Random Matrix Advances in Signal Processing

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High-dimensional data

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From the law of large numbers, as $n \to \infty$,

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with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$.
In reality, one cannot afford $n \to \infty$.

- if $n \gg N$,

$$\mathbf{R}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^{\mathsf{H}}$$

  is a "good" estimator of $\mathbf{R}$.
- if $N/n = O(1)$, and if both (n, N) are large, we can still say, for all (i, j),

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What about the global behaviour? What about the eigenvalue distribution?

Assume $\mathbf{R} = \mathbf{I}_N$ and draw the eigenvalues of $\mathbf{R}_n$ for n, N large.

Empirical and limit spectra of Wishart matrices

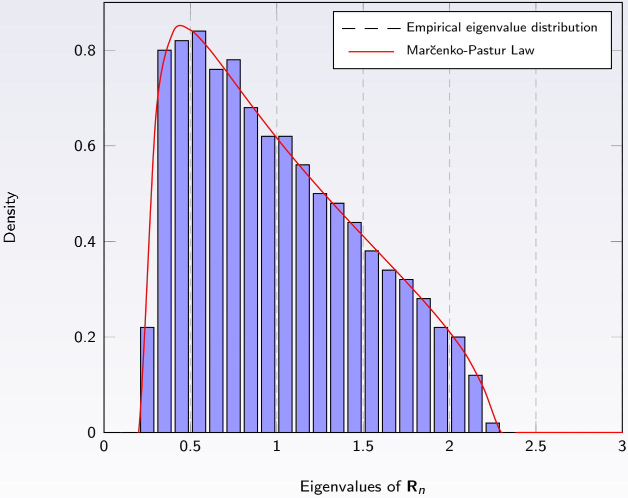

Figure : Histogram of the eigenvalues of $\mathbf{R}_n$ for $n = 2000$, $N = 500$, $\mathbf{R} = \mathbf{I}_N$

Finite size against asymptotic considerations

The field of random matrices is often segmented into

- *Finite-size random matrices*:
  - of interest are: joint entry distributions, ordered eigenvalue distributions, e.s.d., expectation of functionals
  - particularly suitable to small size matrices
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  - however, much problems arise for models more involved than i.i.d. Gaussian
- *Limiting results*:
  - of interest are: limit spectral distributions (l.s.d.), functionals of l.s.d., central limit theorems etc.
  - suitable to large matrices, but often good approximation to smaller matrices
  - much easier to work with than finite size, more flexible (i.i.d., Kronecker, variance profile models, structured matrices)
  - possesses a variety of powerful tools: Stieltjes transform, free probability

Remark: This tutorial will exclusively focus on limiting results.

Why is this useful to wireless communications?

- increasing number of parameters: multi-user systems, multiple concurrent cells, multiple antennas
- matrices with random entries are the basis for MIMO channels, CDMA codes
- it is no longer possible to treat large dimensional problems with classical probability approaches

Example

MIMO channel capacity Call $\mathbf{H} \in \mathbb{C}^{n\times N}$ the realization of a MIMO channel matrix whose entries and distributed according to some random process. We have the per-antenna mutual information

$$C(\sigma^2) = \frac{1}{N} \log\det\left[\mathbf{I}_N + \frac{1}{\sigma^2}\mathbf{H}\mathbf{H}^{\mathsf{H}}\right]$$

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Note that, with $\mathbf{h}_i$ the i^{th} column of $\mathbf{H}$, $\mathbf{H}\mathbf{H}^{\mathsf{H}} = \sum_{i=1}^{N} \mathbf{h}_i \mathbf{h}_i^{\mathsf{H}}$. If $\mathbf{H}$ has i.i.d. entries, then, as both $n, N \to \infty$, $n/N \to c$,

$$C(\sigma^2) \to \int \log \left[1 + \frac{t}{\sigma^2} \right] dF_c(t)$$

with F_c the Marčenko-Pastur law with parameter c.

Why is this useful to signal processing?

- increasing system dimensions: large antenna arrays, large datasets, (not so) large number of snapshots
- need for detection and estimation based on large dimensional random inputs: subspace methods in array processing
- the assumption "sample space $>>$ population space" is less and less valid: large arrays, systems with fast dynamics

Example

MUSIC with "few" samples (or in large arrays) Call $\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)] \in \mathbb{C}^{N \times K}$, N large, K small, the steering vectors to identify and $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \in \mathbb{C}^{N \times n}$ the n samples, taken from

$$y_t = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\sqrt{p}_k s_{k,t} + \sigma w_t.$$

The MUSIC localization function reads $\gamma(\theta) = \mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{U}}_W\hat{\mathbf{U}}_W^{\mathsf{H}}\mathbf{a}(\theta)$ in the "signal vs. noise" spectral decomposition $\mathbf{YY}^{\mathsf{H}} = \hat{\mathbf{U}}_S\hat{\mathbf{\Lambda}}_S\hat{\mathbf{U}}_S^{\mathsf{H}} + \hat{\mathbf{U}}_W\hat{\mathbf{\Lambda}}_W\hat{\mathbf{U}}_W^{\mathsf{H}}$.

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Writing equivalently $\mathbf{A}(\Theta)\mathbf{P}\mathbf{A}(\Theta)^{\mathsf{H}} + \sigma^2\mathbf{I}_N = \mathbf{U}_S\mathbf{\Lambda}_S\mathbf{U}_S^{\mathsf{H}} + \sigma^2\mathbf{U}_W\mathbf{U}_W^{\mathsf{H}}$, as $n, N \to \infty$, $n/N \to c$, from our previous remarks

$$\mathbf{U}_W\mathbf{U}_W^{\mathsf{H}} \not\to \hat{\mathbf{U}}_W\hat{\mathbf{U}}_W^{\mathsf{H}}$$

$\Rightarrow$ Music is NOT consistent in the large N, n regime! We need improved RMT-based solutions.

Marčenko-Pastur law, Semi-circle law, Full circle law...

V. A. Marčenko, L. A. Pastur, "Distributions of eigenvalues for some sets of random matrices", Math USSR-Sbornik, vol. 1, no. 4, pp. 457-483, 1967.

- If $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ has i.i.d. entries of mean 0, variance $1/n$, then (almost surely) $F^{\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}} \Rightarrow F_c$ as $N, n \to \infty$, $N/n \to c$, with F_c the Marčenko-Pastur law with density

$$f_c(x) = (1-c^{-1})^+\delta(x) + \frac{1}{2\pi cx}\sqrt{(x-a)^+(b-x)^+}, \quad a = (1-\sqrt{c})^2,\ b = (1+\sqrt{c})^2.$$

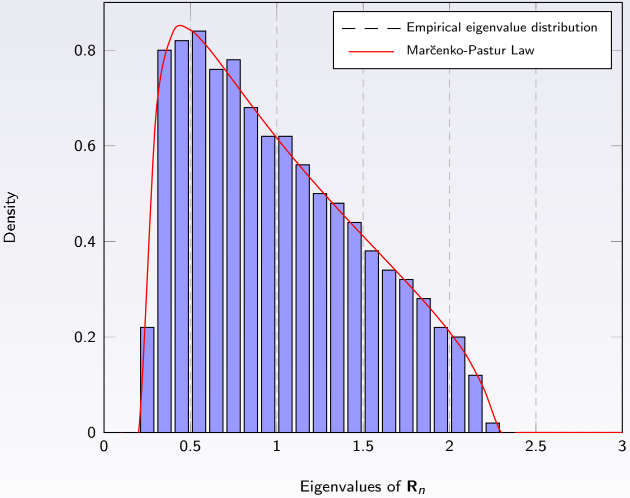
E. Wigner, "Characteristic vectors of bordered matrices with infinite dimensions," The annals of mathematics, vol. 62, pp. 546-564, 1955.

- If $\mathbf{X}_N \in \mathbb{C}^{N\times N}$ is Hermitian with i.i.d. entries of mean 0, variance $1/N$, then (almost surely) $F^{\mathbf{X}_N} \Rightarrow F$ where F has density f the semi-circle law

$$f(x) = \frac{1}{2\pi}\sqrt{(4-x^2)^+}.$$

- If $\mathbf{X}_N \in \mathbb{C}^{N\times N}$ has with i.i.d. 0 mean, variance $1/N$ entries, then asymptotically its complex eigenvalues distribute uniformly on the complex unit circle.

Marčenko-Pastur law



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Semi-circle law

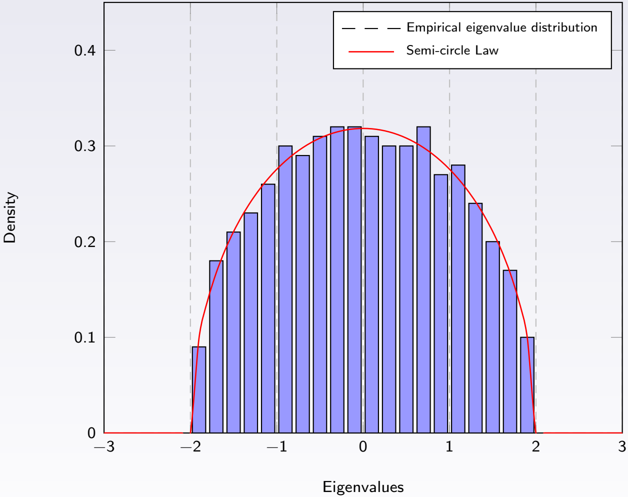

Figure : Histogram of the eigenvalues of Wigner matrices and the semi-circle law, for $N = 500$

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$$\lim_{N \to \infty} \frac{1}{N} \mathrm{tr}\left(\mathbf{X}^{2k+1}\right) = 0$$
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- These are exactly the moments of a semi-cicle distribution!

$$\alpha_{2k} = \frac{1}{\pi} \int_{-2}^{2} x^{2k} \sqrt{4-x^2} dx = -\frac{1}{2\pi} \int_{-2}^{2} \frac{-x}{\sqrt{4-x^2}} x^{2k-1} (4-x^2) dx$$
$$= \frac{1}{2\pi} \int_{-2}^{2} \sqrt{4-x^2} (x^{2k-1}(4-x^2))' dx = 4(2k-1)\alpha_{2k-2} - (2k+1)\alpha_{2k}.$$

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Proof impractical for more involved models

Difficult in general to move from moments to distributions / to compute the moments directly.

Circular law

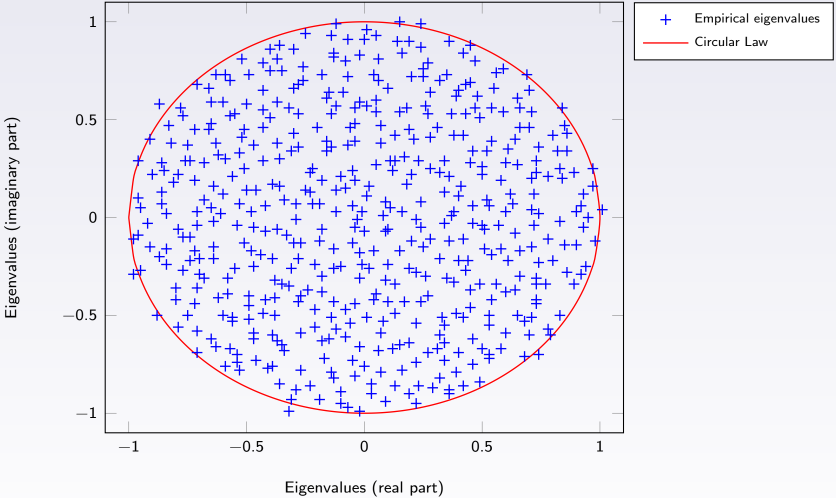

Figure : Eigenvalues of $\mathbf{X}_N$ with i.i.d. standard Gaussian entries, for $N = 500$.

More involved matrix models

- much study has surrounded the Marčenko-Pastur law, the Wigner semi-circle law etc.
- for practical purposes, we often need more general matrix models
  - products and sums of random matrices
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Tools for random matrix theory

To study these models, a consistent powerful mathematical framework is required.

Tools for RMT

Various approaches used to deal with random matrices.

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  - *Orthogonal polynomials and Fredholm determinants*: study hole probability, e.g. extreme eigenvalue distribution through determinantal equations [e.g. Johnstone, Tracy, Widom, Guionnet]

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Definition

Let F be a real probability distribution function. The Stieltjes transform m_F of F is the function defined, for $z \in \mathbb{C}^+$, as

$$m_F(z) = \int \frac{1}{\lambda - z} dF(\lambda)$$

For $a < b$ continuity points of F, denoting $z = x + iy$, we have the inverse formula

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Equivalence $F \leftrightarrow m_F$

Similar to the Fourier transform, knowing m_F is the same as knowing F.

Stieltjes transform and Matrix Spectra

- If F is the e.s.d. of a Hermitian matrix $\mathbf{X} \in \mathbb{C}^{N\times N}$, we might denote $m_{\mathbf{X}} \triangleq m_F$, and

$$m_{\mathbf{X}}(z) = \int \frac{1}{\lambda - z} dF(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{N}\operatorname{tr}\left(\operatorname{diag}(\{\lambda_i\}) - z\mathbf{I}_N\right)^{-1} = \frac{1}{N}\operatorname{tr}\left(\mathbf{X}_N - z\mathbf{I}_N\right)^{-1}$$

- For compactly supported F, $m_F(z)$ is linked to the moments $M_k = E[\frac{1}{N}\operatorname{tr}(\mathbf{X}^k)]$

$$m_F(z) = -\sum_{k=0}^{\infty} M_k z^{-k-1}$$

- m_F defined in general on $\mathbb{C}^+$ but exists everywhere outside the support of F.
- if $\mathbf{X} \in \mathbb{C}^{N\times n}$, the spectral distribution of $\mathbf{X}\mathbf{X}^{\mathrm{H}}$ and $\mathbf{X}^{\mathrm{H}}\mathbf{X}$ only differ by a mass of $|N-n|$ zeros. Say $N \geqslant n$,

$$m_{\mathbf{X}\mathbf{X}^{\mathrm{H}}}(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z} = \frac{1}{N}\sum_{i=1}^{n}\frac{1}{\lambda_i - z} + \frac{1}{N}(N-n)\frac{-1}{z}$$

hence

$$m_{\mathbf{X}\mathbf{X}^{\mathrm{H}}}(z) = \frac{n}{N} m_{\mathbf{X}^{\mathrm{H}}\mathbf{X}} - \frac{N-n}{N}\frac{1}{z}$$

Asymptotic results using the Stieltjes transform

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

Theorem

Let $\underline{\mathbf{B}}_N = \mathbf{X}_N \mathbf{T}_N \mathbf{X}_N^{\mathsf{H}} \in \mathbb{C}^{N \times N}$, *where* $\mathbf{X}_N \in \mathbb{C}^{N \times n}$ *has i.i.d. entries of mean* 0 *and variance* $1/N$, $F^{\mathbf{T}_N} \Rightarrow F^T$ *and* $n/N \to c$. *Then,* $F^{\underline{\mathbf{B}}_N}$ *converges weakly and almost surely to* $\underline{F}$ *with Stieltjes transform*

$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^T(t) - z \right)^{-1}$$

whose solution is unique in the set $\{z \in \mathbb{C}^+, m_{\underline{F}}(z) \in \mathbb{C}^+\}$.

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- in general, no explicit expression for $\underline{F}$.
- the theorem above characterizes also the Stieltjes transform of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N^{\mathsf{H}} \mathbf{X}_N \mathbf{T}_N^{\frac{1}{2}}$ with asymptotic distribution F,

$$m_F = c m_{\underline{F}} + (c-1)\frac{1}{z}$$

  This gives access to the spectrum of the sample covariance matrix model of $\mathbf{y}$, when $\mathbf{y}_i = \mathbf{T}_N^{\frac{1}{2}} \mathbf{x}_i$, $\mathbf{x}_i$ i.i.d., $\mathbf{T}_N = E[\mathbf{y}\mathbf{y}^{\mathsf{H}}]$.

Getting F' from m_F

- Remember that, for $a < b$ real,

$$F'(x) = \lim_{y \to 0} \frac{1}{\pi} \Im[m_F(x + iy)]$$

where m_F is (up to now) only defined on $\mathbb{C}^+$.

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- to plot the density F',
  - *first approach:* span $z = x + iy$ on the line $\{x \in \mathbb{R}, y = \varepsilon\}$ parallel but close to the real axis, solve $m_F(z)$ for each z, and plot $\Im[m_F(z)]$.

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Example (Sample covariance matrix)

For N multiple of 3, let $F^{\mathbf{T}_N}(x) = \frac{1}{3}\mathbf{1}_{x \leqslant 1} + \frac{1}{3}\mathbf{1}_{x \leqslant 3} + \frac{1}{3}\mathbf{1}_{x \leqslant K}$ and let $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}}\mathbf{X}_N^{\mathsf{H}}\mathbf{X}_N\mathbf{T}_N^{\frac{1}{2}}$ with $F^{\mathbf{B}_N} \to F$, then

$$m_F = c m_{\underline{F}} + (c-1)\frac{1}{z}$$

$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^T(t) - z\right)^{-1}$$

We take $c = 1/10$ and alternatively $K = 7$ and $K = 4$.

Spectrum of the sample covariance matrix

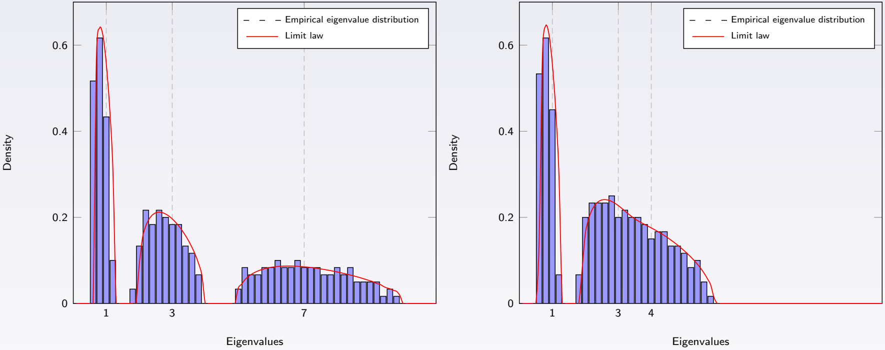

Figure : Histogram of the eigenvalues of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N^{\mathsf{H}} \mathbf{X}_N \mathbf{T}_N^{\frac{1}{2}}$, $N = 3000$, $n = 300$, with $\mathbf{T}_N$ diagonal composed of three evenly weighted masses in (i) 1, 3 and 7 on top, (ii) 1, 3 and 4 at bottom.

