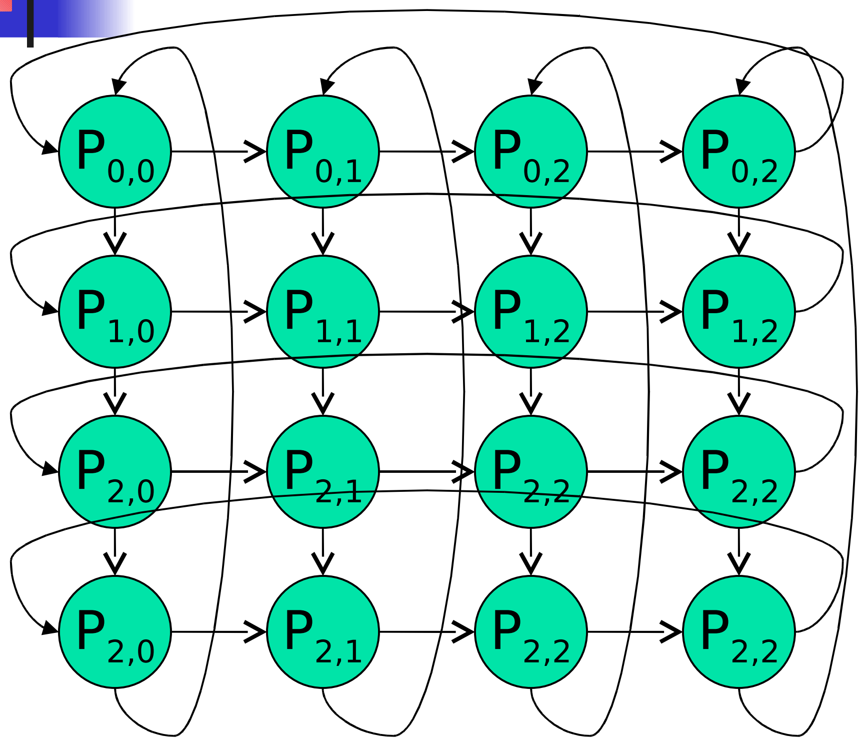
Principles of High Performance Computing (ICS 632)

Algorithms on a Grid of Processors (II)

2-D Matrix Distribution



- We denote by $a_{i,j}$ an element of the matrix
- We denote by $A_{i,j}$ (or A_{ij}) the block of the matrix allocated to $P_{i,j}$

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

The Cannon Algorithm

- This is a very old algorithm
  - From the time of systolic arrays
  - Adapted to a 2-D grid
- The algorithm starts with a redistribution of matrices A and B
  - Called "preskewing"
- Then the matrices are multiplied
- Then the matrices are re-redistributed to match the initial distribution
  - Called "postskewing"

Cannon's Preskewing

- Matrix A: each block row of matrix A is shifted so that each processor in the first processor column holds a diagonal block of the matrix

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{11}	A_{12}	A_{13}	A_{14}
A_{22}	A_{23}	A_{20}	A_{21}
A_{33}	A_{30}	A_{31}	A_{32}

Cannon's Preskewing

- Matrix B: each block column of matrix B is shifted so that each processor in the first processor row holds a diagonal block of the matrix

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

B_{00}	B_{11}	B_{22}	B_{33}
B_{10}	B_{21}	B_{32}	B_{03}
B_{20}	B_{31}	B_{02}	B_{13}
B_{30}	B_{01}	B_{12}	B_{23}

Cannon's Computation

- The algorithm proceeds in q steps
- At each step each processor performs the multiplication of its block of A and B and adds the result to its block of C
- Then blocks of A are shifted to the left and blocks of B are shifted upward
  - Blocks of C never move
- Let's see it on a picture

Cannon's Steps

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{11}	A_{12}	A_{13}	A_{10}
A_{22}	A_{23}	A_{20}	A_{21}
A_{33}	A_{30}	A_{31}	A_{32}

B_{00}	B_{11}	B_{22}	B_{33}
B_{10}	B_{21}	B_{32}	B_{03}
B_{20}	B_{31}	B_{02}	B_{13}
B_{30}	B_{01}	B_{12}	B_{23}

local computation on proc (0,0)

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{01}	A_{02}	A_{03}	A_{00}
A_{12}	A_{13}	A_{10}	A_{11}
A_{23}	A_{20}	A_{21}	A_{22}
A_{30}	A_{31}	A_{32}	A_{33}

