Random Matrices, Robust Estimation, and Applications

(ICASSP'2015, Brisbane)

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Outline

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Context

Baseline scenario: $x_1, \ldots, x_n \in \mathbb{C}^N$ (or $\mathbb{R}^N$) i.i.d. with $E[x_1] = 0$, $E[x_1 x_1^*] = C_N$:

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$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^*.$$

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  or equivalently, in spectral norm

$$\left\| \hat{C}_N - C_N \right\| \xrightarrow{\text{a.s.}} 0.$$

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- For practical N, n with $N \simeq n$, leads to dramatically wrong conclusions
- Even for $N = n/100$.

Setting: $x_i \in \mathbb{C}^N$ i.i.d., $x_1 \sim \mathcal{CN}(0, I_N)$

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- then, joint point-wise convergence

$$\max_{1 \leq i,j \leq N} \left| \left[\hat{C}_N - I_N \right]_{ij} \right| = \max_{1 \leq i,j \leq N} \left| \frac{1}{n} X_{j,\cdot} X_{i,\cdot}^* - \boldsymbol{\delta}_{ij} \right| \xrightarrow{\text{a.s.}} 0.$$

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- however, eigenvalue mismatch

$$0 = \lambda_1(\hat{C}_N) = \ldots = \lambda_{N-n}(\hat{C}_N) \le \lambda_{N-n+1}(\hat{C}_N) \le \ldots \le \lambda_N(\hat{C}_N)$$
$$1 = \lambda_1(I_N) = \ldots = \lambda_{N-n}(I_N) = \lambda_{N-n+1}(\hat{C}_N) = \ldots = \lambda_N(I_N)$$

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$\Rightarrow$ no convergence in spectral norm.

The Marčenko–Pastur law

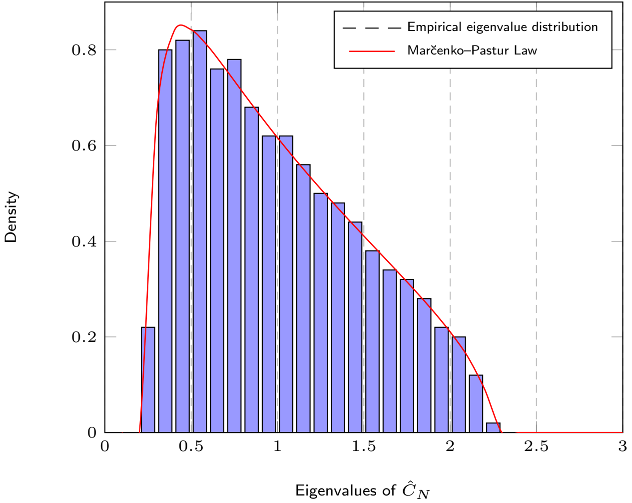

Figure: Histogram of the eigenvalues of $\hat{C}_N$ for $N = 500$, $n = 2000$, $C_N = I_N$.

Definition (Empirical Spectral Density)

Empirical spectral density (e.s.d.) μ_N of Hermitian matrix $A_N \in \mathbb{C}^{N\times N}$ is

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Theorem (Marčenko–Pastur Law **[Marčenko,Pastur'67]**)

$X_N \in \mathbb{C}^{N\times n}$ with i.i.d. zero mean, unit variance entries.
As $N, n \to \infty$ with $N/n \to c \in (0,\infty)$, e.s.d. μ_N of $\frac{1}{n}X_N X_N^*$ satisfies

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weakly, where

- $\mu_c(\{0\}) = \max\{0, 1 - c^{-1}\}$
- *on $(0,\infty)$, μ_c has continuous density f_c supported on $[(1-\sqrt{c})^2, (1+\sqrt{c})^2]$*

$$f_c(x) = \frac{1}{2\pi c x}\sqrt{(x-(1-\sqrt{c})^2)((1+\sqrt{c})^2 - x)}.$$

The Marčenko–Pastur law

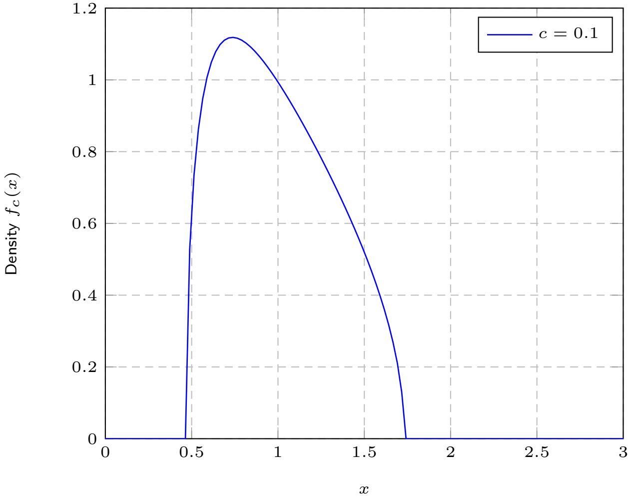
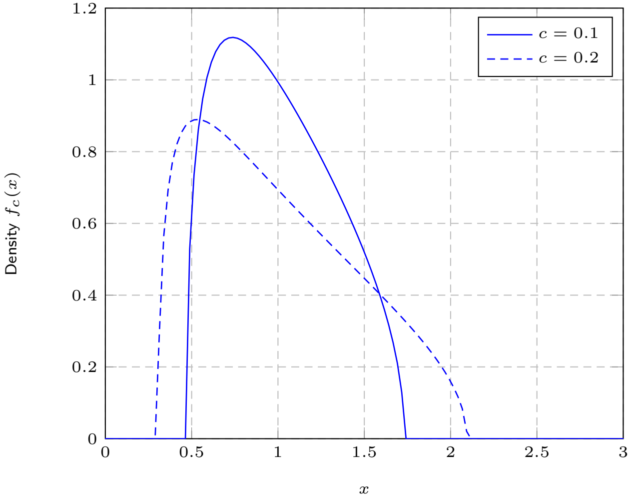


Figure: Marčenko-Pastur law for different limit ratios $c = \lim_{N\to\infty} N/n$.



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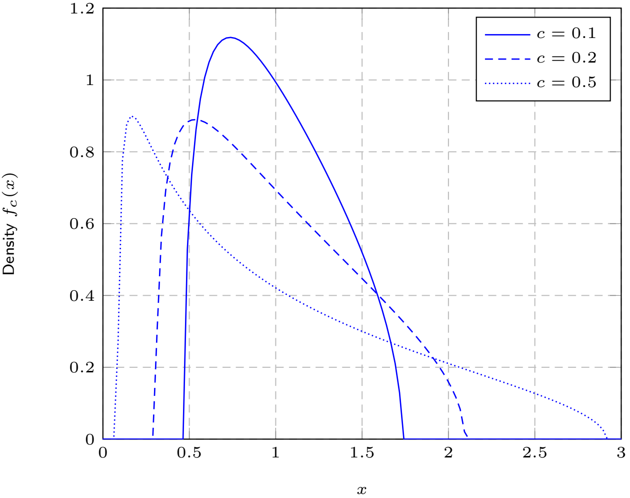

Figure: Marčenko-Pastur law for different limit ratios $c = \lim_{N\to\infty} N/n$.

In Wireless Communications.
Since 1998,

- **[Telatar'98],[Hachem et al.'07]** Mutual information $\mathcal{I}(x,y)$ of multivariate channels (MIMO, CDMA, MAC, etc.):
  $y = Hx + \sigma w$, $H \in \mathbb{C}^{N\times n}$,

$$\mathcal{I}(x,y) = \log\det\left(I_N + \sigma^{-2}HH^*\right) = \int \log(1+\sigma^{-2}t)\mu_N(dt)$$

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- **[Shamai,Verdù'01],[Wagner et al.'12]** Multi-user MIMO rate r of broadcast channels, linear receivers:
  $y_i = H_{i,\cdot}(HH^* + \alpha I_N)^{-1}Hx + \sigma w_i$

$$r = \log\left(1 + \frac{|H_{i,\cdot}(HH^*+\alpha I_N)^{-1}H_{i,\cdot}^*|^2}{\sigma^2 + \|H_{i,\cdot}(HH^*+\alpha I_N)^{-1}H_{-i,\cdot}H_{-i,\cdot}^*\|^2}\right).$$

In Signal Processing.
Since 2005 (mostly),

- **[Cardoso et al.'08],[Bianchi et al.'11]** Hypothesis tests:
  for $i = 1, \ldots, n$, $y_i = \begin{cases} \sigma w_i & , \mathcal{H}_0 \\ a + \sigma w_i & , \mathcal{H}_1 \end{cases}$,

$$\frac{\lambda_{\max}\left(\frac{1}{n}\sum_{i=1}^n y_i y_i^*\right)}{\frac{1}{N}\operatorname{tr}\frac{1}{n}\sum_{i=1}^n y_i y_i^*} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$

$$\frac{|y_n^*(\frac{1}{n-1}\sum_{i=1}^{n-1} y_i y_i^*)^{-1} a|}{\sqrt{y_n^*(\frac{1}{n-1}\sum_{i=1}^{n-1} y_i y_i^*)^{-1} y_n}\sqrt{a^*(\frac{1}{n-1}\sum_{i=1}^{n-1} y_i y_i^*)^{-1} a}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma.$$

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- **[Mestre'08],[Couillet et al.'11]** Subspace and energy estimation:
  for $i = 1, \ldots, n$, $y_i = \sum_{\ell=1}^L \sqrt{p_\ell}\, a(\theta_\ell) + \sigma w_i$,

$$f\left(\frac{1}{n}\sum_{i=1}^n y_i y_i^*\right) \triangleq \hat{p}_\ell \overset{?}{\to} p_\ell$$

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  - former (usually) involves eigenvalues of $\frac{1}{n}\sum_{i=1}^n y_i y_i^*$
  - later (usually) involves eigenspaces of $\frac{1}{n}\sum_{i=1}^n y_i y_i^*$.

Outline

Definition (Stieltjes Transform)

For μ real probability measure of support $\operatorname{supp}(\mu)$, Stieltjes transform m_μ defined, for $z \in \mathbb{C} \setminus \operatorname{supp}(\mu)$, as

$$m_\mu(z) = \int \frac{1}{t - z} \mu(dt).$$

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Property (Inverse Stieltjes Transform)

For $a < b$ continuity points of μ,

$$\mu([a, b]) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_a^b \Im[m_F(x + \imath\varepsilon)] dx$$

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Besides, if μ has a density f at x,

$$f(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \Im[m_F(x + \imath\varepsilon)].$$

Property (Relation to e.s.d.)

If μ e.s.d. of Hermitian $A \in \mathbb{C}^{N \times N}$, (i.e., $\mu = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\delta}_{\lambda_i(A)}$)

$$m_\mu(z) = \frac{1}{N} \operatorname{tr} (A - zI_N)^{-1}$$

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Proof:

$$m_\mu(z) = \int \frac{\mu(dt)}{t - z} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(A) - z} = \frac{1}{N} \mathrm{tr}\, (\mathrm{diag}\{\lambda_i(A)\} - zI_N)^{-1}$$

$$= \frac{1}{N} \mathrm{tr}\, (A - zI_N)^{-1}.$$

The Stieltjes transform

Property (Stieltjes transform and moments)

For compactly supported μ,

$$m_\mu(z) = -\sum_{k=0}^{\infty} M_{\mu,k} z^{-k-1}$$

with $M_{\mu,k} = \int t^k \mu(dt)$.

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If μ e.s.d. of A,

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Property (Stieltjes transform of Gram matrices)

For $X \in \mathbb{C}^{N \times n}$, and

- μ e.s.d. of XX^*
- $\tilde{\mu}$ e.s.d. X^*X

Then

$$m_\mu(z) = \frac{n}{N} m_{\tilde{\mu}}(z) - \frac{N-n}{N} \frac{1}{z}.$$

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Proof:

$$m_\mu(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i(XX^*) - z} = \frac{1}{N}\sum_{i=1}^{n} \frac{1}{\lambda_i(X^*X) - z} + \frac{1}{N}(N-n)\frac{1}{0-z}.$$

Side remark (for wireless communications)

Definition (Shannon Transform)

μ real probability measure with Stieltjes transform m_μ and support $\operatorname{supp}(\mu) \subset \mathbb{R}^+$, then Shannon Transform $\mathcal{V}_\mu$ is

$$\begin{aligned}\mathcal{V}_\mu(x) &= \int_0^\infty \log(1+xt)\mu(dt) \\ &= \int_x^\infty \left(\frac{1}{t} - m_\mu(-t)\right) dt.\end{aligned}$$

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- Fundamental to capacity calculus in wireless communications
- Can be computed from m_μ alone, no need to know μ.

Three fundamental lemmas in all proofs.

Lemma (Resolvent Identity)

For $A, B \in \mathbb{C}^{N \times N}$ invertible,

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

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Corollary

For $t \in \mathbb{C}$, $x \in \mathbb{C}^N$, $A \in \mathbb{C}^{N\times N}$, with A and $A + txx^*$ invertible,

$$(A + txx^*)^{-1}x = \frac{A^{-1}x}{1 + tx^*A^{-1}x}.$$

Three fundamental lemmas in all proofs.