Side remark: the "Shannon"-transform

A. M. Tulino, S. Verdù, "Random matrix theory and wireless communications," Now Publishers Inc., 2004.

Definition

Let F be a probability distribution, m_F its Stieltjes transform, then the Shannon-transform $\mathcal{V}_F$ of F is defined as

$$\mathcal{V}_F(x) \triangleq \int_0^\infty \log(1 + x\lambda) dF(\lambda) = \int_x^\infty \left(\frac{1}{t} - m_F(-t) \right) dt$$

- This quantity is fundamental to wireless communication purposes!
- Note that m_F itself is of interest, not F!

Proof of the Marčenko-Pastur law

V. A. Marčenko, L. A. Pastur, "Distributions of eigenvalues for some sets of random matrices", Math USSR-Sbornik, vol. 1, no. 4, pp. 457-483, 1967.

The theorem to be proven is the following

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ have i.i.d. zero mean variance $1/n$ entries with finite eighth order moments. As $n, N \to \infty$ with $\frac{N}{n} \to c \in (0,\infty)$, the e.s.d. of $\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}$ converges almost surely to a nonrandom distribution function F_c with density f_c given by

$$f_c(x) = (1-c^{-1})^+\delta(x) + \frac{1}{2\pi c x}\sqrt{(x-a)^+(b-x)^+}$$

where $a = (1-\sqrt{c})^2$, and $b = (1+\sqrt{c})^2$.

The Marčenko-Pastur density

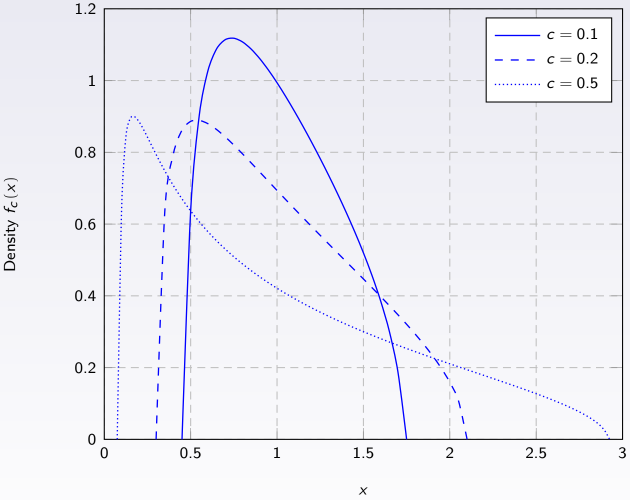

Figure : Marčenko-Pastur law for different limit ratios $c = \lim_{N \to \infty} N/n$.

Diagonal entries of the resolvent

Since we want an expression of m_F, we start by identifying the diagonal entries of the resolvent $(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N)^{-1}$ of $\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}$. Denote

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{y}^{\mathsf{H}} \\ \mathbf{Y} \end{bmatrix}$$

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$$\mathbf{X}_N = \begin{bmatrix} \mathbf{y}^{\mathsf{H}} \\ \mathbf{Y} \end{bmatrix}$$

Now, for $z \in \mathbb{C}^+$, we have

$$\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} = \begin{bmatrix} \mathbf{y}^{\mathsf{H}}\mathbf{y} - z & \mathbf{y}^{\mathsf{H}}\mathbf{Y}^{\mathsf{H}} \\ \mathbf{Y}\mathbf{y} & \mathbf{Y}\mathbf{Y}^{\mathsf{H}} - z\mathbf{I}_{N-1} \end{bmatrix}^{-1}$$

Consider the first diagonal element of $(\mathbf{R}_N - z\mathbf{I}_N)^{-1}$. From the matrix inversion lemma,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}$$

which here gives

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} = \frac{1}{-z - z\mathbf{y}^{\mathsf{H}}(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n)^{-1}\mathbf{y}}$$

Trace Lemma

Z. Bai, J. Silverstein, "Spectral Analysis of Large Dimensional Random Matrices", Springer Series in Statistics, 2009.

To go further, we need the following result,

Theorem

Let $\{\mathbf{A}_N\} \in \mathbb{C}^{N\times N}$ *with bounded spectral norm. Let* $\{\mathbf{x}_N\} \in \mathbb{C}^N$*, be a random vector of i.i.d. entries with zero mean, variance* $1/N$ *and finite* 8^{th} *order moment, independent of* $\mathbf{A}_N$*. Then*

$$\mathbf{x}_N^{\mathsf{H}}\mathbf{A}_N\mathbf{x}_N - \frac{1}{N} tr \mathbf{A}_N \xrightarrow{\text{a.s.}} 0.$$

For large N, we therefore have approximately

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} \simeq \frac{1}{-z - z\frac{1}{N}\mathrm{tr}\left(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n\right)^{-1}}$$

Rank-1 perturbation lemma

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

It is somewhat intuitive that adding *a single column* to $\mathbf{Y}$ won't affect the trace in the limit.

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $N \times N$ *with* $\mathbf{B}$ *Hermitian positive definite, and* $\mathbf{v} \in \mathbb{C}^N$*. For* $z \in \mathbb{C} \setminus \mathbb{R}^-$,

$$\left| \frac{1}{N} tr \left((\mathbf{B} - z\mathbf{I}_N)^{-1} - (\mathbf{B} + \mathbf{v}\mathbf{v}^{\mathsf{H}} - z\mathbf{I}_N)^{-1} \right) \mathbf{A} \right| \leqslant \frac{1}{N} \frac{\|\mathbf{A}\|}{\operatorname{dist}(z, \mathbb{R}^+)}$$

with $\|\mathbf{A}\|$ *the spectral norm of* $\mathbf{A}$*, and* $\operatorname{dist}(z, A) = \inf_{y \in A} \|y - z\|$.

Therefore, for large N, we have approximately,

$$\begin{aligned} \left[\left(\mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N \right)^{-1} \right]_{11} &\simeq \frac{1}{-z - z\frac{1}{N}\operatorname{tr}(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n)^{-1}} \\ &\simeq \frac{1}{-z - z\frac{1}{N}\operatorname{tr}(\mathbf{X}_N^{\mathsf{H}}\mathbf{X}_N - z\mathbf{I}_n)^{-1}} \\ &= \frac{1}{-z - z\frac{n}{N} m_{\underline{F}}(z)} \end{aligned}$$

in which we recognize the Stieltjes transform $m_{\underline{F}}$ of the l.s.d. of $\mathbf{X}_N^{\mathsf{H}}\mathbf{X}_N$.

End of the proof

We have again the relation

$$\frac{n}{N} m_{\underline{F}}(z) = m_F(z) + \frac{N-n}{N}\frac{1}{z}$$

hence

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} \simeq \frac{1}{\frac{n}{N} - 1 - z - zm_F(z)}$$

Note that the choice $(1,1)$ is irrelevant here, so the expression is valid for all pair (i,i). Summing over the N terms and averaging, we finally have

$$m_F(z) = \frac{1}{N}\operatorname{tr}\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \simeq \frac{1}{c - 1 - z - zm_F(z)}$$

which solve a polynomial of second order. Finally

$$m_F(z) = \frac{c-1}{2z} - \frac{1}{2} + \frac{\sqrt{(c-1-z)^2 - 4z}}{2z}.$$

From the inverse Stieltjes transform formula, we then verify that m_F is the Stieltjes transform of the Marčenko-Pastur law.

Related bibliography

- V. A. Marčenko, L. A. Pastur, "Distributions of eigenvalues for some sets of random matrices", Math USSR-Sbornik, vol. 1, no. 4, pp. 457-483, 1967.
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- R. B. Dozier, J. W. Silverstein, "On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices," Journal of Multivariate Analysis, vol. 98, no. 4, pp. 678-694, 2007.
- V. L. Girko, "Theory of Random Determinants," Kluwer, Dordrecht, 1990.
- A. M. Tulino, S. Verdù, "Random matrix theory and wireless communications," Now Publishers Inc., 2004.

Outline

Part 1: Fundamentals of Random Matrix Theory
1.1. The Stieltjes Transform Method
1.2. Extreme Eigenvalues
1.3. The Spiked Model
1.4. Spectrum Analysis and G-estimation

2. Source Detection
2.1. Eigenvalue-based Detection
2.2. Detection in unknown Noise Environment

3. Statistical Inference
3.1. Generic Model
3.1.2. Angle-of-arrival estimation
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3.2. Spiked Model
3.2.1. Spiked G-MUSIC
3.2.2. Local Failure Detection in Sensor Networks

4. Random Matrix Theory and Robust Estimation
4.1. Introduction to Robust Estimation
4.1. Initial Results and Open Problems

A classical pitfall

- Limiting spectral results only say where the "mass" of eigenvalues lies asymptotically. Say $F_N \Rightarrow F$, with $F_N(x) = \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{x \leqslant a_k}$.

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  - more generally, if F_N and $F_N^{(0)}$ are discrete and differ by $o(N)$ bounded masses, $F_N^{(0)} \Rightarrow F$.
- We know that, for $\mathbf{X}_N \in \mathbb{C}^{N \times n}$ with i.i.d. zero mean variance $1/n$,

$$F^{\mathbf{X}_N \mathbf{X}_N^{\mathsf{H}}} \Rightarrow F_c$$

with F_c is the compactly supported Marčenko-Pastur law of parameter $c = \lim_N \frac{N}{n}$.

Question: for very large N, where are the extreme eigenvalues of $\mathbf{X}_N \mathbf{X}_N^{\mathsf{H}}$?

Are there eigenvalues outside the support ?

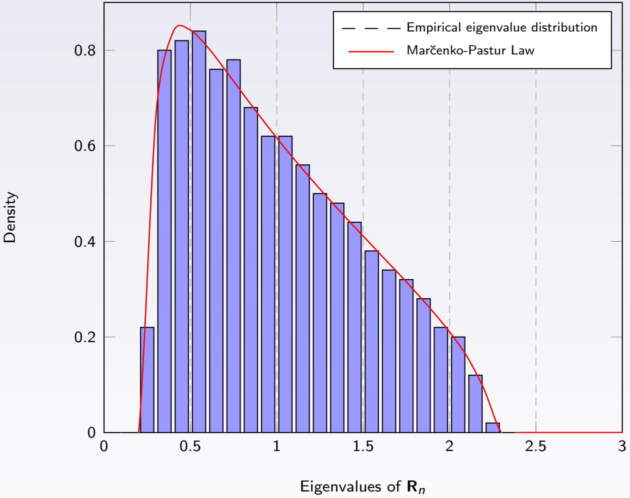

Figure : Histogram of the eigenvalues of $\mathbf{R}_n$ for $n = 2000$, $N = 500$

No eigenvalue outside the support of sample covariance matrices

Z. D. Bai, J. W. Silverstein, "No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices," The Annals of Probability, vol. 26, no.1 pp. 316-345, 1998.

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N \times n}$ with i.i.d. entries with zero mean, variance $1/n$ and 4^{th} order moment of order $O(1/n^2)$. Let $\mathbf{T}_N \in \mathbb{C}^{N \times N}$ be nonrandom and bounded in norm and with $F^{\mathbf{T}_N} \Rightarrow F^T$. We know that

$$F^{\mathbf{B}_N} \Rightarrow F \text{ almost surely}, \quad \mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}.$$

Let F_N be the distribution with $m_N(z)$ solution of

$$\underline{m}_N = -\left(z - \frac{N}{n} \int \frac{\tau}{1 + \tau \underline{m}_N} dF^{\mathbf{T}_N}(\tau) \right)^{-1}, \quad \underline{m}_N(z) = \frac{N}{n} m_N(z) + \frac{N-n}{n} \frac{1}{z}.$$

Choose $N_0 \in \mathbb{N}$ and $[a, b]$, $a > 0$, outside the union of the supports of F and F_N for all $N \geqslant N_0$. Denote $\mathcal{L}_N$ the set of eigenvalues of $\mathbf{B}_N$. Then,

$$P(\mathcal{L}_N \cap [a, b] \neq \emptyset \text{ i.o.}) = 0.$$

How to read the result?

- If $\mathbf{T}_N = \mathbf{I}_N$ for all N, then this result is equivalent to

  "For $[a, b]$ outside the support of the Marčenko-Pastur law, with probability 1, $\mathbf{B}_N$ has no eigenvalue in $[a, b]$ for all large N"

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- If $\mathbf{T}_N$ is not identity,
  - call S the support of the limiting F.
  - for some N_0, take the l.s.d. of $\mathbf{B}_N$ as if $\lim_N F^{\mathbf{T}_N} = F^{\mathbf{T}_{N_0}}$, and call its support S_{N_0}.
  - do the previous for all $N \geqslant N_0$. Call $\mathcal{A} = S \cup \bigcap_{N \geqslant N_0} S_N$.
  - take $[a, b]$ outside $\mathcal{A}$, and pick a random sequence $\mathbf{B}_1, \mathbf{B}_2, \ldots$. The result shows that, for all N large, there is no eigenvalue of $\mathbf{B}_N$ in $[a, b]$.

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- this is very different from taking $[a,b]$ only outside the support of F only!
- this is essential to understand spiked models, discussed later.

No eigenvalue outside the support: which models?

J. W. Silverstein, P. Debashis, "No eigenvalues outside the support of the limiting empirical spectral distribution of a separable covariance matrix," J. of Multivariate Analysis vol. 100, no. 1, pp. 37-57, 2009.

- It has already been shown that (for all large N) there is no eigenvalues outside the support of
  - *Marčenko-Pastur law*: $\mathbf{X}\mathbf{X}^{\mathsf{H}}$, $\mathbf{X}$ i.i.d. with zero mean, variance $1/N$, finite 4^{th} order moment.
  - *Sample covariance matrix*: $\mathbf{T}^{\frac{1}{2}}\mathbf{X}\mathbf{X}^{\mathsf{H}}\mathbf{T}^{\frac{1}{2}}$ and $\mathbf{X}^{\mathsf{H}}\mathbf{T}\mathbf{X}$, $\mathbf{X}$ i.i.d. with zero mean, variance $1/N$, finite 4^{th} order moment.
  - *Doubly-correlated matrix*: $\mathbf{R}^{\frac{1}{2}}\mathbf{X}\mathbf{T}\mathbf{X}^{\mathsf{H}}\mathbf{R}^{\frac{1}{2}}$, $\mathbf{X}$ with i.i.d. zero mean, variance $1/N$, finite 4^{th} order moment.

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J. W. Silverstein, Z.D. Bai, Y.Q. Yin, "A note on the largest eigenvalue of a large dimensional sample covariance matrix," Journal of Multivariate Analysis, vol. 26, no. 2, pp. 166-168, 1988.

- If 4^{th} order moment is infinite,

$$\limsup_N \lambda_{\max}^{\mathbf{X}\mathbf{X}^{\mathsf{H}}} = \infty$$

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  - *Doubly-correlated matrix*: $\mathbf{R}^{\frac{1}{2}}\mathbf{X}\mathbf{T}\mathbf{X}^{\mathsf{H}}\mathbf{R}^{\frac{1}{2}}$, $\mathbf{X}$ with i.i.d. zero mean, variance $1/N$, finite 4^{th} order moment.

J. W. Silverstein, Z.D. Bai, Y.Q. Yin, "A note on the largest eigenvalue of a large dimensional sample covariance matrix," Journal of Multivariate Analysis, vol. 26, no. 2, pp. 166-168, 1988.

- If 4^{th} order moment is infinite,

$$\limsup_N \lambda_{\max}^{\mathbf{X}\mathbf{X}^{\mathsf{H}}} = \infty$$

J. Silverstein, Z. Bai, "No eigenvalues outside the support of the limiting spectral distribution of information-plus-noise type matrices" to appear in Random Matrices: Theory and Applications.

- Only recently, information plus noise models, $\mathbf{X}$ with i.i.d. zero mean, variance $1/N$, finite 4^{th} order moment

$$(\mathbf{X}+\mathbf{A})(\mathbf{X}+\mathbf{A})^{\mathsf{H}}$$

Sketch of Proof

- Proof entirely relies on the Stieltjes transform.
- Up to now, we know $|m_{\mathbf{B}_N}(z) - m_N(z)| \xrightarrow{\text{a.s.}} 0$ for $z \in \mathbb{C} \setminus \mathbb{R}^-$.

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- This is not enough, we need in fact to show: for $z = x + i\sqrt{k}v_N$, $v_N = N^{-1/68}$, $k = 1, \ldots, 34$,

$$\max_{1 \leqslant k \leqslant 34} \sup_{x \in [a,b]} \left| m_{\mathbf{B}_N}(x + ik^{\frac{1}{2}} v_N) - m_N((x + ik^{\frac{1}{2}} v_N) \right| = o(v_N^{67}).$$

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- Expanding the Stieltjes transforms and considering only the imaginary parts, this is

$$\max_{1 \leqslant k \leqslant 34} \sup_{x \in [a,b]} \left| \int \frac{d(F^{\mathbf{B}_N}(\lambda) - F_N(\lambda))}{(x-\lambda)^2 + kv_N^2} \right| = o(v_N^{66})$$

almost surely. Taking successive differences over the 34 values of k, we end up with

$$\sup_{x \in [a,b]} \left| \int \frac{(v_N^2)^{33} d(F^{\mathbf{B}_N}(\lambda) - F_N(\lambda))}{\prod_{k=1}^{34}((x-\lambda)^2 + kv_N^2)} \right| = o(v_N^{66})$$

Consider $a' < a$ and $b' > b$ such that $[a', b']$ is outside the support of F. We then have

$$\sup_{x \in [a,b]} \left| \int \frac{1_{\mathbb{R}^+ \setminus [a',b']}(\lambda) d(F\mathbf{B}_N(\lambda) - F_N(\lambda))}{\prod_{k=1}^{34}((x-\lambda)^2 + kv_N^2)} + \sum_{\lambda_j \in [a',b']} \frac{v_N^{68}}{\prod_{k=1}^{34}((x-\lambda_j)^2 + kv_N^2)} \right| = o(1)$$

almost surely. If, there is one eigenvalue of all $\mathbf{B}_{\phi(N)}$ in $[a, b]$, then one term of the sum is $1/34! > 0$. So the integral must away from zero. But the integral tends to 0. Contradiction.