B_{10}	B_{21}	B_{32}	B_{03}
B_{20}	B_{31}	B_{02}	B_{13}
B_{30}	B_{01}	B_{12}	B_{23}
B_{00}	B_{11}	B_{22}	B_{33}

Shifts

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{01}	A_{02}	A_{03}	A_{00}
A_{12}	A_{13}	A_{10}	A_{11}
A_{23}	A_{20}	A_{21}	A_{22}
A_{30}	A_{31}	A_{32}	A_{33}

B_{10}	B_{21}	B_{32}	B_{03}
B_{20}	B_{31}	B_{02}	B_{13}
B_{30}	B_{01}	B_{12}	B_{23}
B_{00}	B_{11}	B_{22}	B_{33}

local computation on proc (0,0)

The Algorithm

```
Participate in preskewing of A
Partitipate in preskweing of B
For k = 1 to q
   Local C = C + A*B
   Vertical shift of B
   Horizontal shift of A
Participate in postskewing of A
Partitipate in postskewing of B
```

Performance Analysis

- Let's do a simple performance analysis with a 4-port model
  - The 1-port model is typically more complicated
- Symbols
  - n: size of the matrix
  - $q \times q$: size of the processor grid
  - $m = n / q$
  - L: communication start-up cost
  - w: time to do a basic computation (+= . * .)
  - b: time to communicate a matrix element
- $T(m,q) = T_{preskew} + T_{compute} + T_{postskew}$

Pre/Post-skewing times

- Let's consider the horizontal shift
- Each row must be shifted so that the diagonal block ends up on the first column
- On a mono-directional ring:
  - The last row needs to be shifted (q-1) times
  - All rows can be shifted in parallel
  - Total time needed: $(q-1)\ (L + m^2\ b)$
- On a bi-directional ring, a row can be shifted left or right, depending on which way is shortest!
  - A row is shifted at most floor(q/2) times
  - All rows can be shifted in parallel
  - Total time needed: floor(q/2) $(L + m^2\ b)$
- Because of the 4-port assumption, preskewing of A and B can occur in parallel (horizontal and vertical shifts do not interfere)
- Therefore: Tpreskew = Tpostskew = floor(q/2) $(L+m^2b)$

Time for each step

- At each step, each processor computes an mxm matrix multiplication
  - Compute time: m^3 w
- At each step, each processor sends/receives a mxm block in its processor row and its processor column
  - Both can occur simultaneously with a 4-port model
  - Takes time L+ m^2b
- Therefore, the total time for the q steps is: Tcompute = q max (L + m^2b, m^3w)

Cannon Performance Model

- $T(m,n) = 2 * \text{floor}(q/2)\ (L + m^2 b) + q \max(m^3 w, L + m^2 b)$
- This performance model is easily adapted
  - If one assumes mono-directional links, then the "floor(q/2)" above becomes "(q-1)"
  - If one assumes 1-port, there is a factor 2 added in front of communication terms
  - If one assumes no overlap of communication and computation at a

The Fox Algorithm

- This algorithm was originally developed to run on a hypercube topology
  - But in fact it uses a grid, embedded in the hypercube
- This algorithm requires no pre- or post-skewing
- It relies on horizontal broadcasts of the diagonals of matrix A and on vertical shifts of matrix B
- Sometimes called the “multiply-broadcast-roll” algorithm
- Let’s see it on a picture
  - Although it’s a bit awkward to draw because of

Execution Steps...

C_{00}	C_{01}	C_{02}	C_{03}		A_{00}	A_{01}	A_{02}	A_{03}		B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}		A_{10}	A_{11}	A_{12}	A_{13}		B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}		A_{20}	A_{21}	A_{22}	A_{23}		B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}		A_{30}	A_{31}	A_{32}	A_{33}		B_{30}	B_{31}	B_{32}	B_{33}

initial state

C_{00}	C_{01}	C_{02}	C_{03}		A_{00}	A_{00}	A_{00}	A_{00}		B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}		A_{11}	A_{11}	A_{11}	A_{11}		B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}		A_{22}	A_{22}	A_{22}	A_{22}		B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}		A_{33}	A_{33}	A_{33}	A_{33}		B_{30}	B_{31}	B_{32}	B_{33}

Broadcast of A’s 1st diag. (stored in a Separate buffer)

C_{00}	C_{01}	C_{02}	C_{03}		A_{00}	A_{00}	A_{00}	A_{00}		B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}		A_{11}	A_{11}	A_{11}	A_{11}		B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}		A_{22}	A_{22}	A_{22}	A_{22}		B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}		A_{33}	A_{33}	A_{33}	A_{33}		B_{30}	B_{31}	B_{32}	B_{33}

Local computation

Execution Steps...