Lemma (Rank-one perturbation)

For $A, B \in \mathbb{C}^{N\times N}$ Hermitian nonnegative definite, e.s.d. μ of A, $t > 0$, $x \in \mathbb{C}^N$, $z \in \mathbb{C} \setminus \operatorname{supp}(\mu)$,

$$\left| \frac{1}{N} \operatorname{tr} B \left(A + txx^* - zI_N\right)^{-1} - \frac{1}{N} \operatorname{tr} B \left(A - zI_N\right)^{-1} \right| \leq \frac{1}{N} \frac{\|B\|}{\operatorname{dist}(z, \operatorname{supp}(\mu))}$$

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In particular, as $N \to \infty$, if $\limsup_N \|B\| < \infty$,

$$\frac{1}{N} \operatorname{tr} B \left(A + txx^* - zI_N\right)^{-1} - \frac{1}{N} \operatorname{tr} B \left(A - zI_N\right)^{-1} \to 0.$$

Three fundamental lemmas in all proofs.

Lemma (Trace Lemma)

For

- $x \in \mathbb{C}^N$ with i.i.d. entries with zero mean, unit variance, finite eighth moment,
- $A \in \mathbb{C}^{N\times N}$ deterministic (or independent of x), $\limsup_N \|A\| = 0$ (or a.s.),

then

$$\frac{1}{N} x^* A x - \frac{1}{N} \operatorname{tr} A \xrightarrow{\text{a.s.}} 0.$$

Theorem (Marčenko–Pastur Law **[Marčenko,Pastur'67]**)

$X_N \in \mathbb{C}^{N\times n}$ *with i.i.d. zero mean, unit variance entries.*
As $N, n \to \infty$ *with* $N/n \to c \in (0,\infty)$, *e.s.d.* μ_N *of* $\frac{1}{n}X_N X_N^*$ *satisfies*

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Proof of the Marčenko–Pastur law

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- *on* $(0,\infty)$, μ_c *has continuous density* f_c *supported on* $[(1-\sqrt{c})^2, (1+\sqrt{c})^2]$

$$f_c(x) = \frac{1}{2\pi c x}\sqrt{(x-(1-\sqrt{c})^2)((1+\sqrt{c})^2 - x)}.$$

Stieltjes transform approach.

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Proof

- With μ_N e.s.d. of $\frac{1}{n}X_N X_N^*$,

$$m_{\mu_N}(z) = \frac{1}{N}\operatorname{tr}\left(\frac{1}{n}X_N X_N^* - zI_N\right)^{-1} = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{1}{n}X_N X_N^{\mathsf{H}} - zI_N\right)^{-1}\right]_{ii}.$$

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$$X_N = \begin{bmatrix} y^* \\ Y_{N-1} \end{bmatrix} \in \mathbb{C}^{N\times n}$$

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so that, for $\Im[z] > 0$,

$$\left(\frac{1}{n}X_N X_N^{\mathsf{H}} - zI_N\right)^{-1} = \begin{pmatrix} \frac{1}{n}y^*y - z & \frac{1}{n}y^*Y_{N-1} \\ \frac{1}{n}Y_{N-1}y & \frac{1}{n}Y_{N-1}Y_{N-1}^* - zI_{N-1} \end{pmatrix}^{-1}.$$

Proof (continued)

- From block matrix inverse formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(A - BD^{-1}C)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

we have

$$\left[\left(\frac{1}{n}X_N X_N^* - zI_N\right)^{-1}\right]_{11} = \frac{1}{-z - z\frac{1}{n}y^*(\frac{1}{n}Y_{N-1}^* Y_{N-1} - zI_n)^{-1}y}.$$

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- By Trace Lemma, as $N, n \to \infty$

$$\left[\left(\frac{1}{n} X_N X_N^* - z I_N \right)^{-1} \right]_{11} - \frac{1}{-z - z\frac{1}{n} \operatorname{tr} (\frac{1}{n} Y_{N-1}^* Y_{N-1} - z I_n)^{-1}} \xrightarrow{\text{a.s.}} 0.$$

Proof (continued)

- By Rank-1 Perturbation Lemma ($X_N^* X_N = Y_{N-1}^* Y_{N-1} + yy^*$), as $N, n \to \infty$

$$\left[\left(\frac{1}{n} X_N X_N^* - z I_N\right)^{-1}\right]_{11} - \frac{1}{-z - z\frac{1}{n}\operatorname{tr}\left(\frac{1}{n} X_N^* X_N - z I_n\right)^{-1}} \xrightarrow{\text{a.s.}} 0.$$

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$$\left[\left(\frac{1}{n} X_N X_N^* - zI_N\right)^{-1}\right]_{11} - \frac{1}{1 - \frac{N}{n} - z - z\frac{1}{n}\operatorname{tr}\left(\frac{1}{n}X_N X_N^* - zI_N\right)^{-1}} \xrightarrow{\text{a.s.}} 0.$$

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- Repeating for entries $(2, 2), \ldots, (N, N)$, and averaging, we get (for $\Im[z] > 0$)

$$m_{\mu_N}(z) - \frac{1}{1 - \frac{N}{n} - z - z \frac{N}{n} m_{\mu_N}(z)} \xrightarrow{\text{a.s.}} 0.$$

Proof (continued)

- Then $m_{\mu_N}(z) \xrightarrow{\text{a.s.}} m(z)$ solution to

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$$m(z) = \frac{1-c}{2cz} - \frac{1}{2c} + \frac{\sqrt{\left(z - (1+\sqrt{c})^2\right)\left(z - (1-\sqrt{c})^2\right)}}{2cz}.$$

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- Finally, by inverse Stieltjes Transform, for $x > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \Im[m(x + \imath\varepsilon)] = \frac{\sqrt{\left((1+\sqrt{c})^2 - x\right)\left(x - (1-\sqrt{c})^2\right)}}{2\pi c x} 1_{\{x \in [(1-\sqrt{c})^2, (1+\sqrt{c})^2]\}}.$$

And for $x = 0$,

$$\lim_{\varepsilon \downarrow 0} \imath\varepsilon \Im[m(\imath\varepsilon)] = \left(1 - c^{-1}\right) 1_{\{c>1\}}.$$

Theorem (Sample Covariance Matrix Model **[Silverstein,Bai'95]**)

Let $Y_N = C_N^{\frac{1}{2}} X_N \in \mathbb{C}^{N\times n}$, *with*

- $C_N \in \mathbb{C}^{N\times N}$ *nonnegative definite with e.s.d.* $\nu_N \to \nu$ *weakly,*
- $X_N \in \mathbb{C}^{N\times n}$ *has i.i.d. entries of zero mean and unit variance.*

As $N, n \to \infty$, $N/n \to c \in (0,\infty)$, $\tilde{\mu}_N$ *e.s.d. of* $\frac{1}{n} Y_N^* Y_N \in \mathbb{C}^{n\times n}$ *satisfies*

$$\tilde{\mu}_N \xrightarrow{\text{a.s.}} \tilde{\mu}$$

weakly, with $m_{\tilde{\mu}}(z)$, $\Im[z] > 0$, *unique solution with* $\Im[m_{\tilde{\mu}}(z)] > 0$ *of*

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Immediate corollary: For μ_N e.s.d. of $\frac{1}{n} Y_N Y_N^* = \frac{1}{n} \sum_{i=1}^n C_N^{\frac{1}{2}} x_i x_i^* C_N^{\frac{1}{2}}$,

$$\mu_N \xrightarrow{\text{a.s.}} \mu$$

weakly, with $\tilde{\mu} = c\mu + (1-c)\boldsymbol{\delta}_0$.

Side note on other models.
Similar results for multiple matrix models:

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- **Doubly-correlated (or separable variance profile)**: $Y_N = C_N^{\frac{1}{2}} X_N T_N^{\frac{1}{2}}$ (2 fixed point equations)
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  Applications in Section "Robust Estimation and Random Matrices"
- **Information-plus-noise**: $Y_N = A_N + X_N$, A_N deterministic
- **Variance profile**: $Y_N = P_N \odot X_N$ (entry-wise product)
- **Per-column covariance**: $Y_N = [y_1, \ldots, y_n]$, $y_i = C_{N,i}^{\frac{1}{2}} x_i$
- etc.

Retrieving μ (or $\tilde{\mu}$) from $m_{\tilde{\mu}}$.

- Since μ differentiable (unless maybe in zero), recall that

$$f(x) \triangleq \mu'(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \Im[m_\mu(x + \imath\varepsilon)].$$

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- Thus, to plot f,
  - span $z = x + \imath\varepsilon$ for all $x \in \mathbb{R}$ and ε small (say, $\varepsilon = 10^{-3}$)
  - solve $m_{\tilde{\mu}}(x + \imath\varepsilon)$ by fixed-point algorithm (provably convergent) for each x
  - plot $(x, \Im[m_{\tilde{\mu}}(x + \imath\varepsilon)])$ for each x.

Sample Covariance Matrices

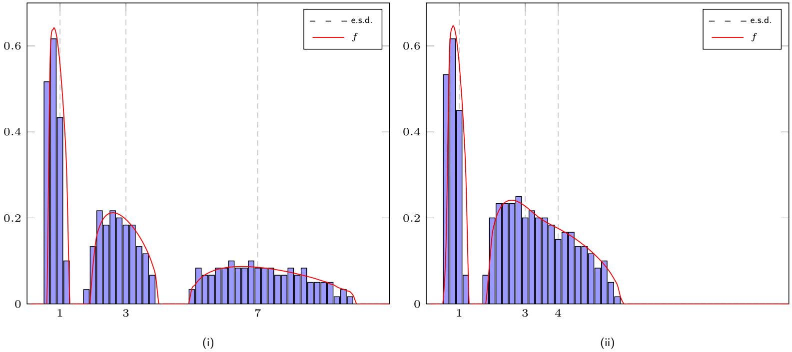

Figure: Histogram of the eigenvalues of $\frac{1}{n}Y_N Y_N^*$, $n = 3000$, $N = 300$, with C_N diagonal with evenly weighted masses in (i) 1, 3, 7, (ii) 1, 3, 4.

Property [Spectral gaps]

Let $Y_N = C_N^{\frac{1}{2}} X_N$ as above. Assume $[a^\mu, b^\mu] \subset \mathbb{R} \setminus \mathrm{supp}(\mu)$, then

$$[a^\nu, b^\nu] \subset \mathbb{R} \setminus \mathrm{supp}(\nu), \text{ where } a^\nu = -1/m_\mu(a^\mu), \; b^\nu = -1/m_\mu(b^\mu).$$

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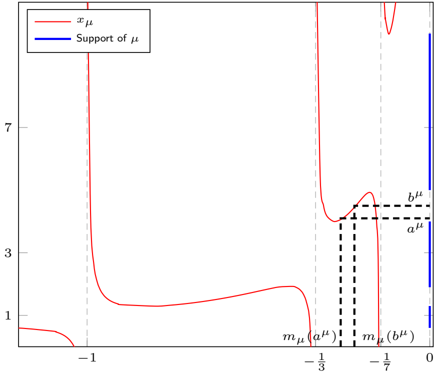

Figure: Function $x_\mu(m)$, extended inverse of $m_\mu(x)$ on real axis, for $\nu = \frac{1}{3}\boldsymbol{\delta}_1 + \frac{1}{3}\boldsymbol{\delta}_3 + \frac{1}{3}\boldsymbol{\delta}_7$.

Theorem (Exact Separation [Bai,Silverstein'99])

Let $Y_N = C_N^{\frac{1}{2}} X_N$ *as above with additionally*

- X_N *has entries of bounded fourth order moment*
- $\max_{1 \leq i \leq n} \operatorname{dist}(\lambda_i(C_N), \operatorname{supp}(\nu)) \to 0$
- $\limsup_N \|C_N\| < \infty$.

Then, letting $[a^\mu, b^\mu] \subset \mathbb{R}^+ \setminus \operatorname{supp}(\mu)$ *with corresponding* $[a^\nu, b^\nu] \subset \mathbb{R}^+ \setminus \operatorname{supp}(\nu)$,

$$\#\left\{\lambda_i\left(\frac{1}{n} Y_N Y_N^*\right) < a^\mu\right\} = \#\left\{\lambda_i(C_N) < a^\nu\right\}$$

for all large n *a.s., except for zero eigenvalues.*

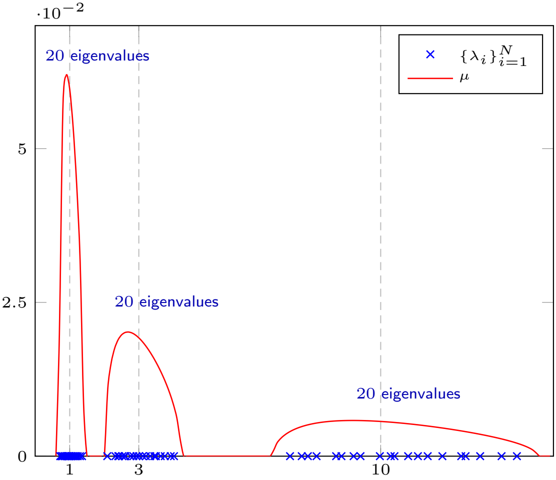

Figure: Eigenvalues of $\frac{1}{n}\mathbf{Y}_N\mathbf{Y}_N^{\mathsf{H}}$ versus μ, $\nu = \frac{1}{3}\boldsymbol{\delta}_1 + \frac{1}{3}\boldsymbol{\delta}_3 + \frac{1}{3}\boldsymbol{\delta}_{10}$, $N = 60$, $n = 600$.