Exact eigenvalue separation

Z. D. Bai, J. W. Silverstein, "Exact Separation of Eigenvalues of Large Dimensional Sample Covariance Matrices," The Annals of Probability, vol. 27, no. 3, pp. 1536-1555, 1999.

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  - says where eigenvalues are not to be found
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  - does not say, as we feel, that (if cluster separation) in cluster k, there are exactly n_k eigenvalues.
- This is in fact the case,

Theorem

Let $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^H \mathbf{T}_N^{\frac{1}{2}}$ with l.s.d. F, $\mathbf{X}_N$ i.i.d., zero mean, variance $1/n$, finite 4^{th} moment, $F^{\mathbf{T}_N} \Rightarrow F^T$, and $\frac{N}{n} \to c$. Consider $0 < a < b$ such that $[a, b]$ is outside the support of F. Denote additionally λ_k's and τ_k's the ordered eigenvalues of $\mathbf{B}_N$ and $\mathbf{T}_N$. Then we have

1. *If $c(1 - F^T(0)) > 1$, then the smallest eigenvalue x_0 of the support of F is positive and $\lambda_N \to x_0$ almost surely, as $N \to \infty$.*
2. *If $c(1 - F^T(0)) \leqslant 1$, or $c(1 - F^T(0)) > 1$ but $[a, b]$ is not contained in $[0, x_0]$, then, almost surely, there exists N_0 such that for all $N \geqslant N_0$,*

$$\lambda_{i_N} > b, \quad \lambda_{i_N+1} < a$$

where i_N is the unique integer such that

$$\tau_{i_N} > -1/m_F(b)$$

$$\tau_{i_N+1} < -1/m_F(a).$$

Consequence of exact separation

- If eigenvalues are found outside the expected clusters, some extra "signal" must have been transmitted.
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- The quantity of eigenvalues in each cluster gives an exact estimate of the multiplicity of the population!
- This is essential for eigen-inference.
- Exact separation is only known for the sample covariance matrix model so far.
- Very recently, extension to information-plus-noise model.

What's the use of all that to signal processing?

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- if $\mathbf{R}_n$ has all eigenvalues inside the *expected* noise support, what can we say?
  - we cannot conclude so far
  - we need to further study the spectrum

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Extreme eigenvalues: Deeper into the spectrum

- In order to derive statistical detection tests, we need more information on the extreme eigenvalues.
- We will study the fluctuations of the extreme eigenvalues (second order statistics)
- However, the Stieltjes transform method is not adapted here!

Distribution of the largest eigenvalues of $\mathbf{X}\mathbf{X}^{\mathsf{H}}$

C. A. Tracy, H. Widom, "On orthogonal and symplectic matrix ensembles," Communications in Mathematical Physics, vol. 177, no. 3, pp. 727-754, 1996.
K. Johansson, "Shape Fluctuations and Random Matrices," Comm. Math. Phys. vol. 209, pp. 437-476, 2000.

Theorem

Let $\mathbf{X} \in \mathbb{C}^{N \times n}$ have i.i.d. Gaussian entries of zero mean and variance $1/n$. Denoting λ_N^+ the largest eigenvalue of $\mathbf{X}\mathbf{X}^{\mathsf{H}}$, then

$$N^{\frac{2}{3}} \frac{\lambda_N^+ - (1+\sqrt{c})^2}{(1+\sqrt{c})^{\frac{4}{3}} c^{\frac{1}{2}}} \Rightarrow X^+ \sim F^+$$

with $c = \lim_N N/n$ and F^+ the Tracy-Widom distribution given by

$$F^+(t) = \exp\left(- \int_t^\infty (x-t)^2 q^2(x) dx \right)$$

with q the Painlevé II function that solves the differential equation

$$q''(x) = xq(x) + 2q^3(x)$$
$$q(x) \sim_{x \to \infty} \mathrm{Ai}(x)$$

in which $\mathrm{Ai}(x)$ is the Airy function.

The law of Tracy-Widom

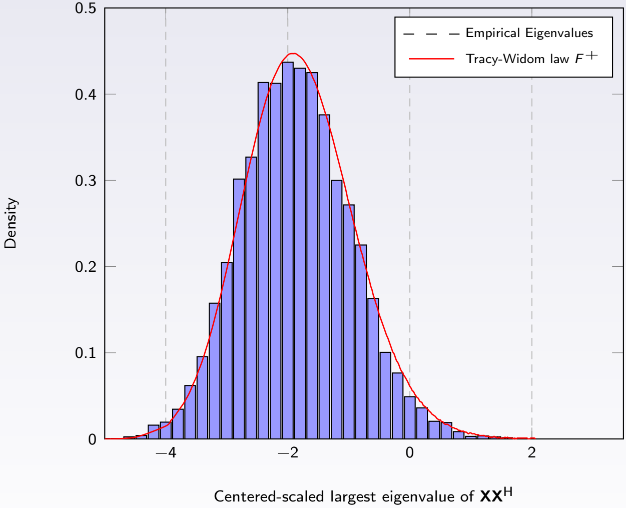

Figure : Distribution of $N^{\frac{2}{3}} c^{-\frac{1}{2}} (1+\sqrt{c})^{-\frac{4}{3}} \left[\lambda_N^+ - (1+\sqrt{c})^2\right]$ against the distribution of X^+ (distributed as Tracy-Widom law) for $N = 500$, $n = 1500$, $c = 1/3$, for the covariance matrix model $\mathbf{XX}^{\mathsf{H}}$. Empirical distribution taken over $10,000$ Monte-Carlo simulations.

Techniques of proof

Method of proof requires very different tools:

- *orthogonal (Laguerre) polynomials*: to write joint *unordered* eigenvalue distribution as a kernel determinant.

$$\rho_N(\lambda_1, \ldots, \lambda_p) = \det_{i,j=1}^{p} K_N(\lambda_i, \lambda_j)$$

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- *Fredholm determinants*: we can write hole probability as a Fredholm determinant.

$$P\left(N^{2/3}\left(\lambda_i - (1+\sqrt{c})^2\right) \in A, i = 1, \ldots, N\right) = 1 + \sum_{k \geqslant 1} \frac{(-1)^k}{k!} \int_{A^c} \cdots \int_{A^c} \det_{i,j=1}^{k} K_N(x_i, x_j) \prod dx_i$$

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- *differential equation tricks*: hole probability in $[t, \infty)$ gives right-most eigenvalue distribution, which is simplified as solution of a Painelvé differential equation: the Tracy-Widom distribution.

$$F^+(t) = e^{-\int_t^\infty (x-t) q(x)^2 dx}, \quad q'' = tq + 2q^3, \; q(x) \sim_{x\to\infty} \mathrm{Ai}(x).$$

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- can be shown to hold for other distributions than Gaussian under mild assumptions

Outline

Part 1: Fundamentals of Random Matrix Theory
1.1. The Stieltjes Transform Method
1.2. Extreme Eigenvalues
1.3. The Spiked Model
1.4. Spectrum Analysis and G-estimation

2. Source Detection
2.1. Eigenvalue-based Detection
2.2. Detection in unknown Noise Environment

3. Statistical Inference
3.1. Generic Model
3.1.2. Angle-of-arrival estimation
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3.2. Spiked Model
3.2.1. Spiked G-MUSIC
3.2.2. Local Failure Detection in Sensor Networks

4. Random Matrix Theory and Robust Estimation
4.1. Introduction to Robust Estimation
4.1. Initial Results and Open Problems

Spiked models

- We can create sample covariance matrix models $\mathbf{T}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{T}_N^{\frac{1}{2}}$ with l.s.d. F ($\mathbf{X}_N$ as usual) for which
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What about the absence of spikes?

The first result

J. Baik, J. W. Silverstein, "Eigenvalues of large sample covariance matrices of spiked population models," Journal of Multivariate Analysis, vol. 97, no. 6, pp. 1382-1408, 2006.

Theorem

Let $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$, *where* $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ *has i.i.d., zero mean and variance* $1/n$ *entries, and* $\mathbf{T}_N \in \mathbb{R}^{N\times N}$ *diagonal given by*

$$\mathbf{T}_N = diag(\underbrace{1+\omega_1,\ldots,1+\omega_1}_{k_1},\ldots,\underbrace{1+\omega_M,\ldots,1+\omega_M}_{k_M},\underbrace{1,\ldots,1}_{N-\sum_{i=1}^M k_i})$$

with $\omega_1 > \ldots > \omega_M > -1$, $c = \lim_N N/n$. *We then have*

- *if* $\omega_j > \sqrt{c}$, $\lambda_{k_1+\ldots+k_{j-1}+i} \xrightarrow{\text{a.s.}} 1+\omega_j + c\frac{1+\omega_j}{\omega_j}$ *(i.e. beyond the Marčenko–Pastur bulk!)*
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- *if* $\omega_{k_j} \in [-\sqrt{c},0)$, $\lambda_{k_1+\ldots+k_{j-1}+i} \xrightarrow{\text{a.s.}} (1-\sqrt{c})^2$ *(i.e. left-edge of the Marčenko–Pastur bulk!)*

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- *for the other eigenvalues, we discriminate over* c:
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Illustration of spiked models

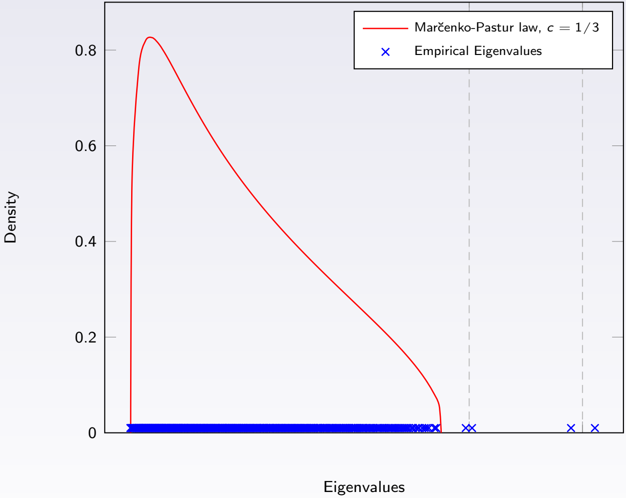

Figure : Eigenvalues of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$, where $F^{\mathbf{T}_N} \Rightarrow 1_{[1,\infty)}$, ...Dimensions: $N = 500$, $n = 1500$.

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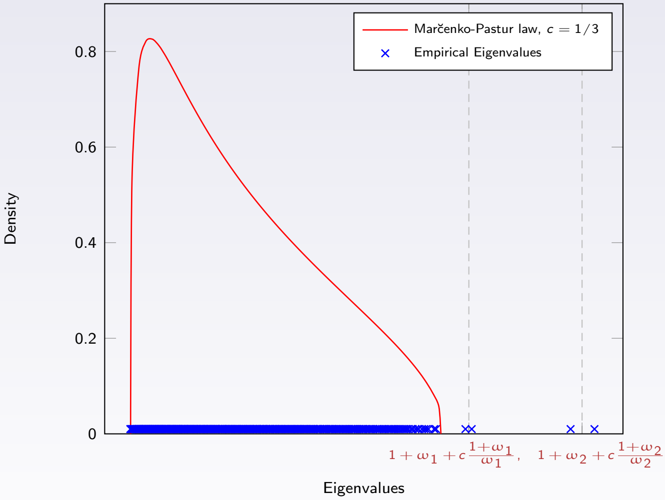

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  - THAT LOOKS LIKE A PARADOX.

General characterization of spiked eigenvalues

- Consider the more general model

$$\mathbf{\Sigma} = (\mathbf{I}_N + \mathbf{P})^{\frac{1}{2}}\mathbf{X}$$

  with, for simplicity
  - $\mathbf{X}$ standard Gaussian
  - $\mathbf{P} = \mathbf{U}\mathbf{\Omega}\mathbf{U}^{\mathsf{H}}$, $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{C}^{N\times r}$, $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \ldots, \omega_r)$, $\omega_1 > \ldots > \omega_r > 0$.
- We can study the convergence properties of
  - $\lambda_1 > \ldots > \lambda_r$, the r largest eigenvalues of $\mathbf{\Sigma}\mathbf{\Sigma}^{\mathsf{H}}$
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- Systematic study based on two ingredients:
  - random matrix tools (the Stieltjes transform method)
  - complex analysis (complex contour integration)

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- Let $x > 0$, then

$$\begin{aligned}\det(\mathbf{\Sigma}\mathbf{\Sigma}^{\mathsf{H}} - x\mathbf{I}_N) &= \det(\mathbf{I}_N + \mathbf{P}) \det(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N + x[\mathbf{I}_N - (\mathbf{I}_N + \mathbf{P})^{-1}]) \\ &= \det(\mathbf{I}_N + \mathbf{P}) \det(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1} \det(\mathbf{I}_N + x\mathbf{P}(\mathbf{I}_N + \mathbf{P})^{-1}(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}).\end{aligned}$$

First order limits on eigenvalues

- We start with a study of the limiting extreme eigenvalues.
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- if x eigenvalue of $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathsf{H}}$ but not of $\mathbf{X}\mathbf{X}^{\mathsf{H}}$, then for n large, $x > (1+\sqrt{c})^2$ (edge of MP law support) and

$$\det(\mathbf{I}_N + x\mathbf{P}(\mathbf{I}_N + \mathbf{P})^{-1}(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}) = \det(\mathbf{I}_r + x\boldsymbol{\Omega}\mathbf{U}^*(\mathbf{I}_N + \mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{\mathsf{H}})^{-1}(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}\mathbf{U}) = 0$$

with $\mathbf{P} = \mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{\mathsf{H}}$, $\mathbf{U} \in \mathbb{C}^{N\times r}$.

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  with $\mathbf{P} = \mathbf{U\Omega U}^{\mathsf{H}}$, $\mathbf{U} \in \mathbb{C}^{N\times r}$.
- due to unitary invariance of $\mathbf{X}$,

$$\mathbf{U}^{\mathsf{H}}(\mathbf{XX}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}\mathbf{U} \xrightarrow{\text{a.s.}} \int (t-x)^{-1} dF^{MP}(t)\mathbf{I}_r \triangleq m(x)\mathbf{I}_r$$

  with F^{MP} the MP law, and $m(x)$ the Stieltjes transform of the MP law (often known for $r = 1$ as trace lemma).

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  with F^{MP} the MP law, and $m(x)$ the Stieltjes transform of the MP law (often known for $r = 1$ as trace lemma).
- finally, we have that the *limiting* solutions ρ_k satisfy $\rho_k m(\rho_k) + (1+\omega_k)\omega_k^{-1} = 0$.
- replacing $m(x)$, this is finally:

$$\lambda_k \xrightarrow{\text{a.s.}} \rho_k \triangleq 1 + \omega_k + c(1+\omega_k)\omega_k^{-1}, \text{ if } \omega_k > \sqrt{c}$$

Generalization of the Tracy-Widom law

J. Baik, G. Ben Arous, S. Péché, "Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices," The Annals of Probability, vol. 33, no. 5, pp. 1643-1697, 2005.

Theorem

Let $\mathbf{X} \in \mathbb{C}^{N\times n}$ have i.i.d. Gaussian entries of zero mean and variance $1/n$ and $\mathbf{T}_N = diag(t_1, \ldots, t_N)$. Assume, for some fixed r, $t_{r+1} = \ldots = t_N = 1$ and $t_1 = \ldots = t_k$ while $t_{k+1}, \ldots, t_r$ lie in a compact subset of $(0, t_1)$. Assume further $c = \lim N/n < 1$. Denoting λ_N^+ the largest eigenvalue of $\mathbf{T}^{\frac{1}{2}}\mathbf{X}\mathbf{X}^{\mathsf{H}}\mathbf{T}^{\frac{1}{2}}$, we have

- *If $t_1 < 1 + \sqrt{\frac{N}{n}}$,*

$$N^{\frac{2}{3}} \frac{\lambda_N^+ - (1+\sqrt{c})^2}{(1+\sqrt{c})^{\frac{4}{3}} c^{\frac{1}{2}}} \Rightarrow X^+ \sim F^+$$

  with F^+ the Tracy-Widom distribution.

- *If $t_1 > 1 + \sqrt{\frac{N}{n}}$,*

$$\left(t_1^2 - \frac{t_1^2 c}{(t_1-1)^2}\right)^{\frac{1}{2}} n^{\frac{1}{2}} \left[\lambda_N^+ - \left(t_1 + \frac{t_1 c}{t_1 - 1}\right)\right] \Rightarrow X_k \sim G_k$$

  for some function G_k that is the distribution of the largest eigenvalue of the $k \times k$ GUE.

$$G_k(x) = \frac{1}{Z_k} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \prod_{1 \leqslant i < j \leqslant k} |\xi_i - \xi_j|^2 \prod_{i=1}^{k} e^{-\frac{1}{2}\xi_i^2} d\xi_1 \ldots d\xi_k$$

  In particular, $G_1(x) = \mathrm{erf}(x)$

Comments on the result

- there exists a "phase transition" when the largest population eigenvalues move from inside to outside $(0, 1 + \sqrt{c})$.

Comments on the result

- there exists a "phase transition" when the largest population eigenvalues move from inside to outside $(0, 1 + \sqrt{c})$.
- more importantly, for $t_1 < 1 + \sqrt{c}$, we still have the same Tracy-Widom,
  - no way to see the spike even when zooming in
  - in fact, simulation suggests that convergence rate to the Tracy-Widom is slower with spikes.

