Markov Chains, Iterated System of Functions and Coupling time for Perfect Simulation

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Outline

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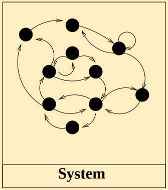
1. Markov chains and simulation
   - Application problems
   - Formalization
   - Simulation and Random Iterated System of Functions
2. Algorithms and Markov chains
   - Visual representation
   - Forward simulation : convergence and bias
   - Backward simulation : coupling time
   - The coupling problem
3. Coupling time and representation
   - Minimize the coupling time
   - Doeblin matrices
   - Binary-Uniform decomposition
4. Future works

Application problems :Modeling and analysis of complex systems

Complex system



Basic model assumptions

System :
- automaton (discrete state space)
- **discrete** or continuous time

Environment : non deterministic
- time homogeneous
- stochastically regular

Problem

Generate "typical" states
- steady-state sampling
- ergodic simulation starting point
- state space exploring techniques

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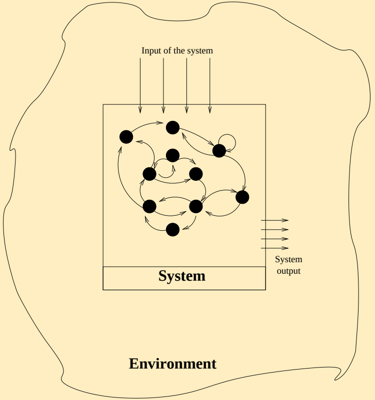

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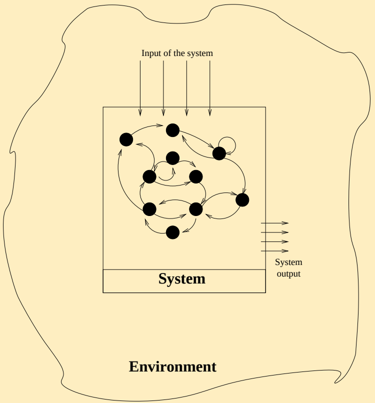

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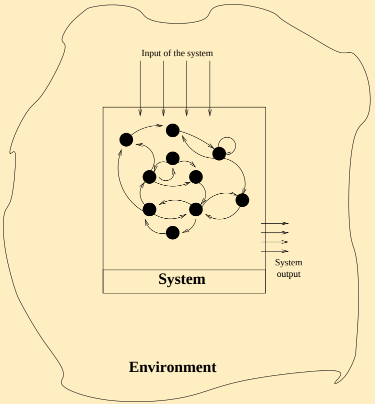

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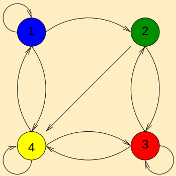
Formalization

Quantification



Stochastic matrix : transition probability

$$P = \frac{1}{12}\begin{bmatrix} 2 & 3 & 0 & 7 \\ 0 & 0 & 1 & 11 \\ 0 & 3 & 6 & 3 \\ 4 & 0 & 7 & 1 \end{bmatrix}$$

Non-negative, if irreducible and aperiodic
Unique probability vector π satisfying $\pi = \pi P$,
$\pi = \frac{1}{350}[46, 47, 142, 115]$

Solving $\pi = \pi P$

Formal methods $N \leqslant 50$
Direct numerical methods $N \leqslant 1000$
Iterative methods with preconditioning $N \leqslant 100,000$
Adapted methods (structured Markov chains) $N \leqslant 1,000,000$
Monte-Carlo simulation $N \geqslant 10^6$

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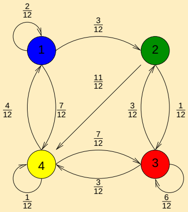

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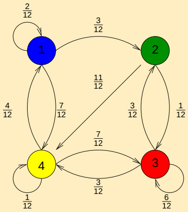

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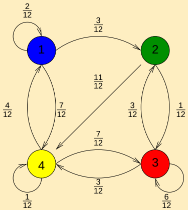

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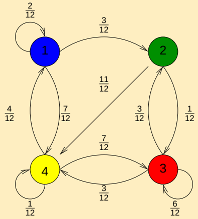
[0, 1] partitionning



Random iterated system of functions

Function	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

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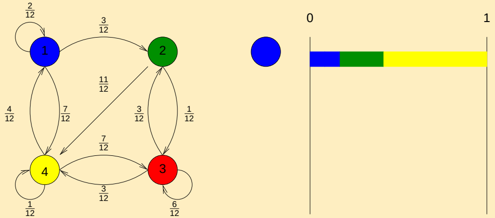