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}
B_{00}	B_{01}	B_{02}	B_{03}

Shift of B

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{01}	A_{01}	A_{01}	A_{01}
A_{12}	A_{12}	A_{12}	A_{12}
A_{23}	A_{23}	A_{23}	A_{23}
A_{30}	A_{30}	A_{30}	A_{30}

B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}
B_{00}	B_{01}	B_{02}	B_{03}

Broadcast of A’s 2nd diag. (stored in a Separate buffer)

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{01}	A_{01}	A_{01}	A_{01}
A_{12}	A_{12}	A_{12}	A_{12}
A_{23}	A_{23}	A_{23}	A_{23}
A_{30}	A_{30}	A_{30}	A_{30}

B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}
B_{00}	B_{01}	B_{02}	B_{03}

Local computation

Fox's Algorithm

```
// No initial data movement
for k = 1 to q  in parallel
   Broadcast A's kth diagonal
   Local C = C + A*B
   Vertical shift of B
// No final data movement
```

- Again note that there is an additional array to store incoming diagonal block
- This is the array we use in the A*B multiplication

Performance Analysis

- You’ll have to do it in a homework assignment
  - Write pseudo-code of the algorithm in more details
  - Write the performance analysis

Snyder’s Algorithm (1992)

- More complex than Cannon’s or Fox’s
- First transposes matrix B
- Uses reduction operations (sums) on the rows of matrix C
- Shifts matrix B

Execution Steps...

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{10}	A_{11}	A_{12}	A_{13}	B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{20}	A_{21}	A_{22}	A_{23}	B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{30}	B_{31}	B_{32}	B_{33}

initial state

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{00}	B_{10}	B_{20}	B_{30}
C_{10}	C_{11}	C_{12}	C_{13}	A_{10}	A_{11}	A_{12}	A_{13}	B_{01}	B_{11}	B_{21}	B_{31}
C_{20}	C_{21}	C_{22}	C_{23}	A_{20}	A_{21}	A_{22}	A_{23}	B_{02}	B_{12}	B_{22}	B_{32}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{03}	B_{13}	B_{23}	B_{33}

Transpose B

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{00}	B_{10}	B_{20}	B_{30}
C_{10}	C_{11}	C_{12}	C_{13}	A_{10}	A_{11}	A_{12}	A_{13}	B_{01}	B_{11}	B_{21}	B_{31}
C_{20}	C_{21}	C_{22}	C_{23}	A_{20}	A_{21}	A_{22}	A_{23}	B_{02}	B_{12}	B_{22}	B_{32}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{03}	B_{13}	B_{23}	B_{33}

Local computation

Execution Steps...

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{01}	B_{11}	B_{21}	B_{31}
B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{32}
B_{00}	B_{10}	B_{20}	B_{30}

Shift B

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{01}	B_{11}	B_{21}	B_{31}
B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{32}
B_{00}	B_{10}	B_{20}	B_{30}

Global sum
on the rows
of C

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{01}	B_{11}	B_{21}	B_{31}
B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{32}
B_{00}	B_{10}	B_{20}	B_{30}

Local
computation

Execution Steps...

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{33}
B_{00}	B_{10}	B_{20}	B_{30}
B_{01}	B_{11}	B_{21}	B_{31}

Shift B

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{33}
B_{00}	B_{10}	B_{20}	B_{30}
B_{01}	B_{11}	B_{21}	B_{31}

Global sum on the rows of C

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{02}	B_{12}	B_{22}	B_{32}
B_{03}	B_{13}	B_{23}	B_{33}
B_{00}	B_{10}	B_{20}	B_{30}
B_{01}	B_{11}	B_{21}	B_{31}

Local computation

The Algorithm

var A,B,C: array[0..m-1][0..m-1] of real

var bufferC: array[0..m-1][0..m-1] of real

Transpose B

MatrixMultiplyAdd(bufferC, A, B, m)

Vertical shifts of B

For k = 1 to q-1

    Global sum of bufferC on proc rows into $C_{i,(i+k-1)\%q}$

    MatrixMultiplyAdd(bufferC, A, B, m)

    Vertical shift of B

Global sum of bufferC on proc rows into $C_{i,(i+k-1)\%q}$

Transpose B

Performance Analysis

- The performance analysis isn't fundamentally different than what we've done so far
- But it's a bit cumbersome
- See the textbook
  - in particular the description of the matrix transposition (see also Exercise 5.1)

Which Data Distribution?