Presence of a spike in previous model

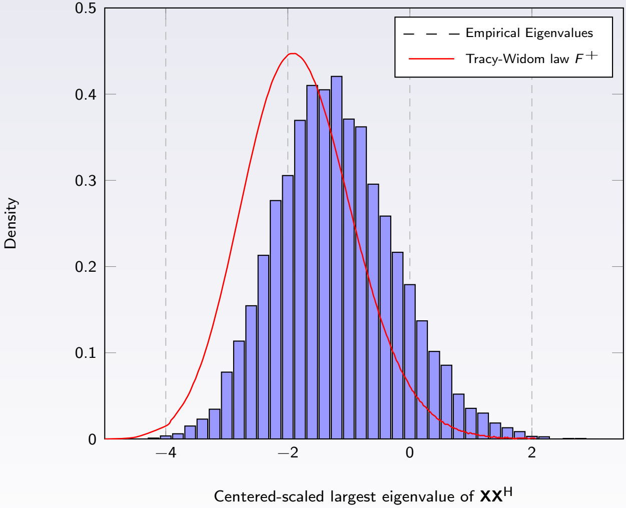

Figure : Distribution of $N^{\frac{2}{3}} c^{-\frac{1}{2}} (1+\sqrt{c})^{-\frac{4}{3}} \left[\lambda_N^+ - (1+\sqrt{c})^2\right]$ against the distribution of X^+ (distributed as Tracy-Widom law) for $N = 500$, $n = 1500$, $c = 1/3$, for the covariance matrix model $\mathbf{T}^{\frac{1}{2}}\mathbf{XX}^\mathsf{H}\mathbf{T}^{\frac{1}{2}}$ with $\mathbf{T}$ diagonal with all entries 1 but for $T_{11} = 1.5$. Empirical distribution taken over 10,000 Monte-Carlo simulations.

Outline

Part 1: Fundamentals of Random Matrix Theory
- 1.1. The Stieltjes Transform Method
- 1.2. Extreme Eigenvalues
- 1.3. The Spiked Model
- 1.4. Spectrum Analysis and G-estimation

2. Source Detection
- 2.1. Eigenvalue-based Detection
- 2.2. Detection in unknown Noise Environment

3. Statistical Inference
- 3.1. Generic Model
  - 3.1.2. Angle-of-arrival estimation
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- 3.2. Spiked Model
  - 3.2.1. Spiked G-MUSIC
  - 3.2.2. Local Failure Detection in Sensor Networks

4. Random Matrix Theory and Robust Estimation
- 4.1. Introduction to Robust Estimation
- 4.1. Initial Results and Open Problems

Stieltjes transform inversion for covariance matrix models

J. W. Silverstein, S. Choi, "Analysis of the limiting spectral distribution of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 295-309, 1995.

- We know for the model $\mathbf{T}_N^{\frac{1}{2}}\mathbf{X}_N$, $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ that, if $F^{\mathbf{T}_N} \Rightarrow F^T$, the Stieltjes transform of the e.s.d. of $\underline{\mathbf{B}}_N = \mathbf{X}_N^{\mathsf{H}}\mathbf{T}_N\mathbf{X}_N$ satisfies $m_{\underline{\mathbf{B}}_N}(z) \xrightarrow{\text{a.s.}} m_{\underline{F}}(z)$, with

$$m_{\underline{F}}(z) = \left(-z - c\int \frac{t}{1+tm_{\underline{F}}(z)}dF^T(t)\right)^{-1}$$

  which is unique on the set $\{z \in \mathbb{C}^+, m_{\underline{F}}(z) \in \mathbb{C}^+\}$.

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- This can be inverted into

$$z_{\underline{F}}(m) = -\frac{1}{m} - c\int \frac{t}{1+tm}dF^T(t)$$

  for $m \in \mathbb{C}^+$.

Stieltjes transform inversion and spectrum characterization

Remember that we can evaluate the spectrum density by taking a complex line close to $\mathbb{R}$ and evaluating $\Im[m_{\underline{F}}(z)]$ along this line. Now we can do better.

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It is shown that

$$\lim_{\substack{z \to x \in \mathbb{R}^* \\ z \in \mathbb{C}^+}} m_{\underline{F}}(z) = m_0(x) \quad \text{exists.}$$

We also have,

- for x_0 inside the support, the density $\underline{f}(x)$ of $\underline{F}$ in x_0 is $\frac{1}{\pi}\Im[m_0]$ with m_0 the unique solution $m \in \mathbb{C}^+$ of
$$[z_{\underline{F}}(m) =] \ x_0 = -\frac{1}{m} - c \int \frac{t}{1+tm} dF^T(t)$$
- let $m_0 \in \mathbb{R}^*$ and $x_{\underline{F}}$ the equivalent to $z_{\underline{F}}$ on the real line. Then "x_0 outside the support of $\underline{F}$" is equivalent to "$x_{\underline{F}}'(m_{\underline{F}}(x_0)) > 0$, $m_{\underline{F}}(x_0) \neq 0$, $-1/m_{\underline{F}}(x_0)$ outside the support of F^T".

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This provides another way to determine the support!. For $m \in (-\infty, 0)$, evaluate $x_{\underline{F}}(m)$. Whenever $x_{\underline{F}}$ decreases, the image is outside the support. The rest is inside.

Another way to determine the spectrum: spectrum to analyze

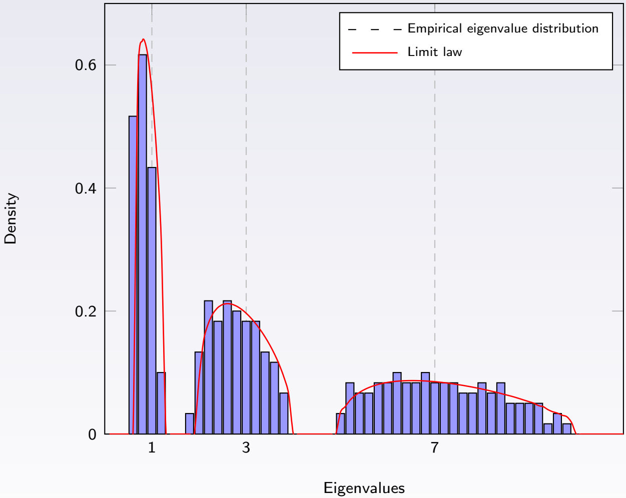

Figure : Histogram of the eigenvalues of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$, $N = 300$, $n = 3000$, with $\mathbf{T}_N$ diagonal composed of three evenly weighted masses in 1, 3 and 7.

Another way to determine the spectrum: inverse function method

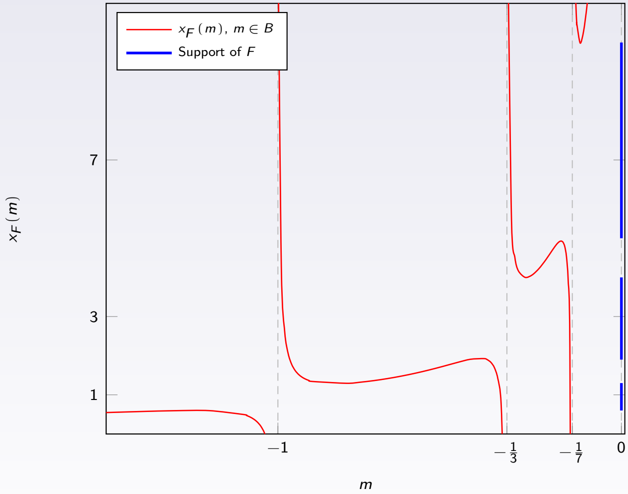

Figure : Stieltjes transform of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$, $N = 300$, $n = 3000$, with $\mathbf{T}_N$ diagonal composed of three evenly weighted masses in 1, 3 and 7. The support of F is read on the vertical axis, whenever m_F is decreasing.

Cluster boundaries in sample covariance matrix models

Xavier Mestre, "Improved estimation of eigenvalues of covariance matrices and their associated subspaces using their sample estimates," IEEE Transactions on Information Theory, vol. 54, no. 11, Nov. 2008.

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N \times n}$ have i.i.d. entries of zero mean, variance $1/n$, and $\mathbf{T}_N$ be diagonal such that $F^{\mathbf{T}_N} \Rightarrow F^T$, as $n, N \to \infty$, $N/n \to c$, where F^T has K masses in $t_1, \ldots, t_K$ with multiplicity $n_1, \ldots, n_K$ respectively. Then the l.s.d. of $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$ has support $\mathcal{S}$ given by

$$\mathcal{S} = [x_1^-, x_1^+] \cup [x_2^-, x_2^+] \cup \ldots \cup [x_Q^-, x_Q^+]$$

with $x_q^- = x_F(m_q^-)$, $x_q^+ = x_F(m_q^+)$, and

$$x_F(m) = -\frac{1}{m} - c\frac{1}{n} \sum_{k=1}^{K} n_k \frac{t_k}{1 + t_k m}$$

with $2Q$ the number of real-valued solutions counting multiplicities of $x_F'(m) = 0$ denoted in order $m_1^- < m_1^+ \leqslant m_2^- < m_2^+ \leqslant \ldots \leqslant m_Q^- < m_Q^+$.

Comments on spectrum characterization

Previous results allows to determine

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Mestre goes further: to determine local separability of the spectrum,

- identify the K inflexion points, i.e. the K solutions $m_1, \dots, m_K$ to

$$x_F''(m) = 0$$

- check whether $x_F'(m_i) > 0$ and $x_F'(m_{i+1}) > 0$
- if so, the cluster in between corresponds to a single population eigenvalue.

Eigeninference: Introduction of the problem

- *Reminder:* for a sequence $\mathbf{x}_1, \ldots, x_n \in \mathbb{C}^N$ of independent random variables,

$$\mathbf{R}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}$$

is an n-consistent estimator of $\mathbf{R} = E[\mathbf{x}_1 \mathbf{x}_1^{\mathrm{H}}]$.

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- If n, N have comparable sizes, this no longer holds.
- Typically, n, N-consistent estimators of the full $\mathbf{R}$ matrix perform very badly.
- If only the eigenvalues of $\mathbf{R}$ are of interest, things can be done. The process of retrieving information about eigenvalues, eigenspace projections, or functional of these is called eigen-inference.

Girko and the G-estimators

V. Girko, "Ten years of general statistical analysis,"
http://www.general-statistical-analysis.girko.freewebspace.com/chapter14.pdf

- Girko has come up with more than 50 N, n-consistent estimators, called G-estimators (Generalized estimators). Among those, we find
  - *G_1-estimator of generalized variance*. For

$$G_1(\mathbf{R}_n) = \alpha_n^{-1}\left[\log\det(\mathbf{R}_n) + \log\frac{n(n-1)^N}{(n-N)\prod_{k=1}^N (n-k)}\right]$$

with α_n any sequence such that $\alpha_n^{-2}\log(n/(n-N)) \to 0$, we have

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- However, Girko's proofs are rarely readable, if existent.

A long standing problem

X. Mestre, "Improved estimation of eigenvalues and eigenvectors of covariance matrices using their sample estimates," IEEE trans. on Information Theory, vol. 54, no. 11, pp. 5113-5129, 2008.

- Consider the model $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$, where $F^{\mathbf{T}_N}$ is formed of a finite number of masses $t_1, \ldots, t_K$.
- It has long been thought the inverse problem of estimating $t_1, \ldots, t_K$ from the Stieltjes transform method was not possible.
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- Only trials were iterative convex optimization methods.
- The problem was partially solved by Mestre in 2008!
- His technique uses elegant complex analysis tools. The description of this technique is the subject of this course.

Reminders

- Consider the sample covariance matrix model $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}} \mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} \mathbf{T}_N^{\frac{1}{2}}$.
- Up to now, we saw:
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  - that there is no eigenvalue outside the support with probability 1 for all large N.
  - that for all large N, when the spectrum is divided into clusters, the number of empirical eigenvalues in each cluster is exactly as we expect.
- these results are of crucial importance for the following.

Inverse problem for sample covariance matrix

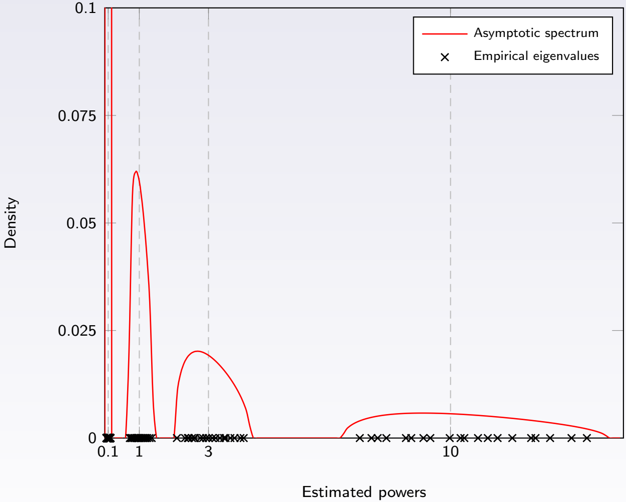

Figure : Empirical and asymptotic eigenvalue distribution of $\frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$ when $\mathbf{P}$ has three distinct entries $P_1 = 1$, $P_2 = 3$, $P_3 = 10$, $n_1 = n_2 = n_3$, $N/n = 10$, $M/N = 10$, $\sigma^2 = 0.1$. Empirical test: $n = 60$.

Eigen-inference for the sample covariance matrix model

X. Mestre, "Improved estimation of eigenvalues and eigenvectors of covariance matrices using their sample estimates," IEEE trans. on Information Theory, vol. 54, no. 11, pp. 5113-5129, 2008.

Theorem

Consider the model $\mathbf{B}_N = \mathbf{T}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{T}_N^{\frac{1}{2}}$, with $\mathbf{X}_N \in \mathbb{C}^{N\times n}$, i.i.d. with entries of zero mean, variance $1/n$, and $\mathbf{T}_N \in \mathbb{R}^{N\times N}$ is diagonal with K distinct entries $t_1, \ldots, t_K$ of multiplicity $N_1, \ldots, N_K$ of same order as n. Let $k \in \{1, \ldots, K\}$. Then, if the cluster associated to t_k is separated from the clusters associated to $k-1$ and $k+1$, as $N, n \to \infty$, $N/n \to c$,

$$\hat{t}_k = \frac{n}{N_k}\sum_{m\in\mathcal{N}_k}(\lambda_m - \mu_m)$$

is an N, n-consistent estimator of t_k, where $\mathcal{N}_k = \{N - \sum_{i=k}^{K} N_i + 1, \ldots, N - \sum_{i=k+1}^{K} N_i\}$, $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $\mathbf{B}_N$ and $\mu_1, \ldots, \mu_N$ are the N solutions of

$$m_{\mathbf{X}_N^{\mathsf{H}}\mathbf{T}_N\mathbf{X}_N}(\mu) = 0$$

or equivalently, $\mu_1, \ldots, \mu_N$ are the eigenvalues of $\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$.

Remarks on Mestre's result

Assuming cluster separation, the result consists in

- taking the empirical *ordered* λ_i's inside the cluster (note that exact separation ensures there are N_k of these!)
- getting the *ordered* eigenvalues $\mu_1, \ldots, \mu_N$ of

$$\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$$

  with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)^{\mathsf{T}}$. Keep only those of index inside $\mathcal{N}_k$.
- take the difference and scale.

How to obtain this result?

- Major trick requires tools from complex analysis

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- Silverstein's Stieltjes transform identity: for the *conjugate* model $\underline{\mathbf{B}}_N = \mathbf{X}_N^* \mathbf{T}_N \mathbf{X}_N$,

$$\underline{m}_N(z) = \left(-z - c \int \frac{t}{1 + t\underline{m}_N(z)} dF^{\mathbf{T}_N}(t) \right)^{-1}$$

with $\underline{m}_N$ the deterministic equivalent of $m_{\underline{\mathbf{B}}_N}$. This is the only random matrix result we need.

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  with $\underline{m}_N$ the deterministic equivalent of $m_{\underline{\mathbf{B}}_N}$. This is the only random matrix result we need.
- Before going further, we need some reminders from complex analysis.

Reminders of complex analysis

- Cauchy integration formula

Theorem

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be holomorphic on U. Let $\gamma \subset U$ be a continuous contour (i.e. closed path). Then, for a **inside** *the surface formed by γ, we have*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a} dz = f(a)$$

while for a **outside** *the surface formed by γ,*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a} dz = 0.$$

Limiting spectrum of the sample covariance matrix

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," J. of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

Reminder:

- If $F^{\mathbf{T}_N} \Rightarrow F^T$, then $m_{\mathbf{B}_N}(z) \xrightarrow{\text{a.s.}} m_F(z)$ such that

$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^T(t) - z \right)^{-1}$$

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$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^T(t) - z \right)^{-1}$$

or equivalently

$$m_{F^T}\left(-1/m_{\underline{F}}(z)\right) = -z m_{\underline{F}}(z) m_F(z)$$

with $m_{\underline{F}}(z) = c m_F(z) + (c-1)\frac{1}{z}$ and $N/n \to c$.

Reminders of complex analysis (2)

- Residue calculus

Theorem

Let γ be a contour on $\mathbb{C}$. For f holomorphic inside γ but on a discrete number of points, to compute the expression

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz$$

one must

1. *determine the poles of f lying inside the surface formed by γ, i.e. those values a such that*

$$\lim_{z\to a}|f(z)|=\infty$$

2. *determine the order of each pole, i.e. the smallest k such that*

$$\lim_{z\to a}|(z-a)^k f(z)|<\infty$$

3. *compute the residues of f at the poles, i.e. evaluate the value*

$$\mathrm{Res}(f,a)\triangleq\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]$$

4. *the integral is then the sum of all residues.*

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz=\sum_{a\in\{\text{ poles of } f\}}\mathrm{Res}(f,a)$$

Complex integration

- From Cauchy integral formula, denoting $\mathcal{C}_k$ a contour enclosing **only** t_k,

$$t_k = \frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{\omega}{\omega - t_k} d\omega$$

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- After the variable change $\omega = -1/m_{\underline{F}}(z)$,

$$t_k = \frac{N}{N_k}\frac{1}{2\pi i}\oint_{\mathcal{C}_{\underline{F},k}} z m_F(z) \frac{m'_{\underline{F}}(z)}{m^2_{\underline{F}}(z)} dz,$$

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- When the system dimensions are large,

$$m_F(z) \simeq m_{\mathbf{B}_N}(z) \triangleq \frac{1}{N}\sum_{k=1}^{N}\frac{1}{\lambda_k - z}, \quad \text{with} \quad (\lambda_1, \ldots, \lambda_N) = \mathrm{eig}(\mathbf{B}_N) = \mathrm{eig}(\mathbf{Y}\mathbf{Y}^{\mathsf{H}}).$$

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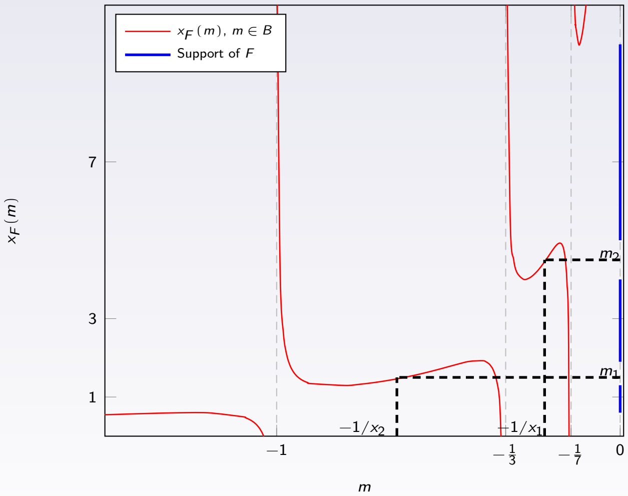
- Dominated convergence arguments then show

$$t_k - \hat{t}_k \xrightarrow{\text{a.s.}} 0 \quad \text{with} \quad \hat{t}_k = \frac{N}{N_k}\frac{1}{2\pi i}\oint_{\mathcal{C}_{\underline{F},k}} z m_{\mathbf{B}_N}(z) \frac{m_{\underline{\mathbf{B}}_N}'(z)}{m_{\underline{\mathbf{B}}_N}^2(z)} dz$$

Understanding the contour change



- **IF** $\mathcal{C}_{\underline{F},k}$ encloses cluster k with real points $m_1 < m_2$
- **THEN** $-1/m_1 = x_1 < t_k < x_2 = -1/m_2$ and $\mathcal{C}_k$ encloses t_k.