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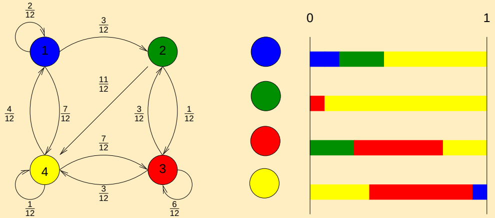

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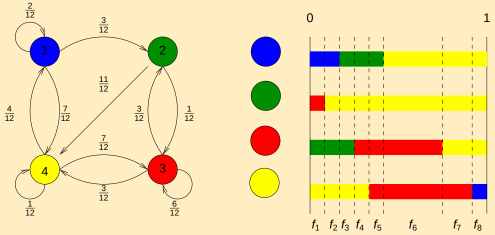

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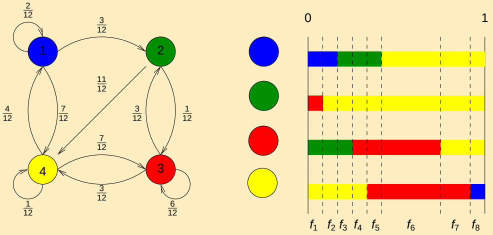

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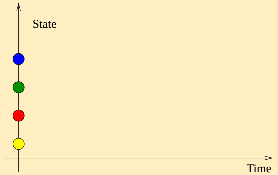
Forward simulation : convergence and bias

Forward iterations



Simulation bias

- choice of the initial state
- bounded error

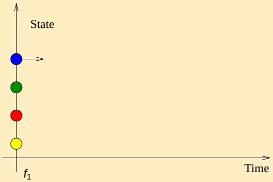
$$||\pi_n - \pi_\infty|| \leqslant C\lambda_2^n.$$

λ_2 second greatest eigenvalue of P

Proposition

The convergence of the forward simulation algorithm does not depend on the RIFS representation

Forward iterations



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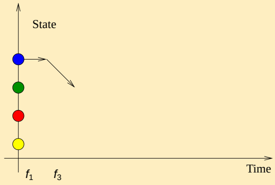

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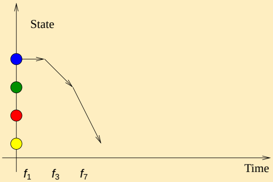

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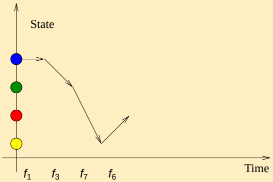

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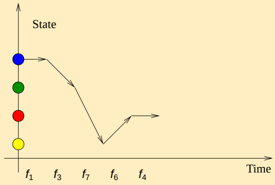

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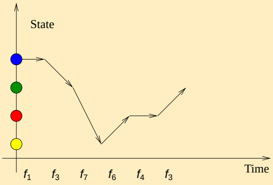

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Forward simulation : avoid initial state dependence

Forward coupling

Example

Always couple in the blue state

Does not guarantee the steady state !

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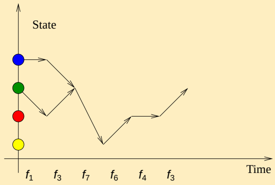

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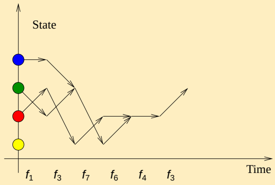

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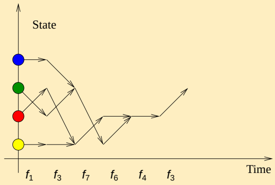

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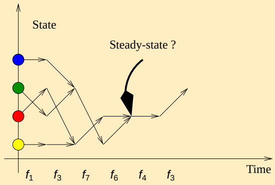

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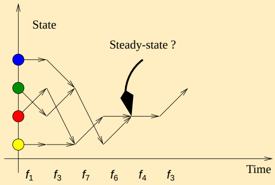

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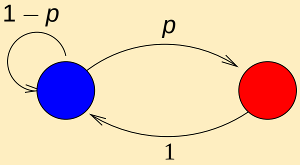

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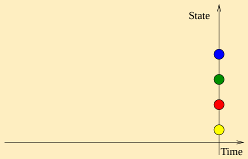

Convergence

When the algorithm stops the generated state is "typical" (stationary distributed).