Estimation of C_N.

- For f complex analytic, estimate of $\int f(t)\nu(dt)$?

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- Reminder: by **[Silverstein,Bai'95]**,

$$m_{\tilde{\mu}}(\omega) = \left(-\omega + c \int \frac{t}{1 + t m_{\tilde{\mu}}(\omega)} \nu(dt) \right)^{-1}$$

or equivalently

$$m_\nu \left(-\frac{1}{m_{\tilde{\mu}}(\omega)} \right) = -\omega m_\mu(\omega) m_{\tilde{\mu}}(\omega).$$

Estimation of C_N.

- Together, with $z = -1/m_{\tilde{\mu}}(\omega)$,

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- By uniform convergence over $\mathcal{C}_\mu$

$$\sup_{z\in\mathcal{C}_\mu} |m_\mu(z) - m_{\mu_N}(z)| \xrightarrow{\text{a.s.}} 0$$

  ($\mu_N = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i(\frac{1}{n}Y_N Y_N^*)}$), we get

$$-\frac{1}{2\pi\imath}\oint_{\mathcal{C}_\mu} f\left(-\frac{1}{m_{\tilde{\mu}_N}(\omega)}\right)\omega m_{\mu_N}(\omega)\frac{m'_{\tilde{\mu}_N}(\omega)}{m_{\tilde{\mu}_N}(\omega)}d\omega \xrightarrow{\text{a.s.}} \int f(t)\nu(dt).$$

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- Since $m_{\tilde{\mu}_N}(\omega) = \frac{1}{N}\sum_{i=1}^N(\lambda_i - z)^{-1}$, computation possible via
  - numerical integrals
  - residue calculus

Estimation of C_N for atomic ν (i.e., $f(t) = t$).

Theorem (Eigen-Inference **[Mestre'08]**)

Let $Y_N = C_N^{\frac{1}{2}} X_N$ with exact separation condition. Assume

$$\nu_N = \sum_{k=1}^{K} \frac{N_k}{N} \boldsymbol{\delta}_{t_k}, \text{ with } \sum_{k=1}^{K} N_k = N, \; N_k/N \to c_k$$

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and there is a cluster in μ that maps $\{t_j\}$ only. Let

$$\hat{t}_j = \frac{n}{N_j} \sum_{m \in \mathcal{N}_j} (\lambda_m - \kappa_m)$$

- $\mathcal{N}_j = \{\sum_{i=1}^{j-1} N_i + 1, \ldots, \sum_{i=1}^{j} N_i\}$
- $\lambda_1 \geq \ldots \geq \lambda_N$ *eigenvalues of* $\frac{1}{n} Y_N Y_N^*$
- $\kappa_1 \geq \ldots \geq \kappa_N$ *eigenvalues of diag*$(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^*$.

Then, as $N, n \to \infty$, $N/n \to c \in (0, \infty)$,

$$\hat{t}_j \xrightarrow{\text{a.s.}} t_j.$$

Outline

Reminder. According to **[Bai,Sil'98]**, asymptotically no eigenvalue of $\frac{1}{n}Y_N Y_N^*$ outside $\mathrm{supp}(\mu)$ if

1. $E[|X_{ij}|^4] < \infty$
2. $\max_{1\leq i\leq n} \mathrm{dist}(\lambda_i(C_N), \mathrm{supp}(\nu)) \to 0$
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Breaking the rules. If we break

- **Rule 1**: Infinitely many eigenvalues may wander away from $\mathrm{supp}(\mu)$.

Spiked Models

Reminder. According to **[Bai,Sil'98]**, asymptotically no eigenvalue of $\frac{1}{n}Y_NY_N^*$ outside $\operatorname{supp}(\mu)$ if

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2. $\max_{1\leq i\leq n} \operatorname{dist}(\lambda_i(C_N), \operatorname{supp}(\nu)) \to 0$
3. $\limsup_N \|C_N\| < \infty$.

Breaking the rules. If we break

- **Rule 1**: Infinitely many eigenvalues may wander away from $\operatorname{supp}(\mu)$.
- **Rule 2**: C_N may create isolated eigenvalues in $\frac{1}{n}Y_NY_N^*$, called spikes.

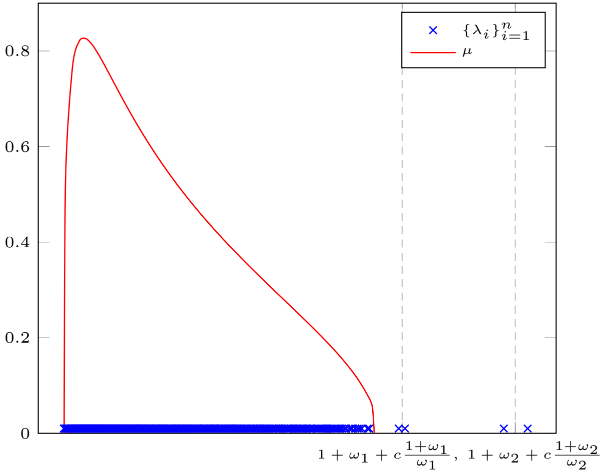

Figure: Eigenvalues of $\frac{1}{n}Y_N Y_N^*$, $\nu_N = \frac{N-4}{N}\boldsymbol{\delta}_1 + \frac{2}{N}\boldsymbol{\delta}_2 + \frac{2}{N}\boldsymbol{\delta}_3$ (hence $\nu = \boldsymbol{\delta}_1$), $N = 500$, $n = 1500$.

Theorem (Eigenvalues [Baik,Silverstein'06])

Let $Y_N = C_N^{\frac{1}{2}} X_N$*, with*

- X_N *with i.i.d. zero mean, unit variance, finite fourth order moment entries*
- $\nu_N = \frac{N - \sum_{m=1}^M k_m}{N} \boldsymbol{\delta}_1 + \sum_{m=1}^M \frac{k_m}{N} \boldsymbol{\delta}_{1+\omega_m}$*, with* $\omega_1 > \ldots > \omega_M$*.*

Theorem (Eigenvalues **[Baik,Silverstein'06]**)

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$$I_N - C_N^{-1} = I_N - (I_N + U\Omega U^*)^{-1} = U(I_K + \Omega^{-1})^{-1}U^*,\ \Omega \in \mathbb{C}^{K\times K},\ K = \sum k_m.$$

Hence

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- As a result, for all large n a.s.,

$$\begin{aligned} 0 &= \det\left(I_K + (I_K + \Omega^{-1})^{-1}U^*(\frac{1}{n}X_N X_N^* - \lambda I_N)^{-1}U\right) \\ &\simeq \prod_{m=1}^{M}\left(1 + \frac{1}{1+\omega_m^{-1}}m_\mu(\lambda)\right)^{k_m} \end{aligned}$$

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- Using Marčenko–Pastur law properties $(m_\mu(z) = (1 - c - z - czm_\mu(z))^{-1})$,

$$\lambda \in \left\{ 1 + \omega_m + c\frac{1+\omega_m}{\omega_m} \right\}_{m=1}^{M}.$$

Theorem (Eigenvectors **[Paul'07]**)

Let $Y_N = C_N^{\frac{1}{2}} X_N$, *with*

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Proof: *Based on Cauchy integral + similar ingredients as eigenvalue proof*

$$a^* U_m U_m^* b = \frac{1}{2\pi\imath} \oint_{\mathcal{C}_m} a^* \left(\frac{1}{n} Y_N Y_N^* - z I_N \right)^{-1} b dz$$

for $\mathcal{C}_m$ *contour circling around* $\lambda_{k_1+\ldots+k_{m-1}+1}, \ldots, \lambda_{k_1+\ldots+k_m}$ *only.*

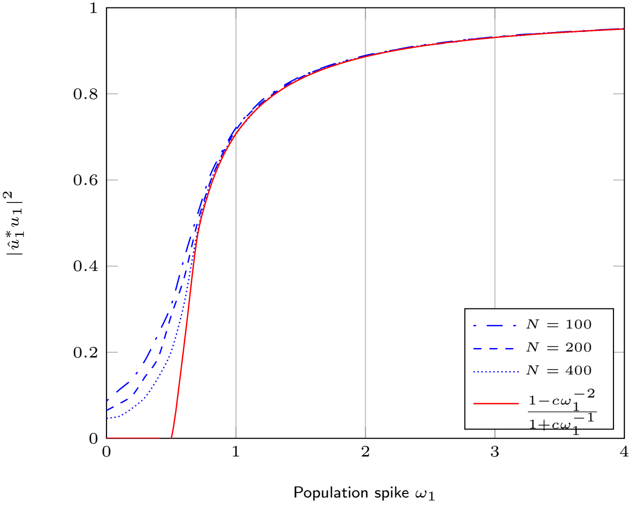

Figure: Simulated versus limiting $|\hat{u}_1^* u_1|^2$ for $Y_N = C_N^{\frac{1}{2}} X_N$, $C_N = I_N + \omega_1 u_1 u_1^*$, $N/n = 1/3$, varying ω_1.

Outline

Theorem (Fluctuations of Linear Statistics **[Bai,Silverstein'04]**)

Let $Y_N = C_N^{\frac{1}{2}} X_N$, X_N *with i.i.d. complex Gaussian zero mean, unit variance entries. Assume* $N(\nu_N - \nu) \xrightarrow{\mathcal{L}} 0$ *and let* $\Delta_{\mu,N} = N(\mu_N - \mu)$*. Then, as* $N/n \to \infty$*,* $N/n = c + o(N^{-1})$*, for* $f_1, \ldots, f_k$ *analytic,*

$$\left(\int f_1(x) \Delta_{\mu,N}(dx), \ldots, \int f_k(x) \Delta_{\mu,N}(dx) \right) \xrightarrow{\mathcal{L}} (X_{f_1}, \ldots, X_{f_k})$$

Gaussian vector with zero mean and covariance matrix

$$\mathrm{Cov}(X_f, X_g) = -\frac{1}{4\pi^2} \oint \oint \frac{f(z_1) g(z_2)}{(m_{\tilde{\mu}}(z_1) - m_{\tilde{\mu}}(z_2))^2} m'_{\tilde{\mu}}(z_1) m'_{\tilde{\mu}}(z_2) dz_1 dz_2.$$

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Several generalizations:

- if $N(\nu_N - \nu) \not\to 0$, valid with $\Delta_{\mu,N} = N(\mu_N - \bar{\mu}_N)$, $\bar{\mu}_N$ deterministic equivalent.

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Theorem (Phase Transition **[Baik,BenArous,Péché'05]**)

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Tracy–Widom Theorem

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- *If* $\omega_1 > \sqrt{c}$,

$$\left(\frac{(1+\omega_1)^2}{c} - \frac{(1+\omega_1)^2}{\omega_1^2} \right)^{\frac{1}{2}} N^{\frac{1}{2}} \left[\lambda_1 - \left(1 + \omega_1 + c \frac{1+\omega_1}{\omega_1} \right) \right] \xrightarrow{\mathcal{L}} G_k$$

with G_k *law of largest eigenvalue of the* $k \times k$ *GUE matrix. In particular,* $G_1(x)$ *real Gaussian distribution function.*

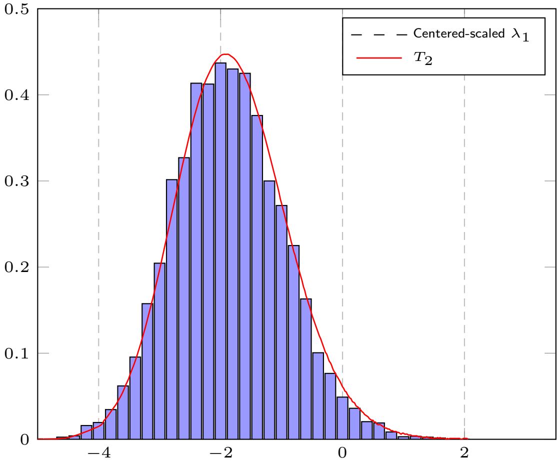

Figure: Distribution of $N^{\frac{2}{3}} c^{-\frac{1}{2}} (1+\sqrt{c})^{-\frac{4}{3}} \left[\lambda_1(\frac{1}{n} X_N X_N^*) - (1+\sqrt{c})^2\right]$ versus Tracy–Widom (T_2), $N = 500$, $n = 1500$.

Outline

Source Detection and Enumeration

Context. Observations $y_1, \ldots, y_n \in \mathbb{C}^N$, independent with

$$y_i = \begin{cases} \sigma w_i & , \mathcal{H}_0 \\ h s_i + \sigma w_i & , \mathcal{H}_1 \end{cases}$$

with $\sigma > 0$ unknown, $s_i \in \mathbb{C}$ random, $h \in \mathbb{C}^N$ unknown, $w_i \sim \mathcal{CN}(0, I_N)$.