Poles and residues

- we find two sets of poles (outside zeros):
  - $\lambda_1, \ldots, \lambda_N$, the eigenvalues of $\mathbf{B}_N$.
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$$m_{\mathbf{B}_N}(w) = \frac{n}{N} m_{\underline{\mathbf{B}}_N}(w) + \frac{n-N}{N} \frac{1}{w}$$

- residue calculus, denote $f(w) = \left(\frac{n}{N} w m_{\underline{\mathbf{B}}_N}(w) + \frac{n-N}{N} \right) \frac{m'_{\underline{\mathbf{B}}_N}(w)}{m_{\underline{\mathbf{B}}_N}(w)^2}$,
  - the λ_k's are poles of order 1 and

$$\lim_{z \to \lambda_k} (z - \lambda_k) f(z) = -\frac{n}{N} \lambda_k$$

  - the μ_k's are also poles of order 1 and by L'Hospital's rule

$$\lim_{z \to \mu_k} (z - \lambda_k) f(z) = \lim_{z \to \mu_k} \frac{n}{N} \frac{(z - \mu_k) z m'_{\underline{\mathbf{B}}_N}(z)}{m_{\underline{\mathbf{B}}_N}(z)} = \frac{n}{N} \mu_k$$

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  - So, finally

$$\hat{t}_k = \frac{n}{N_k}\sum_{m\in\text{contour}}(\lambda_m - \mu_m)$$

Which poles in the contour?

- we now need to determine which poles are in the contour of interest.

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- what about μ_1? the trick is to use the fact that

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{1}{z} dz = 0$$

which leads to

$$\frac{1}{2\pi i} \oint_{\partial \Gamma_k} \frac{m'_{\underline{F}}(w)}{m_{\underline{F}}(w)^2} dw = 0$$

the empirical version of which is

$$\#\{i : \lambda_i \in \Gamma_k\} - \#\{i : \mu_i \in \Gamma_k\}$$

Since their difference tends to 0, there are as many λ_k's as μ_k's in the contour, hence μ_1 is asymptotically in the integration contour.

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Outline

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- 1.1. The Stieltjes Transform Method
- 1.2. Extreme Eigenvalues
- 1.3. The Spiked Model
- 1.4. Spectrum Analysis and G-estimation

2. Source Detection
- 2.1. Eigenvalue-based Detection
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Problem formulation

- We want to test the hypothesis $\mathcal{H}_0$ against $\mathcal{H}_1$,

$$\mathbb{C}^{N\times n} \ni \mathbf{Y} = \begin{cases} \mathbf{h}\mathbf{x}^T + \sigma\mathbf{W} & \text{, information plus noise, hypothesis } \mathcal{H}_1 \\ \sigma\mathbf{W} & \text{, pure noise, hpothesis } \mathcal{H}_0 \end{cases}$$

with $\mathbf{h} \in \mathbb{C}^N$, $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{W} \in \mathbb{C}^{N\times n}$.

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  with $\mathbf{h} \in \mathbb{C}^N$, $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{W} \in \mathbb{C}^{N\times n}$.
- We assume no knowledge whatsoever but that **W** has i.i.d. (non-necessarily Gaussian) entries.

Exploiting the conditioning number

L. S. Cardoso, M. Debbah, P. Bianchi, J. Najim, "Cooperative spectrum sensing using random matrix theory," International Symposium on Wireless Pervasive Computing, pp. 334-338 , 2008.

- under either hypothesis,
  - if $\mathcal{H}_0$, for N large, we expect $F_{\mathbf{YY}^\mathrm{H}}$ close to the Marčenko-Pastur law, of support $[\sigma^2\left(1-\sqrt{c}\right)^2, \sigma^2\left(1+\sqrt{c}\right)^2]$.
  - if $\mathcal{H}_1$, if population spike more than $1+\sqrt{\frac{N}{n}}$, largest eigenvalue is further away.
- the conditioning number of $\mathbf{YY}^\mathrm{H}$ is therefore asymptotically, as $N, n \to \infty$, $N/n \to c$,
  - if $\mathcal{H}_0$,

$$\mathrm{cond}(\mathbf{Y}) \triangleq \frac{\lambda_{\max}}{\lambda_{\min}} \to \frac{\left(1-\sqrt{c}\right)^2}{\left(1+\sqrt{c}\right)^2}$$

  - if $\mathcal{H}_1$,

$$\mathrm{cond}(\mathbf{Y}) \to t_1 + \frac{ct_1}{t_1-1} > \frac{\left(1-\sqrt{c}\right)^2}{\left(1+\sqrt{c}\right)^2}$$

  with $t_1 = \sum_{k=1}^{N} |h_k|^2 + \sigma^2$

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  with $t_1 = \sum_{k=1}^{N} |h_k|^2 + \sigma^2$
- the conditioning number is independent of σ. We then have the decision criterion, whether or not σ is known,

$$\text{decide} \begin{cases} \mathcal{H}_0 : & \text{if } \mathrm{cond}(\mathbf{YY}^\mathsf{H}) \leqslant \frac{\left(1-\sqrt{\frac{N}{n}}\right)^2}{\left(1+\sqrt{\frac{N}{n}}\right)^2} + \varepsilon \\ \mathcal{H}_1 : & \text{otherwise.} \end{cases}$$

for some security margin ε.

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- Advantages:
  - much simpler than finite size analysis
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  - ratio independent of σ, so σ needs not be known
- Drawbacks:
  - only stands for very large N (dimension N for which asymptotic results arise function of σ!)
  - *ad-hoc* method, does not rely on performance criterion.

Generalized likelihood ratio test

P. Bianchi, M. Debbah, M. Maida, J. Najim, "Performance of Statistical Tests for Source Detection using Random Matrix Theory," IEEE Trans. on Information Theory, vol. 57, no. 4, pp. 2400-2419, 2011.

- Alternative generalized likelihood ratio test (GLRT) decision criterion, i.e.

$$C(\mathbf{Y}) = \frac{\sup_{\sigma^2,\mathbf{h}} P_{\mathbf{Y}|\mathbf{h},\sigma^2}(\mathbf{Y},\mathbf{h},\sigma^2)}{\sup_{\sigma^2} P_{\mathbf{Y}|\sigma^2}(\mathbf{Y}|\sigma^2)}.$$

- Denote

$$T_N = \frac{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^{\mathsf{H}})}{\frac{1}{N}\operatorname{tr}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}}$$

To guarantee a maximum false alarm ratio of α,

$$\text{decide} \begin{cases} \mathcal{H}_1 : & \text{if } \left(1-\frac{1}{N}\right)^{(1-N)n} T_N^{-n}\left(1-\frac{\mathbf{T}_N}{N}\right)^{(1-N)n} > \xi_N \\ \mathcal{H}_0 : & \text{otherwise.} \end{cases}$$

for some threshold ξ_N that can be explicitly given as a function of α.

Generalized likelihood ratio test

P. Bianchi, M. Debbah, M. Maida, J. Najim, "Performance of Statistical Tests for Source Detection using Random Matrix Theory," IEEE Trans. on Information Theory, vol. 57, no. 4, pp. 2400-2419, 2011.

- Alternative generalized likelihood ratio test (GLRT) decision criterion, i.e.

$$C(\mathbf{Y}) = \frac{\sup_{\sigma^2,\mathbf{h}} P_{\mathbf{Y}|\mathbf{h},\sigma^2}(\mathbf{Y},\mathbf{h},\sigma^2)}{\sup_{\sigma^2} P_{\mathbf{Y}|\sigma^2}(\mathbf{Y}|\sigma^2)}.$$

- Denote

$$T_N = \frac{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^{\mathsf{H}})}{\frac{1}{N}\operatorname{tr}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}}$$

  To guarantee a maximum false alarm ratio of α,

$$\text{decide} \begin{cases} \mathcal{H}_1 : & \text{if } \left(1-\frac{1}{N}\right)^{(1-N)n} T_N^{-n} \left(1-\frac{\mathbf{T}_N}{N}\right)^{(1-N)n} > \xi_N \\ \mathcal{H}_0 : & \text{otherwise.} \end{cases}$$

  for some threshold ξ_N that can be explicitly given as a function of α.
- Optimal test with respect to GLR.
- Performs better than conditioning number test.

Performance comparison for unknown σ^2, P

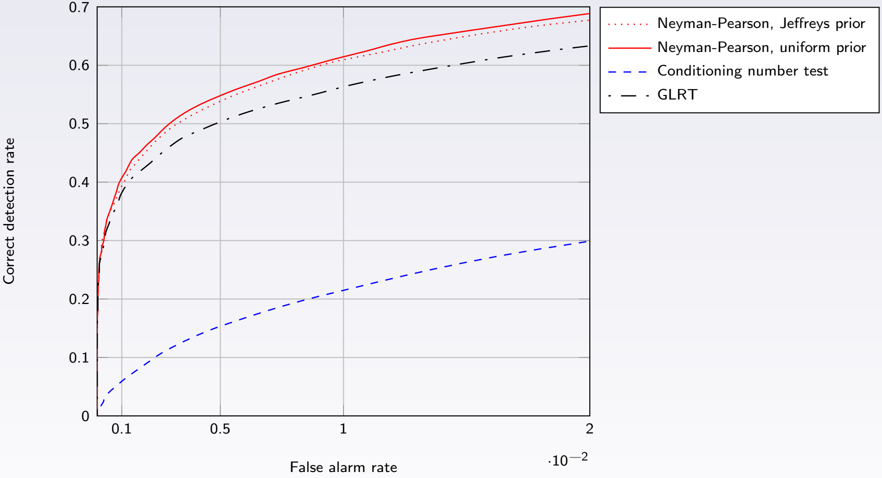

Figure : ROC curve for *a priori* unknown σ^2 of the Neyman-Pearson test, conditioning number method and GLRT, $K = 1$, $N = 4$, $M = 8$, $\text{SNR} = 0 \text{ dB}$. For the Neyman-Pearson test, both uniform and Jeffreys prior, with exponent $\beta = 1$, are provided.

Related biography

- R. Couillet, M. Debbah, “A Bayesian Framework for Collaborative Multi-Source Signal Sensing”, IEEE Transactions on Signal Processing, vol. 58, no. 10, pp. 5186-5195, 2010.
- T. Ratnarajah, R. Vaillancourt, M. Alvo, “Eigenvalues and condition numbers of complex random matrices,” SIAM Journal on Matrix Analysis and Applications, vol. 26, no. 2, pp. 441-456, 2005.
- M. Matthaiou, M. R. McKay, P. J. Smith, J. A. Mossek, “On the condition number distribution of complex Wishart matrices,” IEEE Transactions on Communications, vol. 58, no. 6, pp. 1705-1717, 2010.
- C. Zhong, M. R. McKay, T. Ratnarajah, K. Wong, “Distribution of the Demmel condition number of Wishart matrices,” IEEE Trans. on Communications, vol. 59, no. 5, pp. 1309-1320, 2011.
- L. S. Cardoso, M. Debbah, P. Bianchi, J. Najim, “Cooperative spectrum sensing using random matrix theory,” International Symposium on Wireless Pervasive Computing, pp. 334-338 , 2008.
- P. Bianchi, M. Debbah, M. Maida, J. Najim, “Performance of Statistical Tests for Source Detection using Random Matrix Theory,” IEEE Trans. on Information Theory, vol. 57, no. 4, pp. 2400-2419, 2011.

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1.4. Spectrum Analysis and G-estimation

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Generic inference scenario

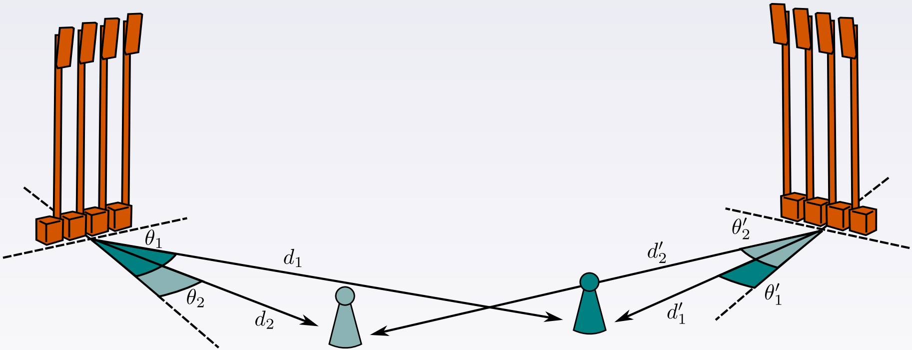

Figure : Signal sensing and angle of arrival detection

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Power estimation: problem Statement

- Consider the model

$$\mathbf{y}^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} \mathbf{H}_k \mathbf{x}_k^{(m)} + \sigma \mathbf{w}^{(m)}$$

and wish to infer $P_1, \ldots, P_K$.

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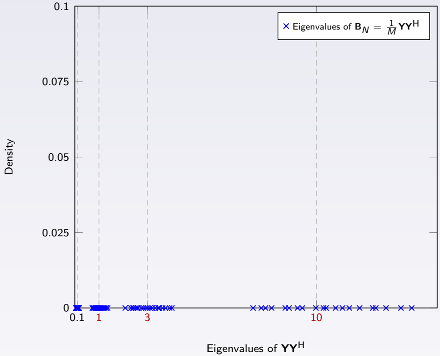
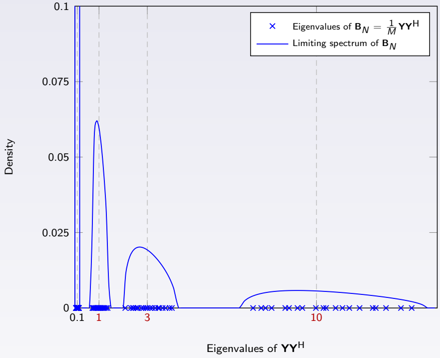
Most information about $P_1, \ldots, P_K$ is contained in the eigenvalues of $\mathbf{B}_N \triangleq \frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$.

From small to large system analysis



Classical approach requires to assume $M >> N$ as well as $N >> n_k$ for each k!

From small to large system analysis



Assuming dimensions N, n_k, M grow large, large dimensional random matrix theory provides

- a link between:
  - **the “observation”:** the limiting spectral distribution (l.s.d.) of $\mathbf{B}_N$;
  - **the “hidden parameters”:** the powers $P_1, \ldots, P_K$, i.e. the l.s.d. of $\mathbf{P}$.
- consistent estimators of the hidden parameters.

Power estimation with random matrices

- **Reminder:** Method consists in:

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- Extending $\mathbf{Y}$ with zeros, our model is a "double sample covariance matrix"

$$\underbrace{\underline{\mathbf{Y}}}_{(N+n)\times M} = \underbrace{\begin{bmatrix} \mathbf{H}\mathbf{P}^{\frac{1}{2}} & \sigma \mathbf{I}_N \\ 0 & 0 \end{bmatrix}}_{(N+n)\times(N+n)} \underbrace{\begin{bmatrix} \mathbf{X} \\ \mathbf{W} \end{bmatrix}}_{(N+n)\times M} .$$

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- Limiting distribution of $\frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$

Theorem (Spectral analysis of $\mathbf{B}_N$)

Let $\mathbf{B}_N = \frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$ with eigenvalues $\lambda_1, \ldots, \lambda_N$. Denote $m_{\underline{\mathbf{B}}_N}(z) \triangleq \frac{1}{M}\sum_{k=1}^{M}\frac{1}{\lambda_k - z}$, with $\lambda_i = 0$ for $i > N$. Then, for $M/N \to c$, $N/n_k \to c_k$, $N/n \to c_0$, for any $z \in \mathbb{C}^+$,

$$m_{\underline{\mathbf{B}}_N}(z) \xrightarrow{\text{a.s.}} m_{\underline{F}}(z)$$

with $m_{\underline{F}}(z)$ the unique solution in $\mathbb{C}^+$ of

$$\frac{1}{m_{\underline{F}}(z)} = -\sigma^2 + \frac{1}{f(z)}\left[\frac{c_0 - 1}{c_0} + m_P\left(-\frac{1}{f(z)}\right)\right], \quad \text{with} \quad f(z) = (c-1)m_{\underline{F}}(z) - czm_{\underline{F}}(z)^2.$$

Stieltjes transform method (2)

R. Couillet, J. W. Silverstein, Z. Bai, M. Debbah, "Eigen-Inference for Energy Estimation of Multiple Sources," *to appear in* IEEE Trans. on Inf. Theory, 2010.

- estimator calculus

Theorem (Estimator of $P_1, \ldots, P_K$)

Let $\mathbf{B}_N \in \mathbb{C}^{N\times N}$ be defined as in Theorem 19, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$, $\lambda_1 < \ldots < \lambda_N$. Assume that asymptotic cluster separability condition is fulfilled for some k. Then, as N, n, $M \to \infty$,

$$\hat{P}_k - P_k \xrightarrow{\text{a.s.}} 0,$$

where

$$\hat{P}_k = \frac{NM}{n_k(M-N)} \sum_{i\in\mathcal{N}_k} (\eta_i - \mu_i)$$

with $\mathcal{N}_k$ the set indexing the eigenvalues in cluster k of F, $\eta_1 < \ldots < \eta_N$ the eigenvalues of $\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$ and $\mu_1 < \ldots < \mu_N$ the eigenvalues of $\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{M}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$.