Finite number of steps

$\Rightarrow$ unbiased generation (perfect)

Coupling condition

Coupling time τ

Proposition

The coupling time of the backward simulation algorithm depends on the RIFS representation

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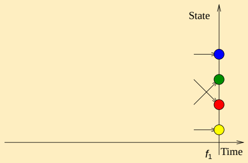

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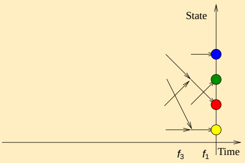

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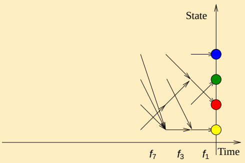

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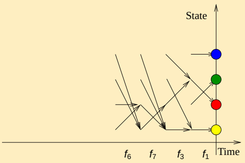

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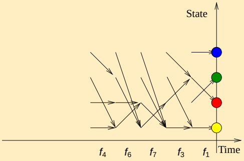

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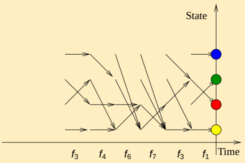

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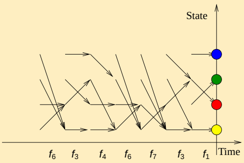

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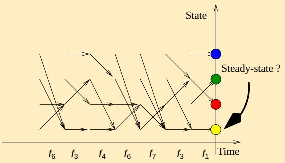

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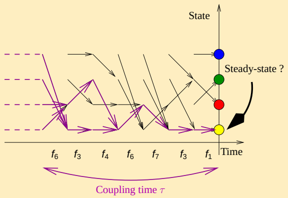

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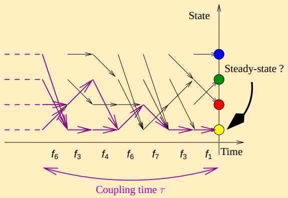

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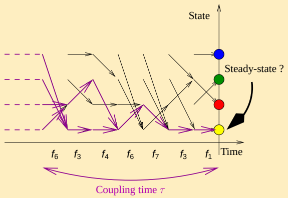

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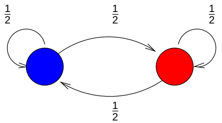

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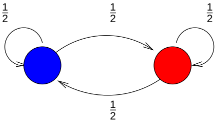
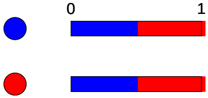
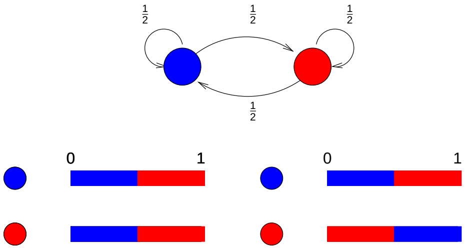




Couples with probability 1

$\tau = 1$

The coupling problem



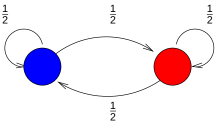
Couples with probability 1

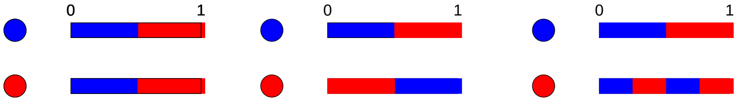
$\tau = 1$

Never couples

$\tau = \infty$

The coupling problem





Couples with probability 1

$\tau = 1$

Never couples

$\tau = \infty$

Couples with probability $\frac{1}{2}$

$\mathbb{E}\tau = 2$

Outline

General problem

Objective

Given a stochastic matrix $P = ((p_{i,j}))$ build a system of function $(f_\theta, \theta \in \Theta)$ and a probability distribution $(p_\theta, \theta \in \Theta)$ such that :

1. the RIFS implements the transition matrix P,
2. ensures coupling in finite time
3. achieve the “best” mean coupling time : tradeoff between
   - choice of the transition function according to $((p_\theta))$,
   - computation of the transition