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GLRT procedure. [Bianchi,Najim,Debbah,Maida'10]

- Hypothesis test for $Y_N = [y_1, \ldots, y_n]$

$$\frac{\sup_{h,s,\sigma} p(Y_N | h, s, \sigma)}{\sup_\sigma p(Y_N | \sigma)} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\lessgtr}} \Gamma.$$

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- Provably equivalent to hypothesis test (with $\lambda_i = \lambda_i(\frac{1}{n} Y_N Y_N^*)$)

$$T(Y_N) \triangleq \frac{\lambda_1}{\frac{1}{N}\sum_{i=1}^N \lambda_i} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\lessgtr}} \gamma$$

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- Setting false alarm rate to η implies $\gamma \geq T_2^{-1}(\eta)$, i.e., test

$$T(Y_N) \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\lessgtr}} (1+\sqrt{c})^2 + N^{-\frac{2}{3}} T_2^{-1}(\eta)(1+\sqrt{c})^{\frac{4}{3}} c^{\frac{1}{2}}.$$

Power Estimation

Context. Estimation of $p_1, \ldots, p_k$ from $y_1, \ldots, y_n$ independent with

$$y_i = \sum_{k=1}^{K} \sqrt{p_k} H_k s_{k,i} + \sigma w_i$$

$H_k \in \mathbb{C}^{N \times N_k}$ i.i.d. zero mean, variance $1/N$, $s_{k,i}$ i.i.d. zero mean, unit variance.

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$$p_m \xleftarrow{\text{a.s.}} \hat{p}_m = \frac{1}{N_m} \sum_{i \in \mathcal{N}_m} \frac{\lambda_i - (c+1)}{2} + \sqrt{(c+1-\lambda_i)^2 - 4c}.$$

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- **Large N_k regime**: $N_k/N \to c_k > 0$, use contour integration method

$$p_m \xleftarrow{\text{a.s.}} \hat{p}_m = \begin{cases} \frac{Nn}{N_m(n-N)} \sum_{i \in \mathcal{N}_m} (\eta_i - \kappa_i) & , n \neq N \\ \frac{N}{N_m(N - \sum_i N_i)} \sum_{i \in \mathcal{N}_m} \left(\sum_{j=1}^{N} \frac{\eta_i}{(\lambda_j - \eta_i)^2} \right)^{-1} & , n = N \end{cases}$$

(under cluster m separability condition) with

- $\eta_1 \geq \ldots \geq \eta_N$ eigenvalues of $\operatorname{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^*$
- $\kappa_1 \geq \ldots \geq \kappa_N$ eigenvalues of $\operatorname{diag}(\boldsymbol{\lambda}) - \frac{1}{n}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^*$

(details in **[Couillet,Silverstein,Bai,Debbah'11]**).

Power Estimation

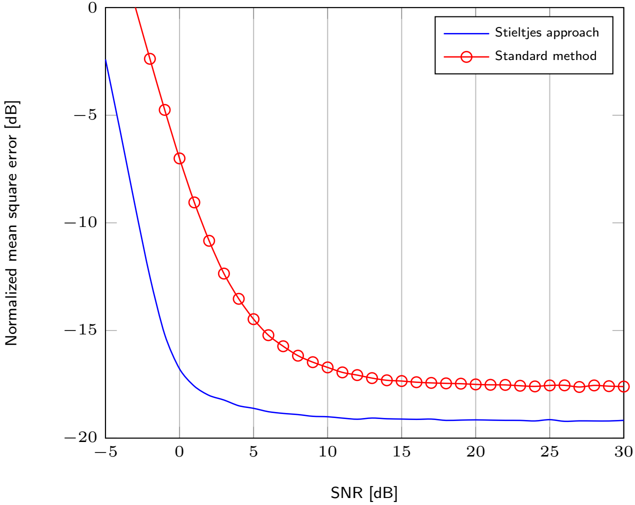

Figure: Estimate NMSE for p_1 (**large N_k regime**), three sources, $p_1 = 1$, $p_2 = 1/4$, $p_3 = 1/16$, $N_1 = N_2 = N_3 = 4$,$N = 24$, $n = 128$. Comparison between standard statistics (assumes $n \gg N \gg N_k$) and Stieltjes transform approach.

Power Estimation

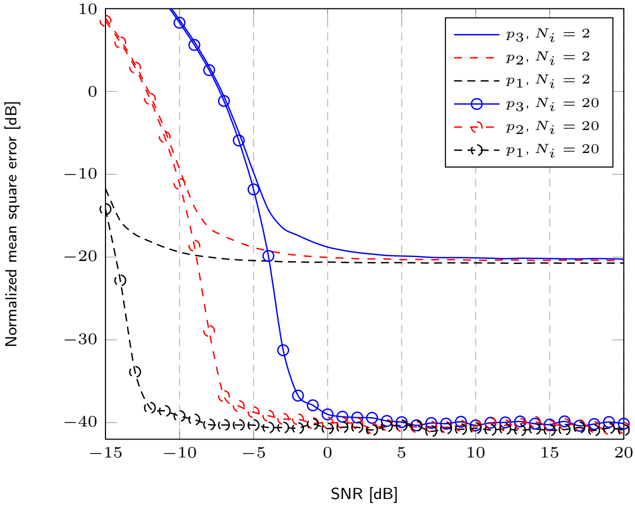

Figure: Estimate NMSE (**large N_k regime**), $p_1 = 10$, $p_2 = 3$, $p_3 = 1$, $N_1 = N_2 = N_3$,$\sum N_k/N = N/n = 1/10$.

Context. Estimate $\theta_1, \ldots, \theta_M$ from $y_1, \ldots, y_n \in \mathbb{C}^N$ independent,

$$y_i = \sum_{m=1}^{M} \sqrt{p_m} a(\theta_m) s_{m,i} + \sigma w_i$$

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- **Spiked (G)-MUSIC**: by $a(\theta)^* \hat{u}_k \hat{u}_k^* a(\theta) \xrightarrow{\text{a.s.}} \frac{1 - c\hat{p}_k^{-2}}{1 + c\hat{p}_k^{-1}} a(\theta)^* u_k u_k^* a(\theta)$,

$$\hat{\theta}_1, \ldots, \hat{\theta}_M = \text{ deep minima of } \left\{ 1 - \sum_{k=M+1}^{N} \frac{1 + c\hat{p}_k^{-1}}{1 - c\hat{p}_k^{-2}} a(\theta)^* \hat{u}_k \hat{u}_k^* a(\theta) \right\}.$$

(details in **[Mestre,Lagunas'08],[Hachem et al.'13]**).

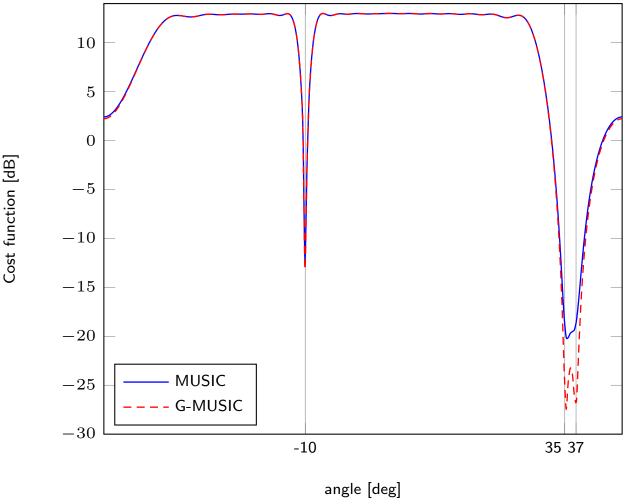
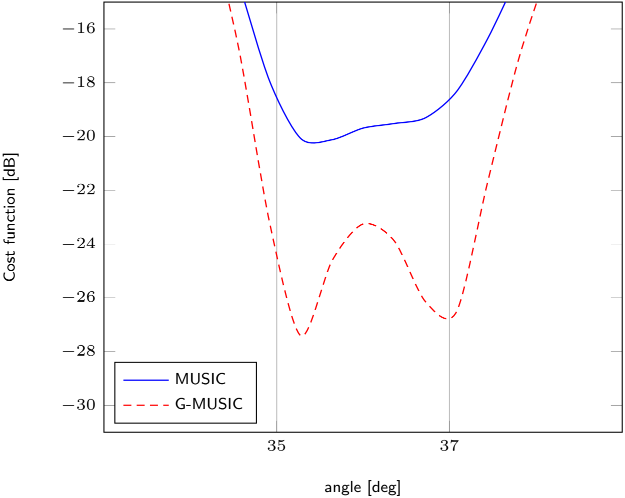


Figure: MUSIC versus G-MUSIC, $M = 3$ sources, $N = 20$, $n = 150$, $\sigma^2 = 0.1$. Angles $\theta_1 = 10^\circ$, $\theta_2 = 35^\circ$, $\theta_3 = 37^\circ$.



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Other SP applications

- **Covariance Matrix Estimation**. Inconsistent in large N, n regime, but
  - **Linear shrinkage**: optimize ρ in estimate $\hat{C}_N = (1-\rho)\frac{1}{n}Y_N Y_N^* + \rho I_N$
  - **Non-linear shrinkage**: optimize $\hat{C}_N = U f(\Lambda) U^*$, with $\frac{1}{n}Y_N Y_N^* = U \Lambda U^*$.

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- **Improved (sparse) PCA**. PCA on $\frac{1}{n}Y_{\mathcal{I}}Y_{\mathcal{I}}^*$, $\mathcal{I} \subset \{1,\ldots,N\}$, $Y_{\mathcal{I}} \in \mathbb{C}^{|\mathcal{I}|\times N}$ such that

$$\left|(\hat{u}_{1,\mathcal{I}}^e)^* u_1\right|^2 \text{ maximum}$$

with $\hat{u}_{1,\mathcal{I}}^e \in \mathbb{C}^N$ PCA vector of $\frac{1}{n}Y_{\mathcal{I}}Y_{\mathcal{I}}^*$ extended with zeros.

- **Toepitz covariance matrices**: Toeplitzification $\mathcal{T}(\frac{1}{n}Y_N Y_N^*)$ consistent in large N, n regime,

$$\left\| \mathcal{T}\left(\frac{1}{n}Y_N Y_N^*\right) - C_N \right\| \xrightarrow{\text{a.s.}} 0$$

(with $[\mathcal{T}(X)]_{ij} = \frac{1}{N}\sum_k [X]_{k,k+(i-j)}$). Many results beyond RMT regime **[Bickel,Levina'08],[Wu,Pouramadi'09],[Vinogradova,Couillet,Hachem'14]**.

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- **Kernel matrices**: used in BSS, spectral clustering, etc.

$$K_{i,j} = k(x_i, x_j) = \begin{cases} x_i^* x_j & \text{, sample covariance matrix} \\ |x_i^* x_j|^2 & \text{, kurtosis-based BSS} \\ \exp(-\|x_i - x_j\|^2) & \text{, Gaussian kernel.} \end{cases}$$

Few results **[ElKaroui'10]**.

Outline

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Baseline scenario: $x_1, \ldots, x_n \in \mathbb{C}^N$ (or $\mathbb{R}^N$) i.i.d. with $E[x_1] = 0$, $E[x_1 x_1^*] = C_N$:

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- **[Pascal'13; Chen'11]** If $N > n$, x_1 elliptical or with outliers, shrinkage extensions

$$\hat{C}_N(\rho) = (1-\rho) \frac{1}{n} \sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N^{-1}(\rho) x_i} + \rho I_N$$

$$\check{C}_N(\rho) = \frac{\check{B}_N(\rho)}{\frac{1}{N} \operatorname{tr} \check{B}_N(\rho)}, \quad \check{B}_N(\rho) = (1-\rho) \frac{1}{n} \sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \check{C}_N^{-1}(\rho) x_i} + \rho I_N$$

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- Application interest:
  - comparison between SCM and robust estimators
  - performance of robust/non-robust estimation methods
  - improvement thereof (by proper parametrization)

Outline of Theoretical Content

- First order convergence:

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- Applications:
  - improved robust covariance matrix estimation
  - improved robust tests / estimators
  - specific examples in statistics at large, array processing, statistical finance, etc.

Outline

Definition (Maronna's Estimator)

For $x_1, \ldots, x_n \in \mathbb{C}^N$ with $n > N$, $\hat{C}_N$ is the solution (upon existence and uniqueness) of

$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^{n} u\left(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

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where $u : [0, \infty) \to (0, \infty)$ is

- non-increasing
- such that $\phi(x) \triangleq xu(x)$ increasing of supremum ϕ_∞ with

$$1 < \phi_\infty < c^{-1}, \; c \in (0, 1).$$

Model Description

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Remark (Correlation Invariance)

For some $C_N \succ 0$, calling $\tilde{x}_i \triangleq C_N^{-\frac{1}{2}} x_i$, $\tilde{C}_N \triangleq C_N^{-\frac{1}{2}} \hat{C}_N C_N^{-\frac{1}{2}}$,

$$\tilde{C}_N = \frac{1}{n} \sum_{i=1}^{n} u \left(\frac{1}{N} \tilde{x}_i^* \tilde{C}_N^{-1} \tilde{x}_i \right) \tilde{x}_i \tilde{x}_i^*$$

If $E[x_i x_i^*] = C_N$, sufficient to assume $E[x_i x_i^*] = I_N$.

Assumption ("Elliptical" Data)

$x_1, \ldots, x_n$ independent,

$$x_i = \sqrt{\tau_i} C_N^{\frac{1}{2}} w_i$$

- $w_i \in \mathbb{C}^N$ isotropic, $\|w_i\|^2 = N$
- $C_N \succ 0$, $\limsup_N \|C_N\| < \infty$
- $\tau_i > 0$ deterministic (or random independent of w_i)

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- $\tau_i > 0$ deterministic (or random independent of w_i)
- for $\tilde{\nu}_n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i}$ and some $m > 0$,

$$\tilde{\nu}_n([0, m)) < 1 - \phi_\infty^{-1} \text{ for all large } n \text{ (a.s.)}$$

- $\int \tau \tilde{\nu}_n(d\tau) \to 1$ (a.s.).