Remarks

- solution is computationally simple, explicit, and the final formula compact.

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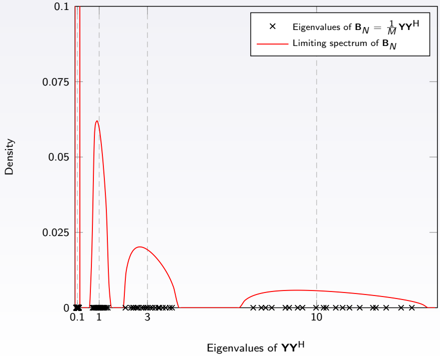
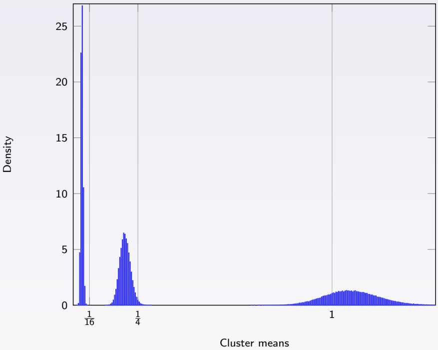
- solution is computationally simple, explicit, and the final formula compact.
- cluster separability condition is fundamental. This requires
  - for all other parameters fixed, the P_k cannot be too close top one another: source separation problem.
  - for all other parameters fixed, σ^2 must be kept low: low SNR undecidability problem.
  - for all other parameters fixed, M/N cannot be too low: sample deficiency issue (not such an issue though).
  - for all other parameters fixed, N/n cannot be too low: diversity issue.
- exact spectrum separability is an essential ingredient (known for very few models to this day).



Simulations



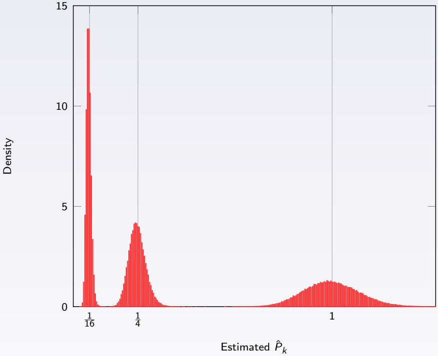


Figure : Histogram of the cluster-mean approach and of $\hat{P}_k$ for $k \in \{1,2,3\}$, $P_1 = 1/16$, $P_2 = 1/4$, $P_3 = 1$, $n_1 = n_2 = n_3 = 4$ antennas per user, $N = 24$ sensors, $M = 128$ samples and $\mathrm{SNR} = 20$ dB.

Performance comparison

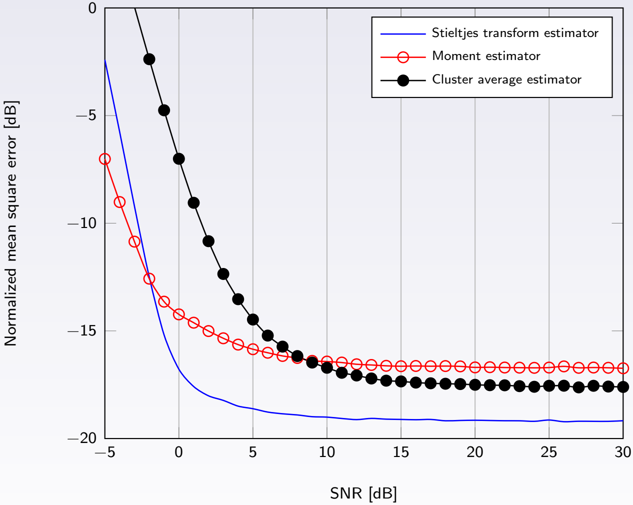


Figure : Normalized mean square error of largest estimated power $\hat{P}_3$, $P_1 = 1/16, P_2 = 1/4, P_3 = 1$, $n_1 = n_2 = n_3 = 4$,$N = 24$, $M = 128$. Comparison between classical, moment and Stieltjes transform approaches.

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Direction-of-arrival estimation: Position of the problem

- We consider the sensor network scenario with:
  - K signal sources
  - an array of N receive antennas, $N > K$
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- Received signal $\mathbf{y}^{(t)} \in \mathbb{C}^N$ at time t

$$\mathbf{y}^{(t)} = \sum_{k=1}^{K} \mathbf{s}(\theta_k) x_k^{(t)} + \sigma \mathbf{w}^{(t)}$$

with $E[s_k] = 0$, $E[|x_k|^2] = P_k$.

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- Therefore

$$E[\mathbf{y}^{(t)} \mathbf{y}^{(y)\mathrm{H}}] \triangleq \mathbf{R} = \mathbf{S}(\Theta)\mathbf{P}\mathbf{S}(\Theta)^{\mathrm{H}} + \sigma^2 \mathbf{I}_N$$

where $\mathbf{S}(\Theta) = [\mathbf{s}(\theta_1), \ldots, \mathbf{s}(\theta_K)] \in \mathbb{C}^{N \times K}$, $\mathbf{P} = \operatorname{diag}(P_1, \ldots, P_K)$.

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- **Objective:** Based on $\mathbf{Y} \triangleq [\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(M)}]$, estimate $\theta_1, \ldots, \theta_K$,

MUSIC method

- Write

$$\mathbf{R} = \begin{pmatrix} \mathbf{E}_W & \mathbf{E}_S \end{pmatrix} \begin{pmatrix} \sigma^2 \mathbf{I}_{N-K} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_S \end{pmatrix} \begin{pmatrix} \mathbf{E}_W^H \\ \mathbf{E}_S^H \end{pmatrix}$$

with $\mathbf{L}_S = \mathrm{diag}(\lambda_{N-K+1}, \ldots, \lambda_N)$, $\mathbf{E}_S = [\mathbf{e}_{N-K+1}, \ldots, \mathbf{e}_N]$ the *signal subspace* and $\mathbf{E}_W = [\mathbf{e}_1, \ldots, \mathbf{e}_{N-K}]$ the *noise subspace*.

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- By definition,

$$\eta(\theta_k) \triangleq \mathbf{s}(\theta_k)^H \mathbf{E}_W \mathbf{E}_W^H \mathbf{s}(\theta_k) = 0$$

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- MUSIC algorithm consists in finding θ such that

$$\hat{\eta}(\theta) \triangleq \mathbf{s}(\theta)^{\mathrm{H}}\hat{\mathbf{E}}_W\hat{\mathbf{E}}_W^{\mathrm{H}}\mathbf{s}(\theta).$$

  reaches a local minimum, with $\hat{\mathbf{E}}_W = [\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_{N-K}] \in \mathbb{C}^{N\times(N-K)}$ the subspace spanned by the $N-K$ smallest eigenvalues of

$$\mathbf{R}_N = \frac{1}{M}\sum_{t=1}^{M}\mathbf{y}^{(t)}\mathbf{y}^{(t)\mathrm{H}}.$$

MUSIC method

- Write
$$\mathbf{R} = \begin{pmatrix}\mathbf{E}_W & \mathbf{E}_S\end{pmatrix} \begin{pmatrix}\sigma^2 \mathbf{I}_{N-K} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_S\end{pmatrix} \begin{pmatrix}\mathbf{E}_W^{\mathsf{H}} \\ \mathbf{E}_S^{\mathsf{H}}\end{pmatrix}$$
with $\mathbf{L}_S = \mathrm{diag}(\lambda_{N-K+1}, \ldots, \lambda_N)$, $\mathbf{E}_S = [\mathbf{e}_{N-K+1}, \ldots, \mathbf{e}_N]$ the *signal subspace* and $\mathbf{E}_W = [\mathbf{e}_1, \ldots, \mathbf{e}_{N-K}]$ the *noise subspace*.
- By definition,
$$\eta(\theta_k) \triangleq \mathbf{s}(\theta_k)^{\mathsf{H}} \mathbf{E}_W \mathbf{E}_W^{\mathsf{H}} \mathbf{s}(\theta_k) = 0$$
- MUSIC algorithm consists in finding θ such that
$$\hat{\eta}(\theta) \triangleq \mathbf{s}(\theta)^{\mathsf{H}} \hat{\mathbf{E}}_W \hat{\mathbf{E}}_W^{\mathsf{H}} \mathbf{s}(\theta).$$
reaches a local minimum, with $\hat{\mathbf{E}}_W = [\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_{N-K}] \in \mathbb{C}^{N \times (N-K)}$ the subspace spanned by the $N-K$ smallest eigenvalues of
$$\mathbf{R}_N = \frac{1}{M} \sum_{t=1}^{M} \mathbf{y}^{(t)} \mathbf{y}^{(t)\mathsf{H}}.$$

Only *M*-consistent!

RMT will provide an (N, M)-consistent procedure.

Result on quadratic forms

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- Starting point: Cauchy integration formula

$$\mathbf{s}(\theta_k)^{\mathsf{H}}\mathbf{E}_W\mathbf{E}_W^{\mathsf{H}}\mathbf{s}(\theta_k) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathbf{s}(\theta_k)\left(\mathbf{R} - z\mathbf{I}_N\right)^{-1}\mathbf{s}(\theta_k)dz$$

with $\mathcal{C}$ circling around σ^2 only (only one pole in $z = \sigma^2$).

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- We then use the result:

Lemma

For $\mathbf{a} \in \mathbb{C}^N$ *deterministic bounded, independent of* $\mathbf{R}_N$,

$$\mathbf{a}^{\mathsf{H}}\left(\mathbf{R}_N - z\mathbf{I}_N\right)^{-1}\mathbf{a} - \mathbf{a}^{\mathsf{H}}\left(\frac{1}{1 + ce_N(z)}\mathbf{R} - z\mathbf{I}_N\right)^{-1}\mathbf{a} \xrightarrow{\text{a.s.}} 0$$

with $e_N(z)$ *solution to*

$$e = \int \frac{t}{\frac{t}{1+ce} - z} dF^{\mathbf{R}}(t).$$

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with $e_N(z)$ solution to

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- By change of variable, dominated convergence arguments, and residue calculus, we conclude.

G-MUSIC

X. Mestre, M. A. Lagunas, "Finite sample size effect on minimum variance beamformers: Optimum diagonal loading factor for large arrays," IEEE Trans. on Signal Processing, vol. 54, no. 1, pp. 69-82, 2006.

Theorem

Under the above conditions,

$$\eta(\theta) - \bar{\eta}(\theta) \xrightarrow{\text{a.s.}} 0$$

as $N, M \to \infty$ *with* $0 < \lim N/M < \infty$, *where*

$$\bar{\eta}(\theta) = \mathbf{s}(\theta)^{\mathsf{H}} \left(\sum_{n=1}^{N} \phi(n) \hat{\mathbf{e}}_n \hat{\mathbf{e}}_n^{\mathsf{H}} \right) \mathbf{s}(\theta)$$

with $\phi(n)$ *defined as*

$$\phi(n) = \begin{cases} 1 + \sum_{k=N-K+1}^{N} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_n - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_n - \hat{\mu}_k} \right) & , \; n \leqslant N-K \\ -\sum_{k=1}^{N-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_n - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_n - \hat{\mu}_k} \right) & , \; n > N-K \end{cases}$$

and with $\mu_1 \leqslant \ldots \leqslant \mu_N$ *the eigenvalues of diag*$(\hat{\boldsymbol{\lambda}}) - \frac{1}{M}\sqrt{\hat{\boldsymbol{\lambda}}}\sqrt{\hat{\boldsymbol{\lambda}}}^{\mathsf{T}}$, $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_N)^{\mathsf{T}}$.

Simulation results

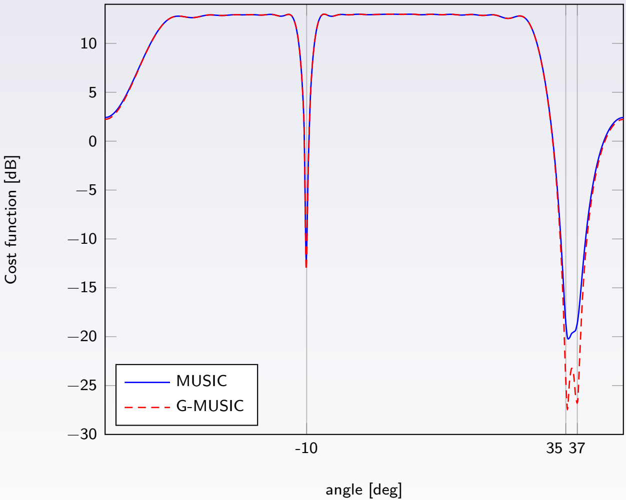

Figure : MUSIC against G-MUSIC for DoA detection of $K = 3$ signal sources, $N = 20$ sensors, $M = 150$ samples, SNR of 10 dB. Angles of arrival of 10°, 35°, and 37°.

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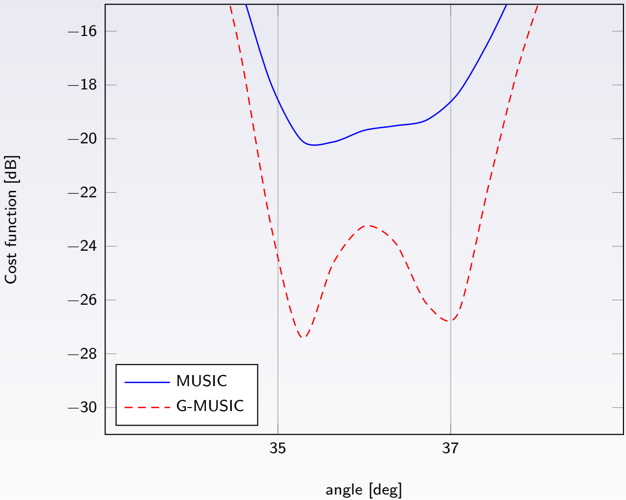

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Part 1: Fundamentals of Random Matrix Theory
1.1. The Stieltjes Transform Method
1.2. Extreme Eigenvalues
1.3. The Spiked Model
1.4. Spectrum Analysis and G-estimation

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2.1. Eigenvalue-based Detection
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3. Statistical Inference
3.1. Generic Model
3.1.2. Angle-of-arrival estimation
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Covariance matrix against spike models

$\rightarrow$ The problems under consideration are of the type

$$\mathbf{Y} = \mathbf{A}(\Theta)\mathbf{X}$$

with $\mathbf{Y} \in \mathbb{C}^{N \times n}$ and

- $\mathbf{X}$ is random with i.i.d. entries
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→ We want to retrieve Θ from the observation $\mathbf{Y}$, when both N and n are large, i.e. derivate (N, n)-consistent estimators

→ As opposed to finite N regime, two RMT approaches:

- $\mathbf{A}(\Theta)$ is a large rank matrix:
  - analysis of the link between $\frac{1}{n}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$ and $\mathbf{A}(\Theta)$
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  → This is the case we already studied.

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  - → Simpler but usually less accurate approach

Localization of small-dimensional sources (1)

$\rightarrow$ We consider the scenario of K sources and an N-antenna array capturing

$$\mathbf{y}^{(m)} = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_k^{(m)} + \mathbf{w}^{(m)}$$

- $s_k^{(m)}$ and $\mathbf{w}^{(m)}$ are random with zero mean and unit variance entries
- $m = 1, \ldots, n$ with N, n large assuming $N/n \rightarrow c > 0$, and K fixed

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→ Spectral decomposition of the population covariance

$$\mathbf{E}[\mathbf{y}\mathbf{y}^{\mathsf{H}}] = \mathbf{A}\mathbf{A}^{\mathsf{H}} + \mathbf{I}_N = \mathbf{U}_S \boldsymbol{\Omega} \mathbf{U}_S^{\mathsf{H}} + \mathbf{I}_N$$

with $\mathbf{U}_S = [\mathbf{u}_1, \ldots, \mathbf{u}_K] \in \mathbb{C}^{N \times K}$ isometric, $\boldsymbol{\Omega} = \operatorname{diag}(\omega_1, \ldots, \omega_K)$, $\omega_1 \geqslant \ldots \geqslant \omega_K$.

- $\mathbf{E}[\mathbf{y}\mathbf{y}^{\mathsf{H}}]$ is a small-rank perturbation of the identity matrix: spike model
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→ Some consequences of the model in the RMT setting (see e.g. Weyl's inequality)

- limiting weak spectrum is the Marcenko–Pastur law!
- up to K eigenvalues can leave the limiting support

Localization of small-dimensional sources (2)

$\rightarrow$ We first need to understand the spectrum of $\frac{1}{n}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$

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- Reminder: If x eigenvalue of $\frac{1}{n}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}$ with $x > (1+\sqrt{c})^2$ (edge of MP law), for all large n,

$$x \triangleq \lambda_k \xrightarrow{\text{a.s.}} \rho_k \triangleq 1 + \omega_k + c(1+\omega_k)\omega_k^{-1}, \ \textit{if}\ \ \omega_k > \sqrt{c}$$

  for some k.

Localization of small-dimensional sources (3)

→ Recall the MUSIC approach: we want to estimate

$$\eta(\theta) = \mathbf{a}(\theta)^{\mathsf{H}} \mathbf{U}_W \mathbf{U}_W^{\mathsf{H}} \mathbf{a}(\theta) \quad (\mathbf{U}_W \in \mathbb{C}^{N \times (N-K)} \text{ such that } \mathbf{U}_W^{\mathsf{H}} \mathbf{U}_S = 0)$$

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→ Instead of this quantity, we start with the study of

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with $\hat{\mathbf{u}}_1,\ldots,\hat{\mathbf{u}}_N$ the eigenvectors belonging to $\lambda_1 \geqslant \ldots \geqslant \lambda_N$.

→ To fall back on known RMT quantities, we use the Cauchy-integral:

$$\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a}(\theta) = -\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i}\mathbf{a}(\theta)^{\mathsf{H}}(\frac{1}{n}\mathbf{Y}\mathbf{Y}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{a}(\theta)dz$$

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with $\mathcal{C}_i$ a contour enclosing λ_i only.