Remarks

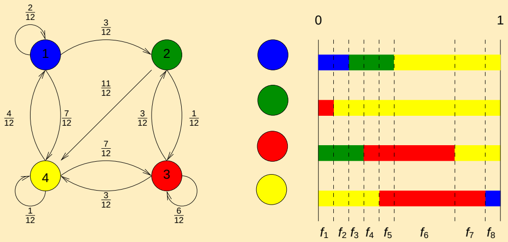
Usual method

$$|\Theta| = \text{number of non-negative elements of } P = \mathcal{O}(n^2)$$

choice in $\mathcal{O}(\log n)$

Non sparse matrices

Rearranging the system



Convergence

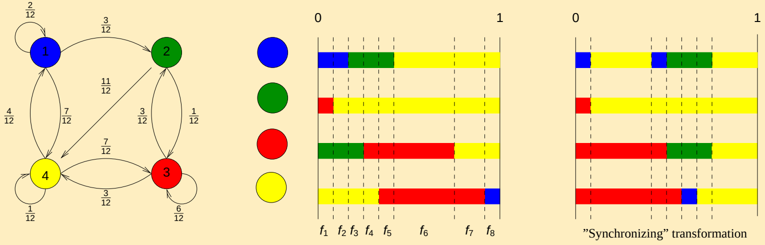
When at least one column is non-negative $\Rightarrow$ one step coupling.
The RIFS ensures coupling and the coupling time τ is upper bounded by a geometric distribution with rate

$$\sum_j \min_i p_{i,j}$$

number of transition functions : could be more than the number of non-negative elements

Non sparse matrices

Rearranging the system



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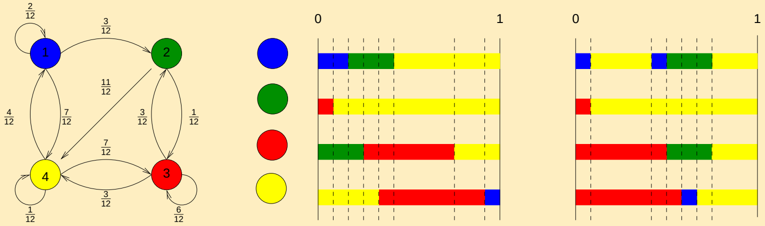


"Synchronizing" transformation

Convergence

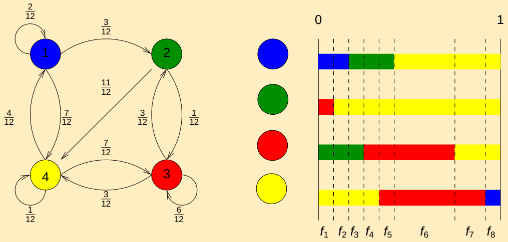
When at least one column is non-negative $\Rightarrow$ one step coupling.
The RIFS ensures coupling and the coupling time τ is upper bounded by a geometric distribution with rate

$$\sum_j \min_i p_{i,j}$$

number of transition functions : could be more than the number of non-negative elements

Sparse matrices

Rearranging the system

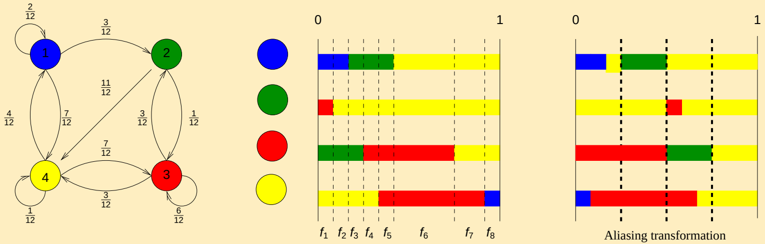


Complexity

M = maximum out degree of states
p_θ uniform on $\{1, \cdots, M\}$, threshold comparison
$\mathcal{O}(1)$ to compute one transition
Combination with "Synchronizing" techniques

Sparse matrices

Rearranging the system



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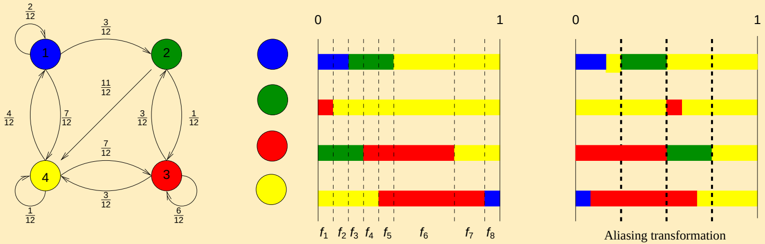
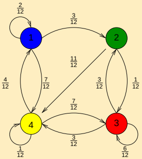