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Fact (Existence and Uniqueness)

By **[Kent&Tyler'91]**, for each $n > N$, $\hat{C}_N$ is a.s. well-defined.

Assumption (Tail Control)

For each $a > b > 0$,

$$\frac{\limsup_n \tilde{\nu}_n([t, \infty))}{\phi(at) - \phi(bt)} \to 0$$

as $t \to \infty$.

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Example: If $u(x) = \frac{\alpha+1}{\alpha+x}$, τ_i i.i.d., sufficient to have $E[\tau_1^{1+\varepsilon}] < \infty$.

Assumption (Random Matrix Regime)

As $n \to \infty$,

$$c_N \triangleq \frac{N}{n} \to c \in (0,1).$$

Definition (v and ψ)

Letting $g(x) = x(1 - c\phi(x))^{-1}$ (on $\mathbb{R}_+$),

$$v(x) \triangleq (u \circ g^{-1})(x) \quad \text{non-increasing}$$
$$\psi(x) \triangleq xv(x) \quad \text{increasing and bounded by } \psi_\infty.$$

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Lemma (Rewriting $\hat{C}_N$)

It holds (with $C_N = I_N$) that

$$\hat{C}_N \triangleq \frac{1}{n}\sum_{i=1}^{n} \tau_i v\left(\tau_i d_i\right) w_i w_i^*$$

with $(d_1, \ldots, d_n) \in \mathbb{R}_+^n$ a.s. unique solution to

$$d_i = \frac{1}{N} w_i^* \hat{C}_{(i)}^{-1} w_i = \frac{1}{N} w_i^* \left(\frac{1}{n} \sum_{j \neq i} \tau_j v(\tau_j d_j) w_j w_j^* \right)^{-1} w_i, \; i = 1, \ldots, n.$$

Remark (Quadratic Form close to Trace)

Random matrix insight: $(\frac{1}{n}\sum_{j\neq i}\tau_j v(\tau_j d_j) w_j w_j^*)^{-1}$ "almost independent" of w_i, so

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for some deterministic sequence $(\gamma_N)_{N=1}^{\infty}$, irrespective of i.

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Lemma (Key Lemma)

Letting $e_i \triangleq \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N)}$ with γ_N unique solution to

$$1 = \int \frac{\psi(\tau\gamma_N)}{1 + c\psi(\tau\gamma_N)} \tilde{\nu}_n(d\tau),$$

we have

$$\max_{1\leq i\leq n} |e_i - 1| \xrightarrow{\text{a.s.}} 0.$$

(Proof in a few slides.)

Theorem (Large dimensional behavior [C,Pascal,Silverstein'13])

With the notations and assumptions above,

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

with

$$\hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^{n} v(\tau_i \gamma_N) x_i x_i^*.$$

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$$\left[\textit{equivalently,} \quad \hat{S}_N = \gamma_N^{-1} \frac{1}{n} \sum_{i=1}^{n} \psi(\tau_i \gamma_N) C_N^{\frac{1}{2}} w_i w_i^* C_N^{\frac{1}{2}} \right]$$

Corollaries

- **Spectral measure**: $\mu_N^{\hat{C}_N} - \mu_N^{\hat{S}_N} \xrightarrow{\mathcal{L}} 0$ a.s. ($\mu_N^X \triangleq \frac{1}{n} \sum_{i=1}^n \boldsymbol{\delta}_{\lambda_i(X)}$)

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- **Local convergence**: $\max_{1 \leq i \leq N} |\lambda_i(\hat{C}_N) - \lambda_i(\hat{S}_N)| \xrightarrow{\text{a.s.}} 0.$

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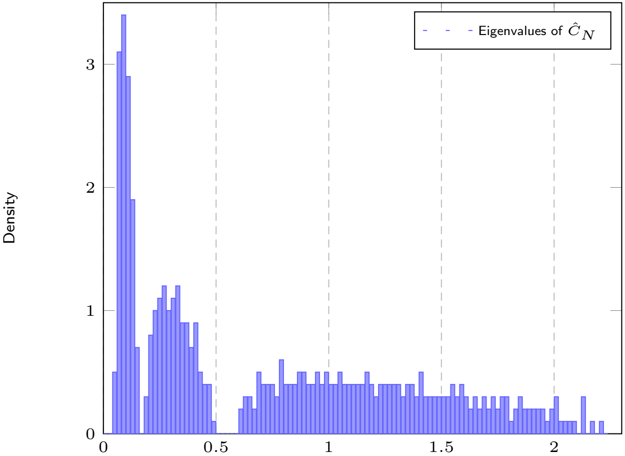
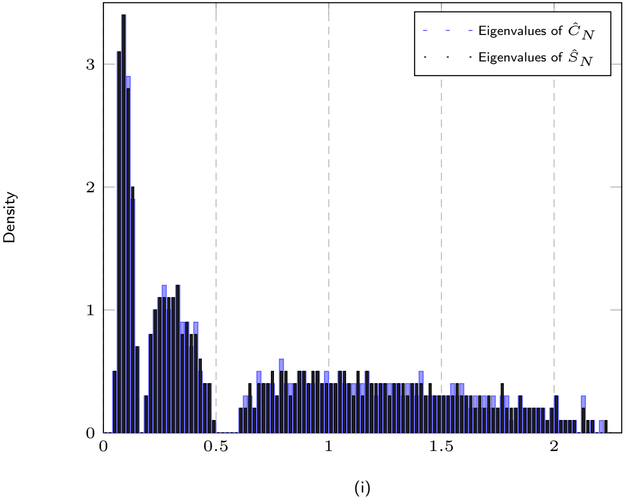
- **Local convergence**: $\max_{1 \leq i \leq N} |\lambda_i(\hat{C}_N) - \lambda_i(\hat{S}_N)| \xrightarrow{\text{a.s.}} 0.$
- **Norm boundedness**: $\limsup_N \|\hat{C}_N\| < \infty$

$\longrightarrow$ Bounded spectrum (unlike SCM!)



(i)

Figure: $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, $\tau_i \sim \Gamma(.5, 2)$ i.i.d.



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Large dimensional behavior

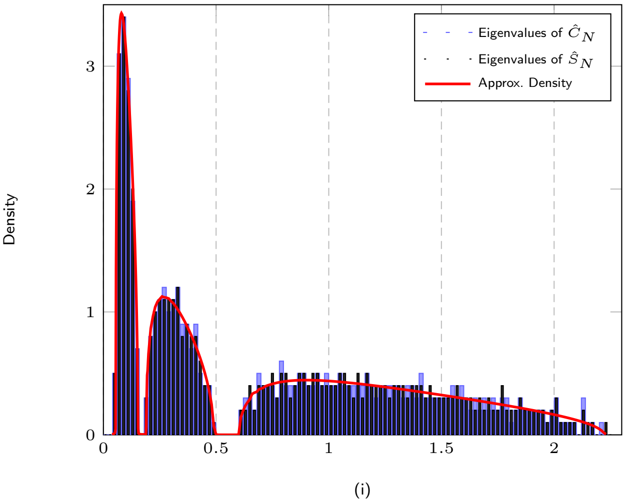

Figure: $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, $\tau_i \sim \Gamma(.5, 2)$ i.i.d.

Proof of the Key Lemma: $\max_i |e_i - 1| \xrightarrow{\text{a.s.}} 0$, $e_i = \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N)}$

Property (Quadratic form and γ_N)

$$\max_{1 \le i \le n} \left| \frac{1}{N} w_i^* \left(\frac{1}{n} \sum_{j \neq i} \tau_j v(\tau_j \gamma_N) w_j w_j^* \right)^{-1} w_i - \gamma_N \right| \xrightarrow{\text{a.s.}} 0.$$

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Proof of the Property

- Uniformity easy (moments of all orders for $[w_i]_j$).
- By a "quadratic form similar to trace" approach, we get

$$\max_{1\leq i\leq n}\left|\frac{1}{N}w_i^*\left(\frac{1}{n}\sum_{j\neq i}\tau_j v(\tau_j\gamma_N)w_jw_j^*\right)^{-1}w_i - m(0)\right| \xrightarrow{\text{a.s.}} 0$$

  with $m(0)$ unique positive solution to **[MarPas'67;SilBai'95]**

$$m(0) = \int \frac{\tau v(\tau\gamma_N)}{1 + c\tau v(\tau\gamma_N)m(0)}\tilde{\nu}_n(d\tau).$$

- γ_N precisely solves this equation, thus $m(0) = \gamma_N$.

Proof of the Key Lemma: $\max_i |e_i - 1| \xrightarrow{\text{a.s.}} 0$, $e_i = \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N)}$

Substitution Trick (case $\tau_i \in [a, b] \subset (0, \infty)$)

Up to relabelling $e_1 \leq \ldots \leq e_n$, use

$$
\begin{aligned}
v(\tau_n \gamma_N) e_n = v(\tau_n d_n) &= v\left(\tau_n \frac{1}{N} w_n^* \left(\frac{1}{n} \sum_{i<n} \tau_i \underbrace{v(\tau_i d_i)}_{= v(\tau_i \gamma_N) e_i} w_i w_i^* \right)^{-1} w_n \right) \\
&\leq v\left(\tau_n e_n^{-1} \frac{1}{N} w_n^* \left(\frac{1}{n} \sum_{i<n} \tau_i v(\tau_i \gamma_N) w_i w_i^* \right)^{-1} w_n \right) \\
&\leq v\left(\tau_n e_n^{-1} (\gamma_N - \varepsilon_n) \right) \text{ a.s., } \varepsilon_n \to 0 \text{ (slow).}
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$$\leq v\left(\tau_n e_n^{-1}(\gamma_N - \varepsilon_n)\right) \text{ a.s., } \varepsilon_n \to 0 \text{ (slow).}$$

Use properties of ψ to get

$$\psi\left(\tau_n\gamma_N\right) \leq \psi\left(\tau_n e_n^{-1}\gamma_N\right)\left(1 - \varepsilon_n\gamma_N^{-1}\right)^{-1}$$

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Use properties of ψ to get

$$\psi\left(\tau_n \gamma_N\right) \leq \psi\left(\tau_n e_n^{-1} \gamma_N\right)\left(1 - \varepsilon_n \gamma_N^{-1}\right)^{-1}$$

Conclusion: If $e_n > 1 + \ell$ i.o., as $\tau_n \in [a,b]$, on subsequence $\begin{cases} \tau_n \to \tau_0 > 0 \\ \gamma_N \to \gamma_0 > 0 \end{cases}$,

$$\psi(\tau_0 \gamma_0) \leq \psi\left(\frac{\tau_0 \gamma_0}{1+\ell}\right), \text{ a contradiction.}$$

General τ_i case

- Control of

$$\Delta_M = \frac{1}{N} w_i^* \left(\frac{1}{n} \sum_{j \neq i} \tau_j v(\tau_j d_j) w_j w_j^* \right)^{-1} w_i - \frac{1}{N} w_i^* \left(\frac{1}{n} \sum_{\substack{j \neq i \\ \tau_j \leq M}} \tau_j v(\tau_j d_j) w_j w_j^* \right)^{-1} w_i.$$

- Rationale: Large M bring small Δ_M but (possibly) large τ_n
  $\longrightarrow$ Relative control between tail of $\tilde{\nu}_n$ and flattening of ψ.

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This concludes the proof.

Assumption (Signal Model)

$x_1, \ldots, x_n$ independent,

$$x_i = \sum_{l=1}^{L} \sqrt{p_l} a_l s_{li} + \sqrt{\tau_i} w_i$$

- $w_i \in \mathbb{C}^N$, τ_i as previously, (for simplicity) $\tilde{\nu}_n \to \tilde{\nu}$
- $s_{li} \in \mathbb{C}$ i.i.d., mean 0, variance 1
- $p_1 \geq \ldots \geq p_L \geq 0$
- $a_1, \ldots, a_L \in \mathbb{C}^N$ deterministic with $\sum_{l=1}^{L} p_l a_l a_l^* \to \mathrm{diag}(p_i)_{i=1}^{L}$.

Spiked Model Extension

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Theorem (Extension of pure-noise model [C'2014])

As $n \to \infty$, under previous assumptions,

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^{n} v(\tau_i \gamma) x_i x_i^*.$$

(same result but different model, $\gamma = \lim_N \gamma_N$)

$\longrightarrow \hat{S}_N$ follows a spiked random matrix model.