→ Woodbury's identity $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$ gives:

$$\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a} = \frac{-1}{2\pi\imath}\oint_{\mathcal{C}_i} \mathbf{a}^{\mathsf{H}}(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}(\frac{\mathbf{X}\mathbf{X}^{\mathsf{H}}}{n} - z\mathbf{I}_N)^{-1}(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}\mathbf{a}dz + \frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \hat{\mathbf{a}}_1^{\mathsf{H}}\widehat{\mathbf{H}}^{-1}\hat{\mathbf{a}}_2dz$$

where $\mathbf{P} = \mathbf{U}_S\boldsymbol{\Omega}\mathbf{U}_S^{\mathsf{H}}$, and

$$\begin{cases} \widehat{\mathbf{H}} & = \mathbf{I}_K + z\boldsymbol{\Omega}(\mathbf{I}_K+\boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{U}_S \\ \hat{\mathbf{a}}_1^{\mathsf{H}} & = z\mathbf{a}(\theta)^{\mathsf{H}}(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}(\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{U}_S \\ \hat{\mathbf{a}}_2 & = \boldsymbol{\Omega}(\mathbf{I}_K+\boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}\mathbf{a}(\theta). \end{cases}$$

Localization of small-dimensional sources (4)

- For large n, the first term has no pole, while the second converges to

$$T_i \triangleq \frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \mathbf{a}_1^{\mathsf{H}}\mathbf{H}^{-1}\mathbf{a}_2 dz, \text{ with } \begin{cases} \mathbf{H} & = \mathbf{I}_K + zm(z)\boldsymbol{\Omega}(\mathbf{I}_K + \boldsymbol{\Omega})^{-1} \\ \mathbf{a}_1^{\mathsf{H}} & = zm(z)\mathbf{a}^*(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}\mathbf{U}_S \\ \mathbf{a}_2 & = m(z)\boldsymbol{\Omega}(\mathbf{I}_K + \boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}\mathbf{a} \end{cases}$$

which after development is

$$T_i = \sum_{\ell=1}^{K} \frac{1}{1+\omega_\ell}\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \frac{zm^2(z)}{\frac{1+\omega_\ell}{\omega_\ell} + zm(z)} dz.$$

Localization of small-dimensional sources (4)

- For large n, the first term has no pole, while the second converges to

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- Using residue calculus, the sole pole is in ρ_i and we find

$$\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a}_i \xrightarrow{\text{a.s.}} \frac{1-c\omega_i^{-2}}{1+c\omega_i^{-1}}\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{u}_i\mathbf{u}_i^{\mathsf{H}}\mathbf{a}(\theta).$$

Localization of small-dimensional sources (5)

→ We now conclude

$$\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{U}_W\mathbf{U}_W^{\mathsf{H}}\mathbf{a}(\theta) = \mathbf{a}(\theta)^{\mathsf{H}}\mathbf{a}(\theta) - \sum_{k=1}^{K}\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{u}_k\mathbf{u}_k^{\mathsf{H}}\mathbf{a}(\theta)$$

where

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→ The ω_k are however unknown. But they can be estimated from

$$\lambda_k \xrightarrow{\text{a.s.}} \rho_k = 1 + \omega_k + c(1+\omega_k)\omega_k^{-1}$$

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$$\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{U}_W\mathbf{U}_W^{\mathsf{H}}\mathbf{a}(\theta) \simeq \mathbf{a}(\theta)^{\mathsf{H}}\mathbf{a}(\theta) - \sum_{k=1}^{K}\frac{1+c\hat{\omega}_k^{-1}}{1-c\hat{\omega}_k^{-2}}\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_k\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{a}(\theta)$$

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→ We then obtain another (N,n)-consistent MUSIC estimator, only valid for K finite!

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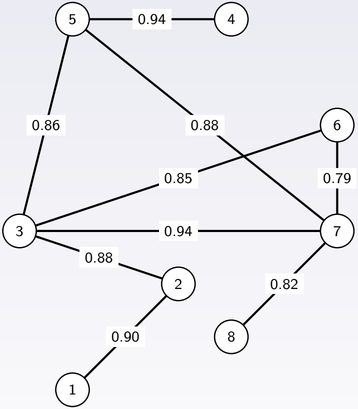
Problem statement



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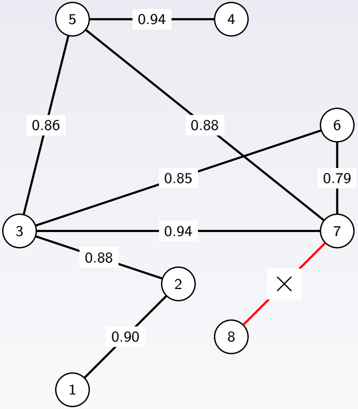

- Localize local failures based on observations from a sensor network.
- Focus on failures modeled as small rank perturbations of large random matrices.

Target

- Systems with failures modeled by small rank perturbations
- Observation matrix $\mathbf{\Sigma} = [\mathbf{s}_1, \ldots, \mathbf{s}_n] \in \mathbb{C}^{N \times n}$ modeled by

$$\mathbf{\Sigma} = (\mathbf{I}_N + \mathbf{P}_k)^{\frac{1}{2}} \mathbf{X}$$

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- Subspace approach for:
  - detecting a failure: decide between $\mathcal{H}_0$ and $\bar{\mathcal{H}}_0$
  - diagnosing a failure: upon failure detection, decide on the most probable $\mathcal{H}_k$.

Example 1

Node failure in sensor networks

- Consider the model

$$\mathbf{y} = \mathbf{H}\theta + \sigma\mathbf{w}$$

with $\mathbf{H} \in \mathbb{C}^{N\times p}$ deterministic, $\theta \sim \mathcal{CN}(0, \mathbf{I}_p)$, $\mathbf{w} \sim \mathcal{CN}(0, \mathbf{I}_N)$.

  - In particular $E[\mathbf{y}] = 0$ and $E[\mathbf{y}\mathbf{y}^{\mathsf{H}}] = \mathbf{R} \triangleq \mathbf{H}\mathbf{H}^{\mathsf{H}} + \sigma^2\mathbf{I}_N$
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  - With $\mathbf{s} = \mathbf{R}^{-\frac{1}{2}}\mathbf{y}$, $E[\mathbf{s}\mathbf{s}^{\mathsf{H}}] = \mathbf{I}_N$.
- Upon failure of sensor k, $\mathbf{y}$ becomes
$$\mathbf{y}' = (\mathbf{I}_N - \mathbf{e}_k\mathbf{e}_k^{\mathsf{H}})\mathbf{H}\theta + \sigma_k\mathbf{e}_k\mathbf{e}_k^{\mathsf{H}}\theta' + \sigma\mathbf{w}$$
for some noise variance σ_k^2.
  - Now $E[\mathbf{y}'] = 0$ and $E[\mathbf{y}'\mathbf{y}'^{\mathsf{H}}] = (\mathbf{I}_N - \mathbf{e}_k\mathbf{e}_k^{\mathsf{H}})\mathbf{H}\mathbf{H}^{\mathsf{H}}(\mathbf{I}_N - \mathbf{e}_k\mathbf{e}_k^{\mathsf{H}}) + \sigma_k^2\mathbf{e}_k\mathbf{e}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N$.
  - With now $\mathbf{s} = \mathbf{R}^{-\frac{1}{2}}\mathbf{y}'$,
$$E[\mathbf{s}\mathbf{s}^{\mathsf{H}}] = \mathbf{I}_N + \mathbf{P}_k$$
with
$$\mathbf{P}_k = -\mathbf{R}^{-\frac{1}{2}}\mathbf{H}\mathbf{H}^{\mathsf{H}}\mathbf{e}_k\mathbf{e}_k^{\mathsf{H}}\mathbf{R}^{-\frac{1}{2}} + \mathbf{R}^{-\frac{1}{2}}\mathbf{e}_k\left[(\mathbf{e}_k^{\mathsf{H}}\mathbf{H}\mathbf{H}^{\mathsf{H}}\mathbf{e}_k + \sigma_k^2)\mathbf{e}_k^{\mathsf{H}}\mathbf{R}^{-\frac{1}{2}} - \mathbf{e}_k^{\mathsf{H}}\mathbf{H}\mathbf{H}^{\mathsf{H}}\mathbf{R}^{-\frac{1}{2}}\right]$$
of rank-2 (image of $\mathbf{P}_k$ in $\mathrm{Span}(\mathbf{R}^{-\frac{1}{2}}\mathbf{e}_k, \mathbf{R}^{-\frac{1}{2}}\mathbf{H}\mathbf{H}^{\mathsf{H}}\mathbf{e}_k)$)

Example 2

Sudden parameter change detection in sensor networks

- Upon sudden change of parameter θ_k,

$$\mathbf{y}' = \mathbf{H}(\mathbf{I}_p + \alpha_k \mathbf{e}_k \mathbf{e}_k^*)\theta + \mu_k \mathbf{H}\mathbf{e}_k + \sigma \mathbf{w}$$

- Then

$$E[\mathbf{y}'\mathbf{y}'^{\mathsf{H}}] = \mathbf{H}(\mathbf{I}_p + [\mu_k^2 + (1+\alpha_k)^2 - 1]\mathbf{e}_k\mathbf{e}_k^{\mathsf{H}})\mathbf{H}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N.$$

- With $\mathbf{R} = \mathbf{H}\mathbf{H}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N$ and $\mathbf{s} = \mathbf{R}^{-\frac{1}{2}}\mathbf{y}'$,

$$E[\mathbf{s}\mathbf{s}^{\mathsf{H}}] = \mathbf{I}_N + \mathbf{P}_k$$

with

$$\mathbf{P}_k = [\mu_k^2 + (1+\alpha_k)^2 - 1]\mathbf{R}^{-\frac{1}{2}}\mathbf{H}\mathbf{e}_k\mathbf{e}_k^{\mathsf{H}}\mathbf{H}^{\mathsf{H}}\mathbf{R}^{-\frac{1}{2}}.$$

of rank-1.

Eigenvalue and eigenvectors statistics: Method

- Consider the model

$$\mathbf{\Sigma} = (\mathbf{I}_N + \mathbf{P})^{\frac{1}{2}}\mathbf{X}$$

with, for simplicity

  - $\mathbf{X}$ standard Gaussian
  - $\mathbf{P} = \mathbf{U}\mathbf{\Omega}\mathbf{U}^{\mathsf{H}}$, $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{C}^{N \times r}$, $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \ldots, \omega_r)$, $\omega_1 > \ldots > \omega_r > 0$.

- Convergence properties of
  - $\lambda_1 > \ldots > \lambda_r$, the r largest eigenvalues of $\mathbf{\Sigma}\mathbf{\Sigma}^{\mathsf{H}}$
  - $\mathbf{u}_i^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{u}_i$, with $\hat{\mathbf{u}}_i$ the eigenvector associated to λ_i.

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- Study based on two ingredients
  - the Stieltjes transform method
  - complex analysis

First order limits

- (Reminder) The *limiting* ρ_k are given by:

$$\lambda_k \xrightarrow{\text{a.s.}} \rho_k \triangleq 1 + \omega_k + c(1 + \omega_k)\omega_k^{-1}, \; \textit{if} \;\; \omega_k > \sqrt{c}$$

- Consider ω_i and its corresponding eigenvector $\mathbf{u}_i$, then

$$\mathbf{u}_i^{\mathsf{H}} \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^{\mathsf{H}} \mathbf{u}_i \xrightarrow{\text{a.s.}} \zeta_i \triangleq \frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-1}}.$$

Fluctuations

Second order behaviour for the joint variable

$$\left(\left(\sqrt{N}(\lambda_i - \rho_i)\right)_{i=1}^r, \left(\sqrt{N}(\mathbf{u}_i^H \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H \mathbf{u}_i - \zeta_i)\right)_{i=1}^r\right)$$

R. Couillet, W. Hachem, "Local failure detection and diagnosis in large sensor networks", IEEE Transactions on Information Theory, arXiv Preprint 1107.1409.

Theorem

Under the conditions above, assuming $\omega_i > \sqrt{c}$ for each $i \in \{1, \ldots, r\}$,

$$\left(\left(\sqrt{N}(\lambda_i - \rho_i)\right)_{i=1}^r, \left(\sqrt{N}(\mathbf{u}_i^H \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H \mathbf{u}_i - \zeta_i)\right)_{i=1}^r\right) \Rightarrow \mathcal{N}\left(0, \begin{bmatrix} C(\rho_1) & & \\ & \ddots & \\ & & C(\rho_r) \end{bmatrix}\right)$$

where

$$C(\rho_i) \triangleq \begin{bmatrix} \frac{c^2(1+\omega_i)^2}{(c+\omega_i)^2(\omega_i^2-c)}\left(c\frac{(1+\omega_i)^2}{(c+\omega_i)^2}+1\right) & \frac{(1+\omega_i)^3c^2}{(\omega_i+c)^2\omega_i} \\ \frac{(1+\omega_i)^3c^2}{(\omega_i+c)^2\omega_i} & \frac{c(1+\omega_i)^2(\omega_i^2-c)}{\omega_i^2} \end{bmatrix}.$$

Reminder: Fluctuations at the edge of the bulk

- The previous theorem holds for $\omega_i > \sqrt{c}$, i.e. "strong perturbations"

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- The previous theorem holds for $\omega_i > \sqrt{c}$, i.e. "strong perturbations"
- For $\omega_i < \sqrt{c}$, the eigenvalue fluctuations are:

Theorem

If $0 \leqslant \omega_i < \sqrt{c}$,

$$N^{\frac{2}{3}}(1+\sqrt{c})^{-\frac{4}{3}}c^{-\frac{1}{2}}(\lambda_i - (1+\sqrt{c})^2) \Rightarrow T_2$$

where T_2 *is the complex Tracy-Widom distribution function.*

Failure detection and localization

- The proposed subspace procedure is a two-step approach:
  - **Failure detection procedure**, $\mathcal{H}_0$ vs. $\bar{\mathcal{H}}_0$: We evaluate the statistics of λ_1 against the Tracy-Widom law for a false alarm rate η,

$$\lambda_1' \underset{\bar{\mathcal{H}}_0}{\overset{\mathcal{H}_0}{\lessgtr}} (T_2)^{-1}(1-\eta)$$

where $\lambda_1' \triangleq N^{\frac{2}{3}}(1+\sqrt{c_N})^{-\frac{4}{3}} c_N^{-\frac{1}{2}}(\lambda_1 - (1+\sqrt{c_N})^2)$.

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  - **Failure diagnosis**, selection of $\mathcal{H}_k$: We evaluate the joint statistics of λ_i, $\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{u}_{k,i}$ for each $k \in \{1,\ldots,K\}$, and obtain the maximum-likelihood test,

$$\hat{k} = \arg \max_{1 \leqslant k \leqslant K} \prod_{i=1}^{r} f\left(\left(\left(\sqrt{N}(\lambda_i - \rho_{k,i})\right)_{i=1}^{r}, \left(\sqrt{N}(\mathbf{u}_{k,i}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{u}_{k,i} - \zeta_{k,i})\right)_{i=1}^{r}\right); C(\rho_{k,i})\right)$$

    with $f(x;\mathbf{R})$ the Gaussian density with zero mean and variance $\mathbf{R}$, and indices k corresponding to hypothesis $\mathcal{H}_k$.

Results

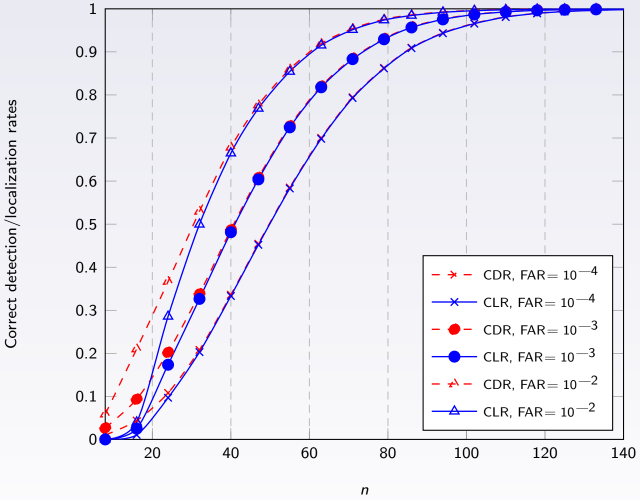

Figure : Simulation of sensor failure in an $N = 10$ node network. Correct detection (CDR) and localization (CLR) rates for different false alarm rates (FAR) and different n.

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Parameter estimation and sample covariance matrix

P.J. Huber, “Robust Statistics”, 1981.

→ Many statistical inference techniques rely on the sample covariance matrix (SCM) taken from i.i.d. observations $x_1, \ldots, x_n$ of a r.v. $x \in \mathbb{C}^N$.

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$$\hat{S}_N \triangleq \frac{1}{n} XX^* \xrightarrow{\text{a.s.}} C_N \text{ as } n \to \infty.$$

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  - The SCM $\hat{S}_N$ is the ML estimate of C_N for Gaussian x
    → One therefore expects $\hat{\theta}$ to closely approximate θ for all finite n.
- This approach however has two limitations:
  - if N, n are of the same order of magnitude,
  $$\|\hat{S}_N - C_N\| \not\to 0 \text{ as } N, n \to \infty, \; N/n \to c > 0, \text{ so that in general } |\hat{\theta} - \theta| \not\to 0$$
    → This motivated the introduction of G-estimators.
  - if x is not Gaussian, but has heavier tails, $\hat{S}_N$ is a poor estimator for C_N.
    → This motivated the introduction of robust estimators.