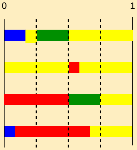
Complexity

M = maximum out degree of states
p_θ uniform on $\{1, \cdots, M\}$, threshold comparison
$\mathcal{O}(1)$ to compute one transition
Combination with "Synchronizing" techniques

Uniform-binary decomposition

Uniform superposition

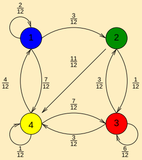


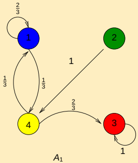


Decomposition

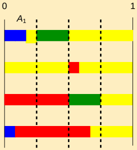
$$P = \frac{1}{M} \sum_{i=1}^{M} P_i, \quad P_i \text{ : stochastic matrix with at most 2 non negative elements per row}$$

Uniform superposition





A_1



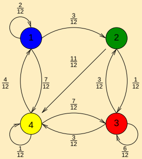
Aliasing transformation

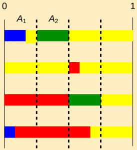
Decomposition

$$P = \frac{1}{M}\sum_{i=1}^{M} P_i, \quad P_i \;:\; \text{stochastic matrix with at most 2 non negative elements per row}$$

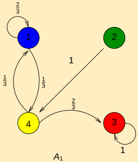
Uniform-binary decomposition

Uniform superposition





Aliasing transformation



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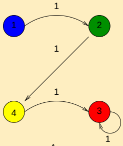


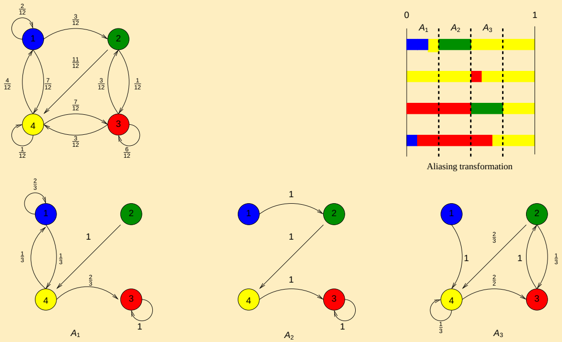
A_2

Decomposition

$$P = \frac{1}{M}\sum_{i=1}^{M} P_i, \quad P_i \text{ : stochastic matrix with at most 2 non negative elements per row}$$

Uniform-binary decomposition

Uniform superposition



Decomposition

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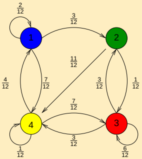
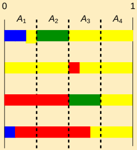
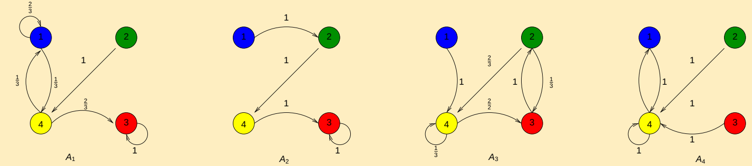
Uniform-binary decomposition

Uniform superposition





Aliasing transformation



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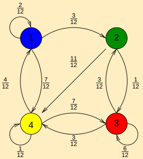
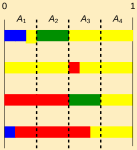
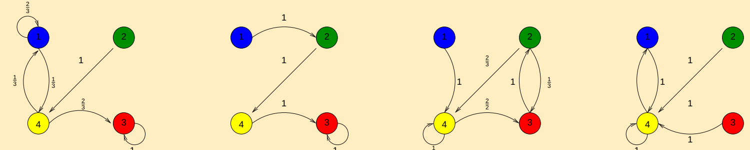
Uniform-binary decomposition

Uniform superposition





Aliasing transformation



A_1 A_2 A_3 A_4

Decomposition

$$P = \frac{1}{M} \sum_{i=1}^{M} P_i, \quad P_i \text{ : stochastic matrix with at most 2 non negative elements per row}$$

Outline

Future works

Complexity

- find optimal representation
- find minimal representation

explore heuristics

Applications

performance evaluation

- queueing networks (software PSI2)
- dense or sparse matrices (software PSI) state space : $\simeq 2^{32}$

Fundamental properties

monotonicity of the functions

- find partial order such that the RIFS is monotone

-find the optimal order

$\Rightarrow$ Reduction by n of the simulation time

coupling time computation

- link with matrix properties