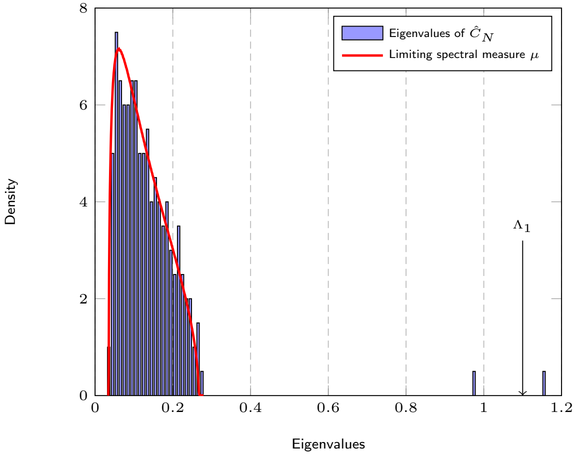
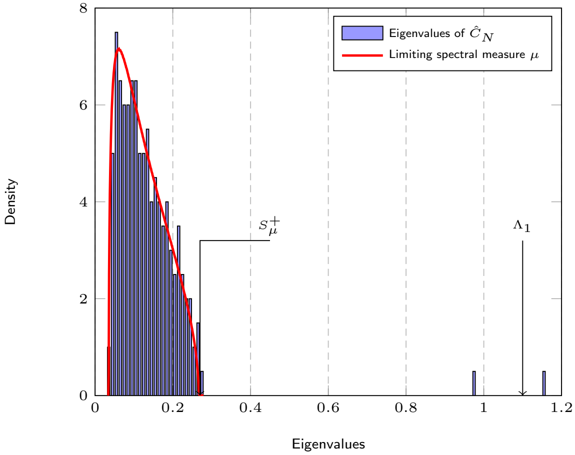


Figure: Eigenvalues of $\hat{C}_N$, $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Student-t impulsions.

Spiked Model Extension

$\longrightarrow$ But eigenvalues allowed to wander away from limiting support.



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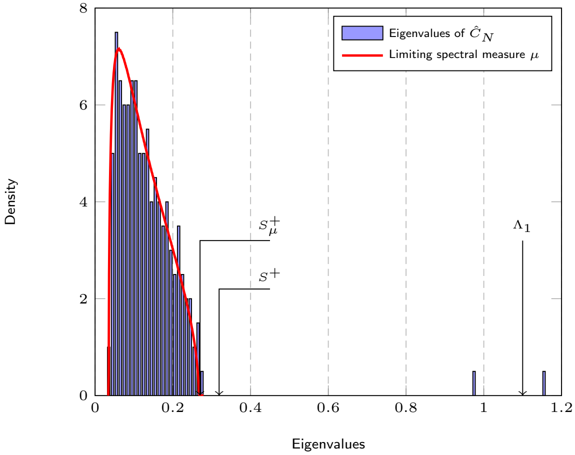
Spiked Model Extension

$\longrightarrow$ Noise eigenvalues are bounded by some S^+.



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Spiked Model Extension

$\longrightarrow$ To be compared versus SCM $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$

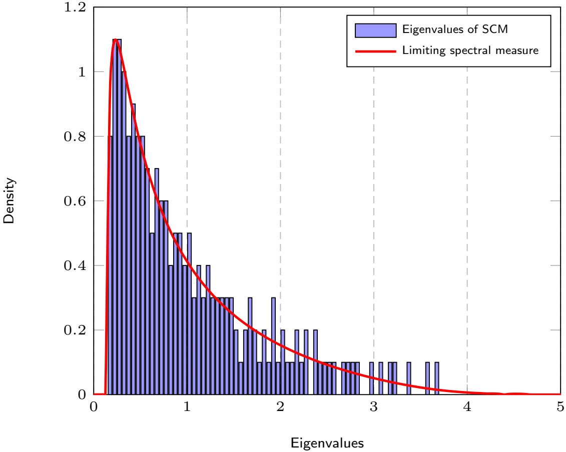

Figure: Eigenvalues of $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Sudent-t impulsions.

Spiked Model Extension

Some important remarks:

- If $p_1 = \ldots = p_L = 0$, noise-only model and

$$\limsup_N \|\hat{C}_N\| = \limsup_N \left\| \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\tau_i \gamma)}{\gamma} w_i w_i^* \right\| \leq S^+ \triangleq \frac{\phi_\infty (1+\sqrt{c})^2}{(1-c\phi_\infty)\gamma}.$$

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- If $p_1 \geq \ldots \geq p_L > 0$, informative spikes if $\det(\hat{S}_N - xI_N)$ has solutions beyond S^+ (and not S_μ^+!), i.e., if

$$p_l > p_- \triangleq \lim_{x \downarrow S^+} -c \left(\int \frac{\delta(x) v(t\gamma)}{1+\delta(x) t v(t\gamma)} \tilde{\nu}(dt) \right)^{-1}$$

with $\delta(x)$, $x > S_\mu^+$, unique solution to

$$\delta(x) = c \left(-x + \int \frac{t v(t\gamma)}{1+\delta(x) t v(t\gamma)} \tilde{\nu}(dt) \right)^{-1}.$$

Theorem (Spiked estimation, known $\tilde{\nu}$ [C'2014])

With the SVD $AA^* = \sum_{l=1}^{L} q_l u_l u_l^*$ *and* $\hat{C}_N = \sum_{i=1}^{N} \hat{\lambda}_i \hat{u}_i \hat{u}_i^*$ *(*$\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_N$*),*

Extreme eigenvalues. *For each* j *with* $p_j > p_-$*,*

$$\hat{\lambda}_j \xrightarrow{\text{a.s.}} \Lambda_j > S^+ \text{ a.s., where } -c\left(\int \frac{\delta(\Lambda_j)v(\tau\gamma)}{1+\delta(\Lambda_j)\tau v(\tau\gamma)}\tilde{\nu}(d\tau)\right)^{-1} = p_j.$$

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Bilinear form estimation. *For* $a, b \in \mathbb{C}^N$, $\|a\| = \|b\| = 1$, *and* j *with* $p_j > p_-$,

$$\sum_{k, p_k = p_j} a^* u_k u_k^* b - \sum_{k, p_k = p_j} w_k a^* \hat{u}_k \hat{u}_k^* b \xrightarrow{\text{a.s.}} 0$$

where

$$w_k \triangleq \int \frac{v(t\gamma)\tilde{\nu}(dt)}{\left(1+\delta(\hat{\lambda}_k)tv(t\gamma)\right)^2}\left[\int \frac{v(t\gamma)\tilde{\nu}(dt)}{1+\delta(\hat{\lambda}_k)tv(t\gamma)}\left(1-\frac{1}{c}\int \frac{\delta(\hat{\lambda}_k)^2 t^2 v(t\gamma)^2 \tilde{\nu}(dt)}{\left(1+\delta(\hat{\lambda}_k)tv(t\gamma)\right)^2}\right)\right]^{-1}$$

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$$\max_{\tau_j < M} |\tau_j - \hat{\tau}_j| \xrightarrow{\text{a.s.}} 0, \ \hat{\tau}_i \triangleq \frac{1}{\hat{\gamma}_n} \frac{1}{N} x_i^* \hat{C}_{(i)}^{-1} x_i.$$

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for the corresponding $\hat{w}_k = f(\{\hat{\tau}_i\}, \hat{\delta}(\hat{\lambda}_k))$.

$\longrightarrow$ Application to angle estimation with

$$a_l = a(\theta_l), \ \theta_l \in [0, 2\pi)$$

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Corollary (Robust G-MUSIC)

Define $\hat{\eta}_{\mathrm{RG}}(\theta)$ *and* $\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta)$ *as*

$$\hat{\eta}_{\mathrm{RG}}(\theta) = 1 - \sum_{k=1}^{|\{p_j > p_-\}|} w_k a(\theta)^* \hat{u}_k \hat{u}_k^* a(\theta)$$

$$\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta) = 1 - \sum_{k=1}^{|\{p_j > p_-\}|} \hat{w}_k a(\theta)^* \hat{u}_k \hat{u}_k^* a(\theta)$$

Then, for each j *with* $p_j > p_-$,

$$\hat{\theta}_j \xrightarrow{\mathrm{a.s.}} \theta_j$$

$$\hat{\theta}_j^{\mathrm{emp}} \xrightarrow{\mathrm{a.s.}} \theta_j$$

where

$$\hat{\theta}_j \triangleq \mathrm{argmin}_{\theta \in V(\theta_j)} \left\{ \hat{\eta}_{\mathrm{RG}}(\theta) \right\}$$

$$\hat{\theta}_j^{\mathrm{emp}} \triangleq \mathrm{argmin}_{\theta \in V(\theta_j)} \left\{ \hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta) \right\}.$$

Spiked Model Extension

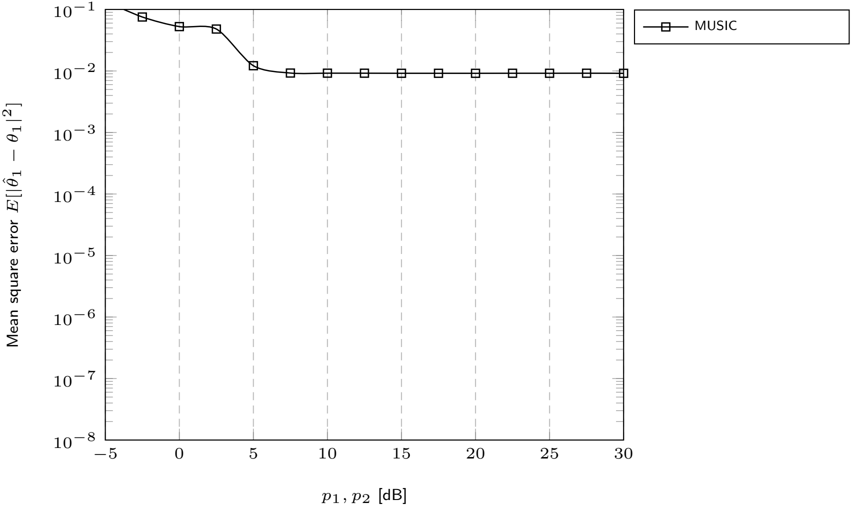
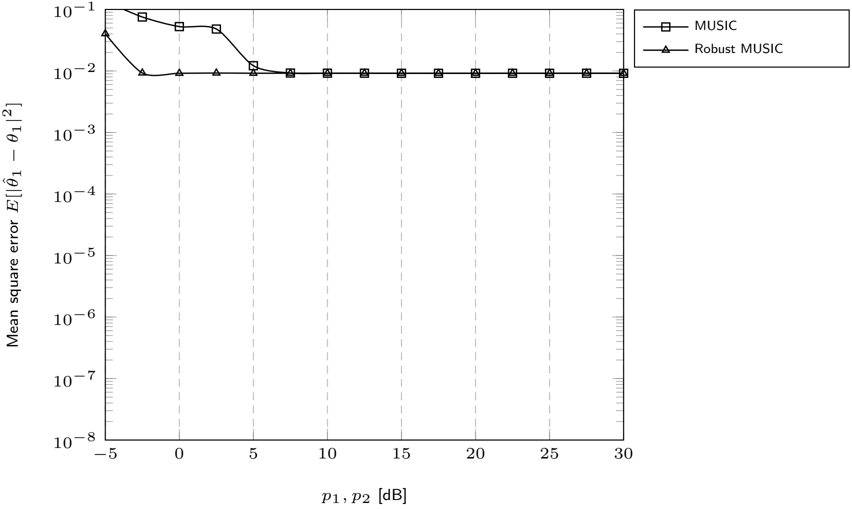


Figure: MSE for estimate of $\theta_1 = 10^\circ$, $N = 20$, $n = 100$, $L = 2$ sources at 10° and 12°, Student-t impulsions, $u(x) = (1 + \alpha)/(\alpha + x)$ with $\alpha = 0.2$, $p_1 = p_2$.



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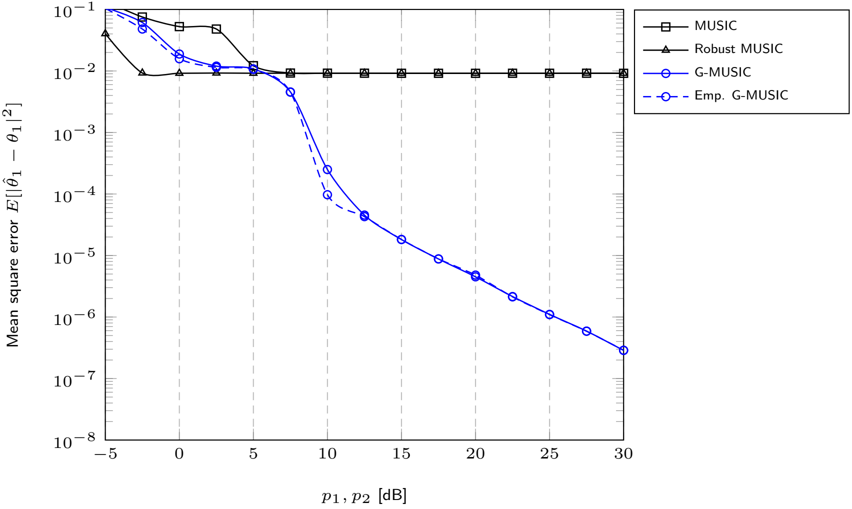

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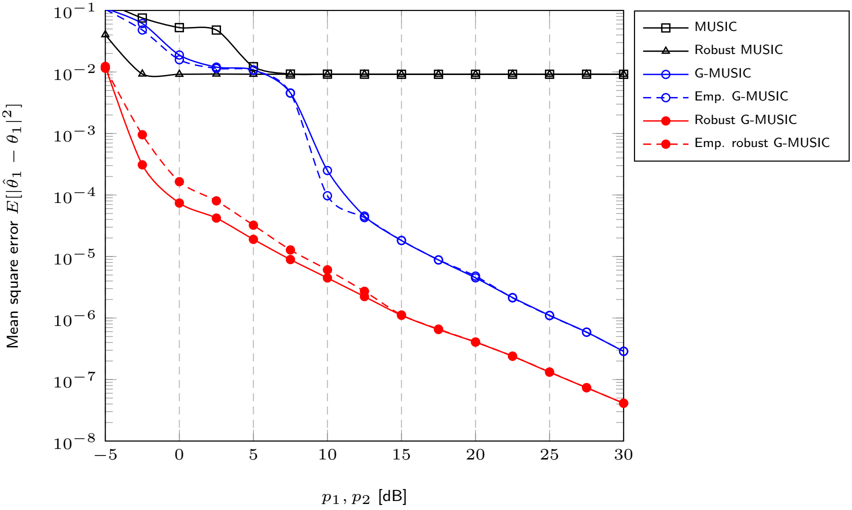