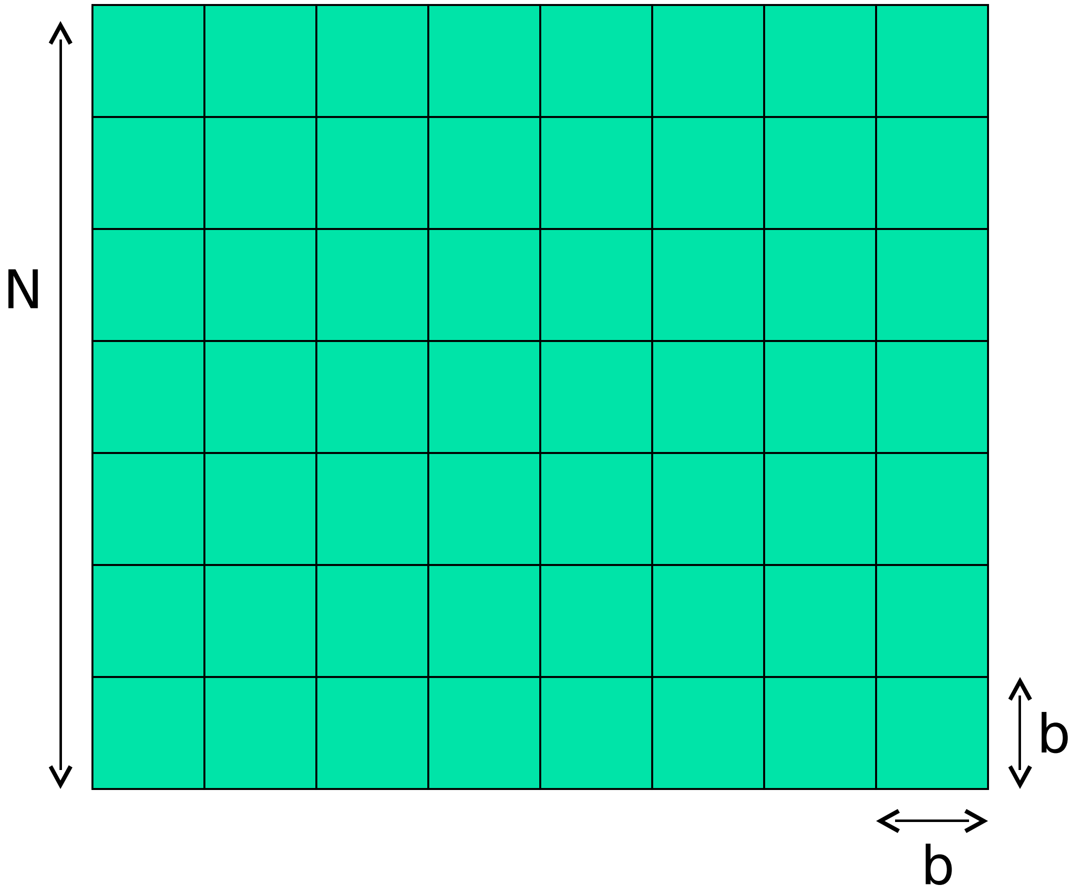
- So far we've seen:
  - Block Distributions
  - 1-D Distributions
  - 2-D Distributions
  - Cyclic Distributions
- One may wonder what a good choice is for a data distribution?
- Many people argue that a good "Swiss Army knife" is the "2-D block cyclic distribution