Reminders on robust estimation

J. T. Kent, D. E. Tyler, "Redescending M-estimates of multivariate location and scatter", 1991.
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→ The objectives of robust estimators:

- Replace the SCM $\hat{S}_N$ by another estimate $\hat{C}_N$ of C_N which:
  - rejects (or downscales) observations deterministically
  - or rejects observations inconsistent with the full set of observations

→ **Example**: Huber estimator, $\hat{C}_N$ defined as solution of

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} \beta_i x_i x_i^* \text{ with } \beta_i = \alpha \min\left\{1, \frac{k^2}{\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i}\right\} \text{ for some } \alpha > 1, k^2 \text{ function of } \hat{C}_N.$$

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  - or rejects observations inconsistent with the full set of observations

  → **Example**: Huber estimator, $\hat{C}_N$ defined as solution of

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} \beta_i x_i x_i^* \text{ with } \beta_i = \alpha \min\left\{1, \frac{k^2}{\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i}\right\} \text{ for some } \alpha > 1, k^2 \text{ function of } \hat{C}_N.$$

- Provide scale-free estimators of C_N:

  → **Example**: Tyler's estimator: if one observes $x_i = \tau_i z_i$ for unknown scalars τ_i,

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i} x_i x_i^*$$

  - existence and uniqueness of $\hat{C}_N$ defined up to a constant.
  - few constraints on $x_1, \ldots, x_n$ ($N+1$ of them must be linearly independent)

Reminders on robust estimation

→ The objectives of robust estimators:

- replace the SCM $\hat{S}_N$ by the ML estimate for C_N.
  → **Example**: Maronna's estimator for elliptical x

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

with $u(s)$ such that

(i) $u(s)$ is continuous and non-increasing on $[0, \infty)$
(ii) $\phi(s) = su(s)$ is non-decreasing, bounded by $\phi_\infty > 1$. Moreover, $\phi(s)$ increases where $\phi(s) < \phi_\infty$.

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  (note that Huber's estimator is compliant with Maronna's estimators)
  - existence is not too demanding
  - uniqueness imposes constraints on $N, n, u(s)$, e.g. $\phi_\infty > \frac{n}{n-N}$. Inconsistent with random matrix regime!
  - consistency result: $\hat{C}_N \to C_N$ **if** $u(s)$ meets the ML estimator for C_N.

Robust Estimation and RMT

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Robust RMT estimation

Can we study the performance of estimators based on the $\hat{C}_N$?

- what are the spectral properties of $\hat{C}_N$?
- can we generate RMT-based estimators relying on $\hat{C}_N$?

Outline

Part 1: Fundamentals of Random Matrix Theory
1.1. The Stieltjes Transform Method
1.2. Extreme Eigenvalues
1.3. The Spiked Model
1.4. Spectrum Analysis and G-estimation

2. Source Detection
2.1. Eigenvalue-based Detection
2.2. Detection in unknown Noise Environment

3. Statistical Inference
3.1. Generic Model
3.1.2. Angle-of-arrival estimation
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3.2. Spiked Model
3.2.1. Spiked G-MUSIC
3.2.2. Local Failure Detection in Sensor Networks

4. Random Matrix Theory and Robust Estimation
4.1. Introduction to Robust Estimation
4.1. Initial Results and Open Problems

Some first answers

→ Recall that

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{N} u\left(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

for some i.i.d. $x_1, \ldots, x_n$ taken from a random vector x, and for some function $u(s)$.

→ For x Gaussian and $u(s)$ of Maronna or Tyler type, simulations suggest:

- Marčenko-Pastur/Bai-Silverstein distribution, i.e. up to some constant α

  $$F^{\alpha \hat{C}_N} - F^{\hat{S}_N} \Rightarrow 0 \text{ as } N, n \to \infty,\ N/n \to c > 0$$

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⇒ This is what we are going to prove.

Then, what happens to $\hat{C}_N$ when no concentration result occurs?

⇒ So far, we have no general answer to this question!

Some first answers (2)

$\rightarrow$ Main difficulties for handling $\hat{C}_N$:

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→ Main difficulties for handling $\hat{C}_N$:

- $\hat{C}_N$ does not always exist/is not always unique.
  - sometimes, uniqueness results inconsistent with random matrix regime
- Contrary to classical RMT, the column vectors $\sqrt{u(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i)}x_i$ are not independent
  - difficult to find an angle to reuse previous results
- In general, it is already difficult to show that both $\|\hat{C}_N\|$ and $\|\hat{C}_N^{-1}\|$ remain bounded as $N, n \to \infty$, $N/n \to c > 0$.

Robust model

Assumptions

- Assumptions on $u(s)$,
  (i) $u(s)$ is continuous and non-increasing on $[0, \infty)$
  (ii) $\phi(s) = su(s)$ is non-decreasing, bounded by $\phi_\infty > 1$. Moreover, $\phi(s)$ increases where $\phi(s) < \phi_\infty$.
- Assumptions on $x_1, \ldots, x_n$,
  - $x_i = A_N y_i \in \mathbb{C}^N$, $y_i \in \mathbb{C}^M$ has independent entries with
    - $E[y_{i,j}] = 0$
    - $E[y_{i,j}^2] = 0$, $E[|y_{i,j}|^2] = 1$
    - $\sup_{i,j} E[|y_{i,j}|^{8+\eta}] < \infty$.
  - With $c_N = N/n$, $\bar{c}_N = M/N \geqslant 1$,

$$0 < \liminf_n c_N \leqslant \limsup_n c_N < 1, \quad \limsup_n \bar{c}_N < \infty$$

  - Denoting $C_N = A_N A_N^*$,

$$0 < \liminf_N \{\lambda_1(C_N)\} \leqslant \limsup_N \{\lambda_N(C_N)\} < \infty$$

Robust SCM estimator in the RMT regime

R. Couillet, F. Pascal, J. W. Silverstein, "Robust M-Estimation for Array Processing: A Random Matrix Approach", (submitted to) IEEE Trans. on Information Theory, 2013.

Theorem

Assume the above and consider the fixed-point equation in $Z \in \mathbb{C}^{N \times N}$,

$$Z = \frac{1}{n} \sum_{i=1}^{n} u\left(\frac{1}{N} x_i^* Z^{-1} x_i\right) x_i x_i^* . \tag{1}$$

Then,

(I) *Equation (1) has a unique solution $\hat{C}_N$ for all large N a.s., defined as*

$$\hat{C}_N = \lim_{t \to \infty} Z^{(t)}$$

where

$$\begin{cases} Z^{(0)} & = I_N \\ Z^{(t+1)} & = \frac{1}{n} \sum_{i=1}^{n} u\left(\frac{1}{N} x_i^* (Z^{(t)})^{-1} x_i\right) x_i x_i^*, \; t \in \mathbb{N}. \end{cases}$$

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(II) *Defining $\hat{C}_N = I_N$ when* (1) *does not have a unique solution,*

$$\left\| \phi^{-1}(1)\hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0.$$

Robust statistical inference in RMT regime

→ From Theorem 1,

- Weak convergence results on $\hat{S}_N$ propagate to $\hat{C}_N$;
- No eigenvalues and exact separation results propagate to $\hat{S}_N$;
- First order results on spiked models as well, etc.
- Irrelevant of underlying distribution of x, as opposed to the finite N regime

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- RMT-based statistical estimators using $\hat{S}_N$ can be replaced by identical estimators using $\hat{C}_N$
- e.g. Mestre's DoA estimator

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→ Theorem 1 however does not say anything about second order results.

- Current investigation: CLT on linear statistics for $\hat{C}_N$, for x with i.i.d. entries.
- This should provide the asymptotic performance comparison between robust-RMT estimators and traditional RMT estimators.
- So far, it seems that limiting variance depends mostly on C_N, c, $u'(\phi^{-1}(1))$, and the kurtosis of the entries of x.

Robust G-MUSIC estimator

→ Consider the model

$$x_t = \sum_{k=1}^{K} \sqrt{p_k} s(\theta_k) z_{k,t} + \sigma w_t = A_N y_t, \quad A_N \triangleq \begin{bmatrix} S(\Theta) P^{\frac{1}{2}} & \sigma I_N \end{bmatrix}, \text{ with}$$

- $S(\Theta) = [s(\theta_1), \ldots, s(\theta_K)]$ deterministic bounded norm steering vectors,
- $P = \mathrm{diag}(p_1, \ldots, p_K)$ diagonal of powers,
- $y_t = (z_{1,t}, \ldots, z_{K,t}, w_t^{\mathsf{T}})^{\mathsf{T}} \in \mathbb{C}^{N+K}$, signals and noise vector.

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→ From the above results and Mestre's G-MUSIC,

Theorem (Robust G-MUSIC)

Denote $E_W \in \mathbb{C}^{N \times (N-K)}$ the "noise subspace" of C_N, $\hat{e}_k$ the eigenvector of $\hat{C}_N$ with eigenvalue $\hat{\lambda}_k \triangleq \lambda_k(\hat{C}_N)$. Then, as $N, n \to \infty$ and K fixed,

$$\gamma(\theta) - \hat{\gamma}(\theta) \xrightarrow{\text{a.s.}} 0, \quad \gamma(\theta) = s(\theta)^* E_W E_W^* s(\theta), \quad \hat{\gamma}(\theta) = \sum_{i=1}^{N} \beta_i s(\theta)^* \hat{e}_i \hat{e}_i^* s(\theta)$$

and

$$\beta_i = \begin{cases} 1 + \sum_{k=N-K+1}^{N} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , \; i \leqslant N-K \\ -\sum_{k=1}^{N-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right) & , \; i > N-K \end{cases}$$

with $\hat{\mu}_1 \leqslant \ldots \leqslant \hat{\mu}_N$ the eigenvalues of $\operatorname{diag}(\hat{\boldsymbol{\lambda}}) - \frac{1}{n}\sqrt{\hat{\boldsymbol{\lambda}}}\sqrt{\hat{\boldsymbol{\lambda}}}^{\mathsf{T}}$, $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_N)^{\mathsf{T}}$.

Results

→ The interest of the above robust-DoA scheme is to:

- handle noise that is "only well-approximated by Gaussian"
- handle model based on bursts of errors on individual antennas
- handle noise distributions with heavier-than-Gaussian tails in radars with distributed antennas (e.g. MIMO radars)

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→ Some strong limitations:

- cannot handle distributions with heavier-than-Gaussian tails in classical radars
  - this would impose to choose x e.g. elliptically distributed
  - our proof technique collapses here
- cannot handle scale-free detectors/estimators, with $u(s) = 1/s$

Simulation results: The Gaussian noise reference

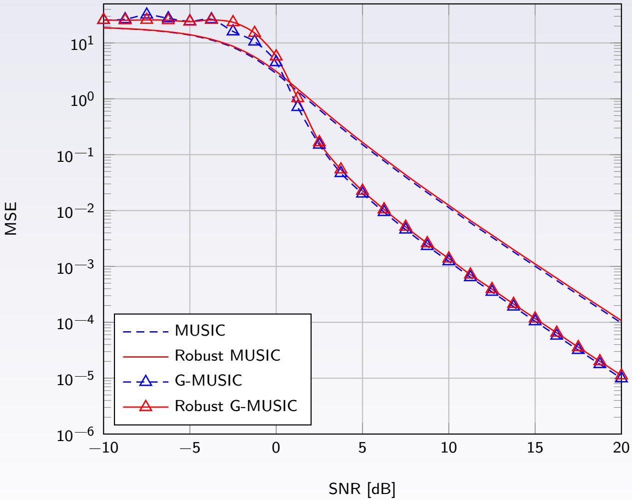

Figure : MSE performance of the various MUSIC estimators for $K = 1$, Gaussian noise, $N = 10$, and $n = 50$, $u(s) = (1 + \nu')/(s + \nu')$, $\nu' = 0.5$.

Simulation results: Close-to-Gaussian noise with i.i.d. Student entries

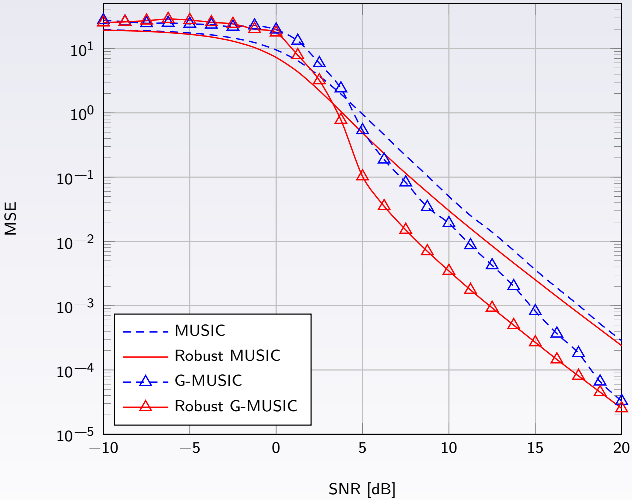

Figure : MSE performance of the various MUSIC estimators for $K = 1$, Student-t noise with $\nu = 5$, $N = 10$, and $n = 50$, $u(s) = (1 + \nu')/(s + \nu')$, $\nu' = 0.5$.

Simulation results: Far-from-Gaussian noise with i.i.d. Student entries

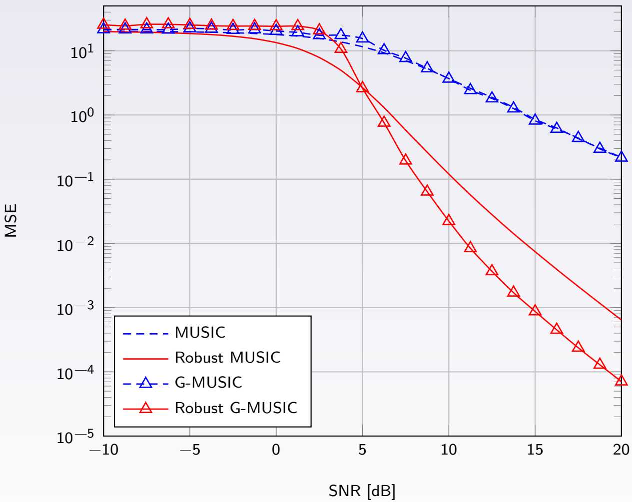

Figure : MSE performance of the various MUSIC estimators for $K = 1$, Student-t noise with $\nu = 2.5$, $N = 10$, and $n = 50$, $u(s) = (1 + \nu')/(s + \nu')$, $\nu' = 0.5$.

Simulation results: Resolution power

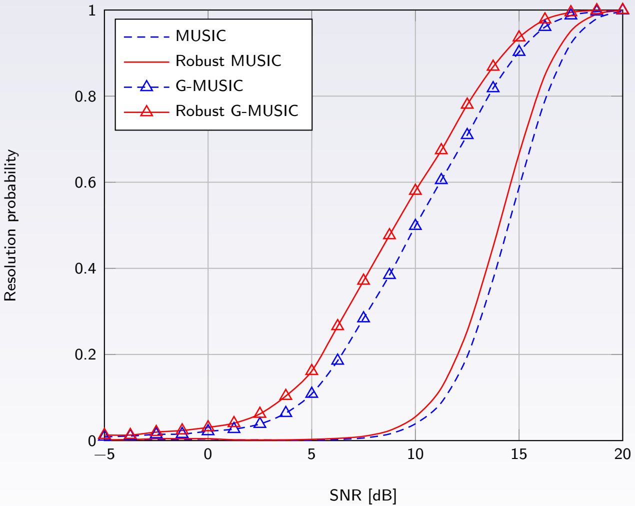

Figure : Resolution performance of the various MUSIC estimators, $\theta_1 = 10^\circ$, $\theta_2 = 15^\circ$, Student-t noise with $\nu = 5$, $N = 10$, and $n = 50$, $u(s) = (1 + \nu')/(s + \nu')$, $\nu' = 0.5$.

Sketch of proof

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→ Take $(d_1, \ldots, d_n)$, $d_i = \frac{1}{N} x_i^* \hat{C}_N^{-1} x_i$ with $\hat{C}_N$ the (almost surely) unique solution:

- We assume $d_1 \leqslant \ldots \leqslant d_n$;
- We also define $D = \operatorname{diag}(u(d_1), \ldots, u(d_n))$.

→ $u(s)$ is non-increasing, so

$$XDX^* \succeq u(d_n) XX^*$$

so that

$$\frac{1}{u(d_n)} \hat{S}_N^{-1} \succeq \hat{C}_N^{-1}$$

and then

$$\frac{1}{u(d_n)} \frac{1}{N} x_n^* \hat{S}_N^{-1} x_n \geqslant d_n$$

from which

$$\phi(d_n) \leqslant \frac{1}{N} x_n^* \hat{S}_N^{-1} x_n$$

→ Proceeding similarly for d_1, and using ϕ non-decreasing, we conclude, for all i

$$\frac{1}{N} x_1^* \hat{S}_N^{-1} x_1 \leqslant \phi(d_1) \leqslant \phi(d_i) \leqslant \phi(d_n) \leqslant \frac{1}{N} x_n^* \hat{S}_N^{-1} x_n$$

Sketch of proof: Convergence

→ It is then possible to show that "the $\frac{1}{N}x_i^*\hat{S}_N^{-1}x_i$ concentrate" as $n \to \infty$, so that

$$\max_{i \leqslant n} |\phi(d_i) - 1| \xrightarrow{\text{a.s.}} 0.$$

→ But $\phi_\infty > 1$ so that ϕ invertible in a neighborhood of 1, and

$$\max_{i \leqslant n} |d_i - \phi^{-1}(1)| \xrightarrow{\text{a.s.}} 0$$

so that

$$\max_{i \leqslant n} \left| u(d_i) - \frac{1}{\phi^{-1}(1)} \right| \xrightarrow{\text{a.s.}} 0$$

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→ We then conclude with

$$\min_{i \leqslant n} \left\{ u(d_i) - \frac{1}{\phi^{-1}(1)} \right\} \frac{1}{n} XX^* \preceq \frac{1}{n} \sum_{i=1}^{n} \left(u(d_i) - \frac{1}{\phi^{-1}(1)} \right) x_i x_i^* \preceq \max_{i \leqslant n} \left\{ u(d_i) - \frac{1}{\phi^{-1}(1)} \right\} \frac{1}{n} XX^*$$

which entails, along with the a.s. boundedness of $\| \frac{1}{n} XX^* \|$,

$$\left\| \hat{C}_N - \frac{1}{\phi^{-1}(1)} \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

Related biography

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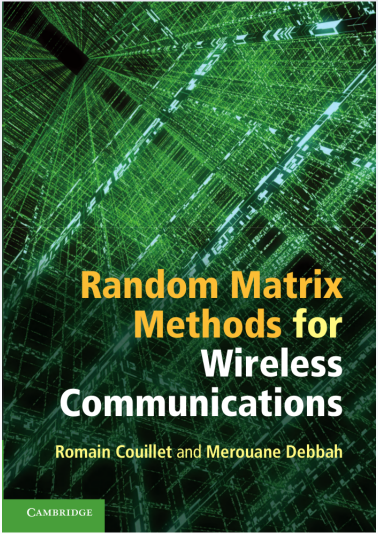
To know more about all this



The end

Thank you