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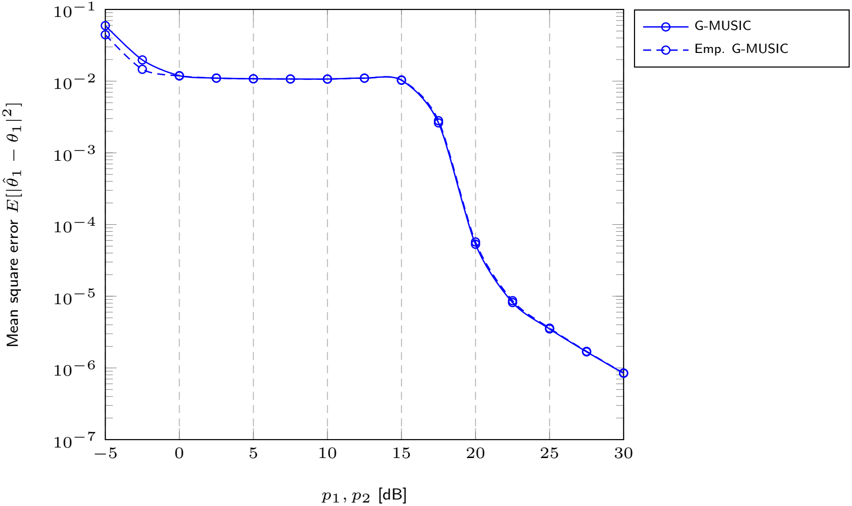

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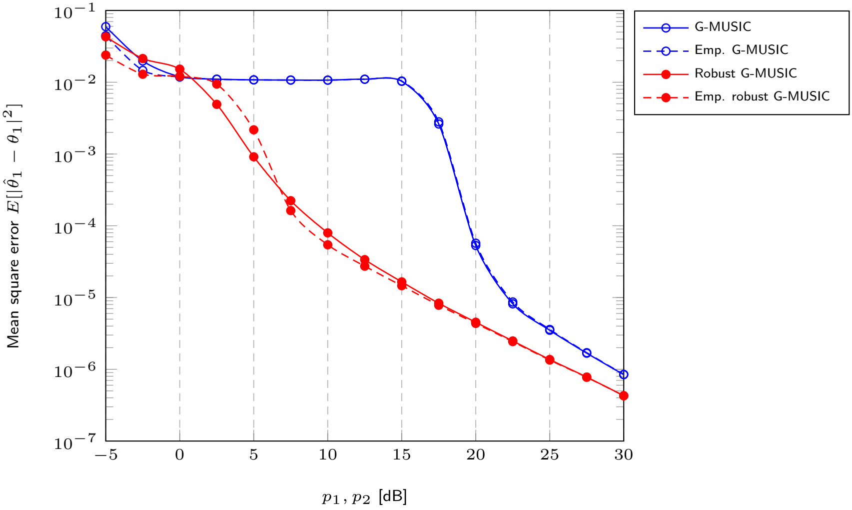

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Outlier Data

Original setting of Huber

Assumption (Outlying Data)

Observation set

$$X = \left[x_1, \ldots, x_{(1-\varepsilon_n)n}, a_1, \ldots, a_{\varepsilon_n n}\right]$$

where $x_i \sim \mathcal{CN}(0, C_N)$ and $a_1, \ldots, a_{\varepsilon_n n} \in \mathbb{C}^N$ deterministic with

$$\limsup_n \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} \frac{1}{N} a_i^* C_N^{-1} a_i < \infty$$

(or only a.s. if a_i random).

Theorem (Outlier Rejection [Morales-Jimenez,C,McKay'14])

As $n \to \infty$,

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq v\left(\gamma_N\right) \frac{1}{n} \sum_{i=1}^{(1-\varepsilon_n)n} x_i x_i^* + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} v\left(\alpha_{i,n}\right) a_i a_i^*$$

with γ_N and $\alpha_{1,n}, \ldots, \alpha_{\varepsilon_n n,n}$ unique positive solutions to

$$\gamma_N = \frac{1}{N} \mathrm{tr} C_N \left(\frac{(1-\varepsilon)v(\gamma_N)}{1+cv(\gamma_N)\gamma_N} C_N + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} v\left(\alpha_{i,n}\right) a_i a_i^* \right)^{-1}$$

$$\alpha_{i,n} = \frac{1}{N} a_i^* \left(\frac{(1-\varepsilon)v(\gamma_N)}{1+cv(\gamma_N)\gamma_N} C_N + \frac{1}{n} \sum_{j \neq i}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^* \right)^{-1} a_i, \;\; i = 1, \ldots, \varepsilon_n n.$$

Outlier Data

- For $\varepsilon_n n = 1$,

$$\hat{S}_N = v\left(\frac{\phi^{-1}(1)}{1-c}\right)\frac{1}{n}\sum_{i=1}^{n-1} x_i x_i^* + \left(v\left(\frac{\phi^{-1}(1)}{1-c}\frac{1}{N}a_1^* C_N^{-1} a_1\right) + o(1)\right) a_1 a_1^*$$

Outlier rejection relies on $\frac{1}{N}a_1^* C_N^{-1} a_1 \lessgtr 1$.

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- For $a_i \sim \mathcal{CN}(0, D_N)$, $\varepsilon_n \to \varepsilon \geq 0$,

$$\hat{S}_N = v(\gamma_n)\frac{1}{n}\sum_{i=1}^{(1-\varepsilon_n)n} x_i x_i^* + v(\alpha_n)\frac{1}{n}\sum_{i=1}^{\varepsilon_n n} a_i a_i^*$$

$$\gamma_n = \frac{1}{N}\mathrm{tr}\, C_N \left(\frac{(1-\varepsilon)v(\gamma_n)}{1+cv(\gamma_n)\gamma_n} C_N + \frac{\varepsilon v(\alpha_n)}{1+cv(\alpha_n)\alpha_n} D_N\right)^{-1}$$

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For $\varepsilon_n \to 0$,

$$\hat{S}_N = v\left(\frac{\phi^{-1}(1)}{1-c}\right)\frac{1}{n}\sum_{i=1}^{(1-\varepsilon_n)n} x_i x_i^* + \frac{1}{n}\sum_{i=1}^{\varepsilon_n n} v\left(\frac{\phi^{-1}(1)}{1-c}\frac{1}{N}\operatorname{tr} D_N C_N^{-1}\right) a_i a_i^*$$

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Outlier Data

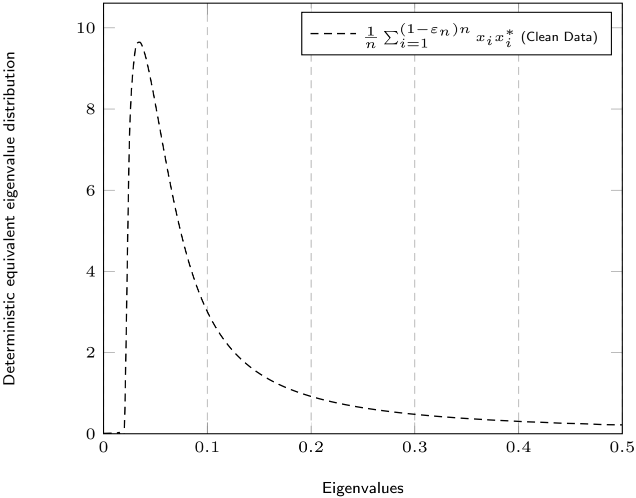
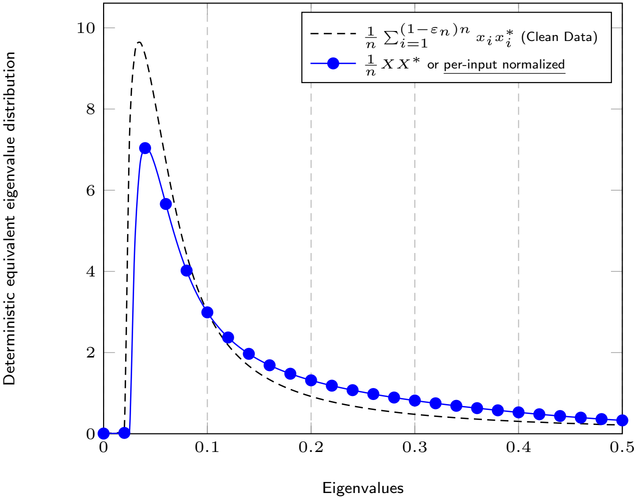


Figure: Limiting eigenvalue distributions. $[C_N]_{ij} = .9^{|i-j|}$, $D_N = I_N$, $\varepsilon = .05$.



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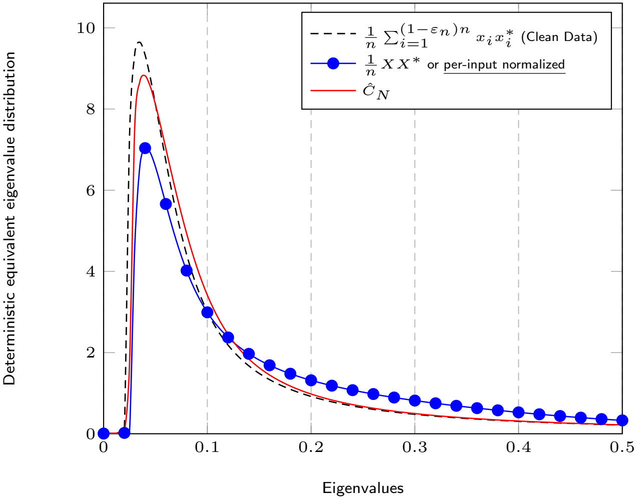

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Outline

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- Generalize robust estimators to $N > n$

Shrinkage Estimators

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Assumption (Pure-noise model)

Independent $x_1, \ldots, x_n \in \mathbb{C}^N$,

$$x_i = \sqrt{\tau}_i z_i$$

with

- $\tau_i > 0$ arbitrary
- $z_i \sim \mathcal{CN}(0, C_N)$, $\limsup_N \|C_N\| < \infty$
- $\nu_n \triangleq \frac{1}{N} \sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i(C_N)} \to \nu$.

Two estimators in the literature

Definition (Abramovich–Pascal estimate)

For $\rho \in (\max\{0, 1-n/N\}, 1]$, unique solution $\hat{C}_N(\rho)$ to

$$\hat{C}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n}\frac{x_i x_i^*}{\frac{1}{N}x_i^*\hat{C}_N^{-1}(\rho)x_i} + \rho I_N.$$

Property: $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}(\rho) = 1$.

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Property: $\frac{1}{N} \operatorname{tr} \hat{C}_N^{-1}(\rho) = 1$.

Definition (Chen estimate)

For $\rho \in (0, 1]$, unique solution $\check{C}_N(\rho)$ to

$$\check{C}_N(\rho) = \frac{\check{B}_N(\rho)}{\frac{1}{N} \operatorname{tr} \check{B}_N(\rho)}$$

$$\check{B}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \check{C}_N^{-1}(\rho) x_i} + \rho I_N.$$

Property: $\frac{1}{N} \operatorname{tr} \check{C}_N(\rho) = 1$.

Theorem (Abramovich–Pascal estimator [C,McKay'14])

For $\hat{\mathcal{R}}_\varepsilon = [\varepsilon + \max\{0, 1 - c^{-1}\}, 1]$ *as* $n \to \infty$ *with* $N/n \to c \in (0, \infty)$,

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} \left\| \hat{C}_N(\rho) - \hat{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

with

$$\hat{S}_N(\rho) = \frac{1}{\hat{\gamma}(\rho)} \frac{1 - \rho}{1 - (1 - \rho)c} \frac{1}{n} \sum_{i=1}^{n} z_i z_i^* + \rho I_N$$

and $\hat{\gamma}(\rho)$ *unique positive solution to*

$$1 = \int \frac{t}{\rho \hat{\gamma}(\rho) + (1 - \rho)t} \nu(dt).$$

Theorem (Chen estimator [C,McKay'14])

Letting $\check{\mathcal{R}}_\varepsilon = [\varepsilon, 1]$, *as* $n \to \infty$ *with* $N/n \to c \in (0, \infty)$,

$$\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \left\| \check{C}_N(\rho) - \check{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\check{S}_N(\rho) = \frac{1-\rho}{1-\rho+T_\rho} \frac{1}{n} \sum_{i=1}^{n} z_i z_i^* + \frac{T_\rho}{1-\rho+T_\rho} I_N$$

in which $T_\rho = \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)$ *with, for all* $x > 0$,

$$F(x;\rho) = \frac{1}{2}(\rho - c(1-\rho)) + \sqrt{\frac{1}{4}(\rho - c(1-\rho))^2 + (1-\rho)\frac{1}{x}}$$

and $\check{\gamma}(\rho)$ *unique positive solution to*

$$1 = \int \frac{t}{\rho\check{\gamma}(\rho) + \frac{1-\rho}{(1-\rho)c + F(\check{\gamma}(\rho);\rho)} t} \nu(dt).$$

Corollary (Model Equivalence)

For $\rho \in (0,1]$, there exists a unique $(\hat{\rho},\check{\rho})$ such that

$$\frac{\hat{S}_N(\hat{\rho})}{\lim_N \frac{1}{N}\operatorname{tr}\hat{S}_N(\hat{\rho})} = \check{S}_N(\check{\rho}) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n} z_i z_i^* + \rho I_N.$$

Besides, $\rho \mapsto \hat{\rho}$ and $\rho \mapsto \check{\rho}$ are continuously increasing and onto.