The 2-D block cyclic distribution

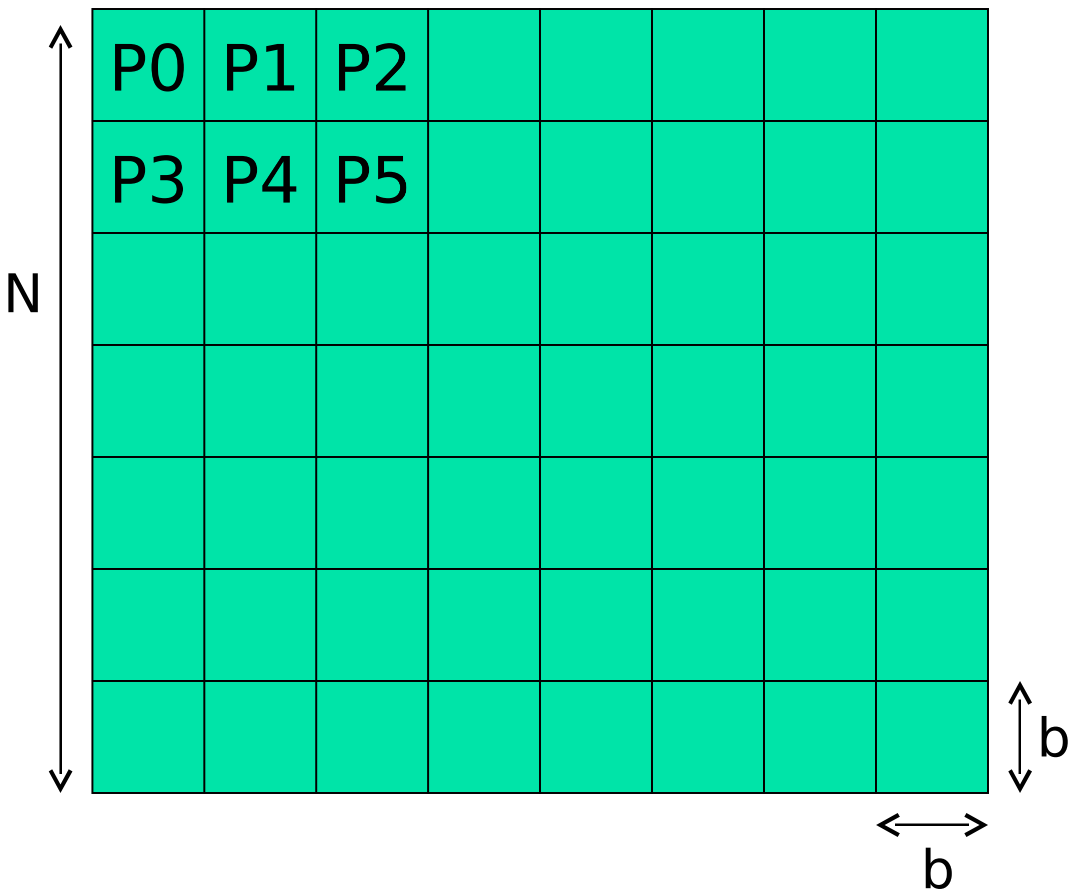
- Goal: try to have all the advantages of both the horizontal and the vertical 1-D block cyclic distribution
  - Works whichever way the computation "progresses"
    - left-to-right, top-to-bottom, wavefront, etc.
- Consider a number of processors p = r * c
  - arranged in a rxc matrix
- Consider a 2-D matrix of size NxN
- Consider a block size b (which divides N)

The 2-D block cyclic distribution

P0	P1	P2
P3	P4	P5



The 2-D block cyclic distribution



P0	P1	P2
P3	P4	P5

The 2-D block cyclic distribution

P0	P1	P2	P0	P1	P2	P0	P1
P3	P4	P5	P3	P4	P5	P3	P4
P0	P1	P2	P0	P1	P2	P0	P1
P3	P4	P5	P3	P4	P5	P3	P4
P0	P1	P2	P0	P1	P2	P0	P1
P3	P4	P5	P3	P4	P5	P3	P4
P0	P1	P2	P0	P1	P2	P0	P1

N

b

b

P0	P1	P2
P3	P4	P5

- Slight load imbalance
  - Becomes negligible with many blocks
- Index computations had better be implemented in separate functions
- Also: functions that tell a process who its neighbors are
- Overall, requires a whole infrastructure, but many think you can't go wrong with this distribution

Conclusion

- All the algorithms we have seen in the semester can be implemented on a 2-D block cyclic distribution
- The code ends up much more complicated
- But one may expect several benefits “for free”
- The ScaLAPAK library recommends to use the 2-D block cyclic distribution
  - Although its routines support all other distributions