Consequence: both estimators equivalent in limit to Ledoit–Wolf on z_i (not x_i).

Uniform convergence allows for optimization over ρ.

Proposition (Optimal Frobenius-norm Shrinkage)

For each ρ, define

$$\hat{D}_N(\rho) = \frac{1}{N}\operatorname{tr}\left(\frac{\hat{C}_N(\rho)}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)} - C_N\right)^2$$

$$\check{D}_N(\rho) = \frac{1}{N}\operatorname{tr}\left(\check{C}_N(\rho) - C_N\right)^2$$

$$D^\star = c\frac{M_{\nu,2} - 1}{c + M_{\nu,2} - 1} \quad (M_{\nu,2},\ \text{order-2 moment})$$

$$\rho^\star = \frac{c}{c + M_{\nu,2} - 1}$$

and $\hat{\rho}^\star, \check{\rho}^\star$ unique solutions to

$$\frac{\hat{\rho}^\star}{\frac{1}{\hat{\gamma}(\hat{\rho}^\star)}\frac{1-\hat{\rho}^\star}{1-(1-\hat{\rho}^\star)c} + \hat{\rho}^\star} = \frac{T_{\check{\rho}^\star}}{1 - \check{\rho}^\star + T_{\check{\rho}^\star}} = \rho^\star.$$

Then,

$$\inf_{\rho\in\hat{\mathcal{R}}_\varepsilon} \hat{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star, \quad \inf_{\rho\in\check{\mathcal{R}}_\varepsilon} \check{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star$$

$$\hat{D}_N(\hat{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star, \quad \check{D}_N(\check{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star.$$

Proposition (Optimal Frobenius-norm shrinkage estimate)

Let $\hat{\rho}_N, \check{\rho}_N$ be solutions to

$$\frac{\hat{\rho}_N}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho}_N)} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}$$

$$\frac{\check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}}{1 - \check{\rho}_N + \check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}.$$

Then

$$\hat{D}_N(\hat{\rho}_N) \xrightarrow{\text{a.s.}} D^\star$$
$$\check{D}_N(\check{\rho}_N) \xrightarrow{\text{a.s.}} D^\star.$$

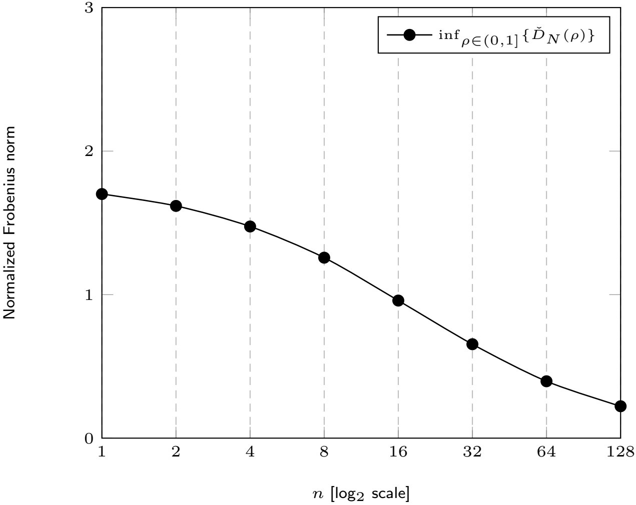
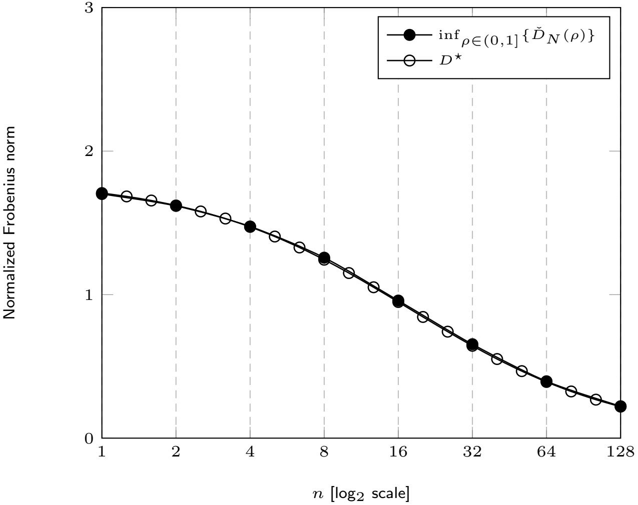


Figure: Optimal shrinkage, $N = 32$, $[C_N]_{ij} = .7^{|i-j|}$; $\check{\rho}_O$ clairvoyant estimator of (Chen et al., 2011) assuming $\hat{C}_N(\rho) \simeq (1-\rho)\frac{1}{n}\sum_i \frac{x_i x_i^*}{\frac{1}{N} x_i^* C_N{}^{-1} x_i} + \rho I_N$.

Optimal asymptotic shrinkage



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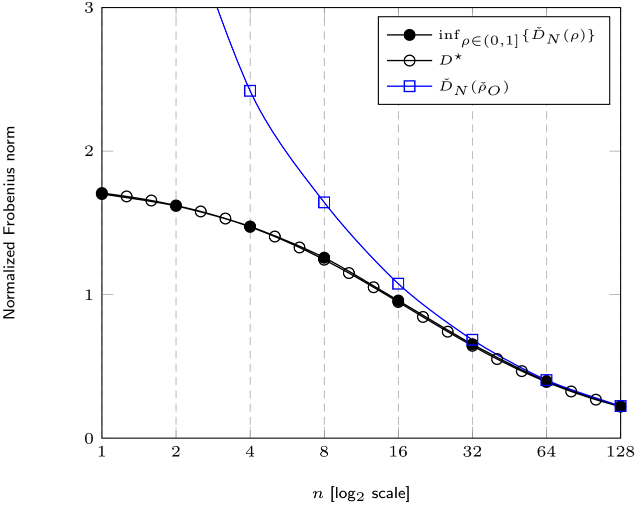

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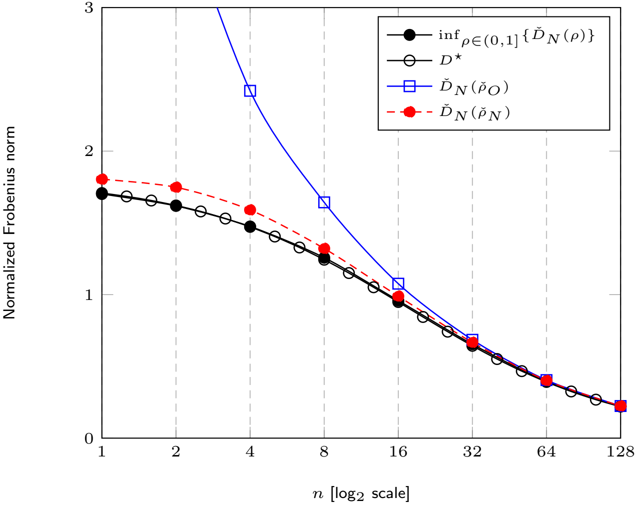

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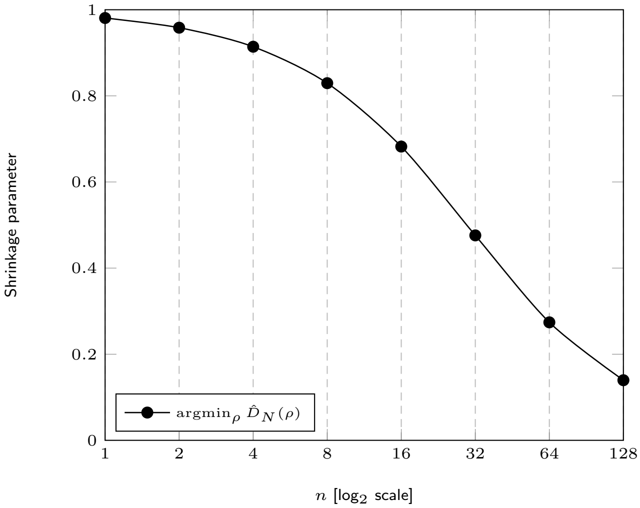

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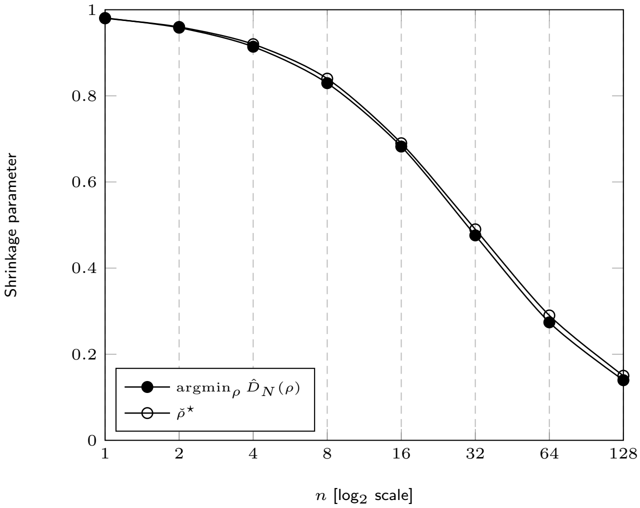

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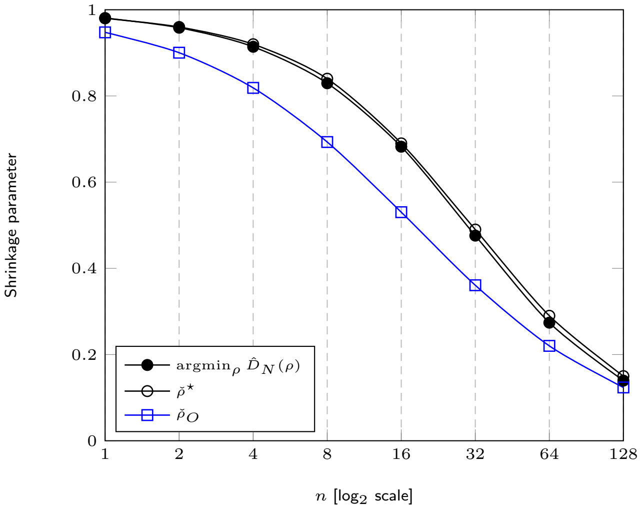

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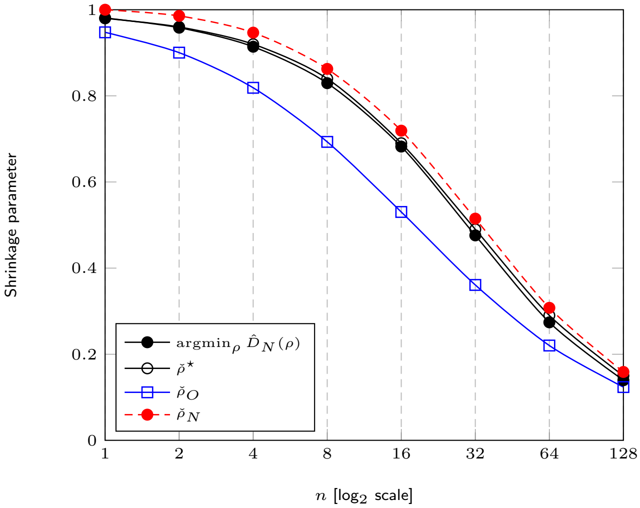

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Outline

Fluctuations of $\hat{C}_N$

Context (about $\sup_\rho \|\hat{C}_N(\rho) - \hat{S}_N(\rho)\| \xrightarrow{\text{a.s.}} 0$)

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Theorem (Fluctuation of bilinear forms [C,Kammoun,Pascal'14])

Let $a,b\in\mathbb{C}^N$ *with* $\|a\|=\|b\|=1$. *Then, as* $n\to\infty$, $N/n\to c\in(0,\infty)$, *for all* $\varepsilon>0$, $k\in\mathbb{Z}$,

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(with $\varepsilon < \frac{1}{2}$, desired result)

Proof idea

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$$a^* \hat{C}_N^{-1} b - a^* \hat{S}_N^{-1} b = a^* \hat{C}_N^{-1} \left(\frac{1-\rho}{1-(1-\rho)c_N} \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{\gamma_N} - \frac{1}{d_i} \right] z_i z_i^* \right) \hat{S}_N^{-1} b$$

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$$\tilde{d}_i(\rho) = \frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i = \frac{1}{N} z_i^* \left(\frac{1-\rho}{1-(1-\rho)c_N} \frac{1}{n} \sum_{j \neq i}^{n} \frac{z_j z_j^*}{\gamma_N} + \rho I_N \right)^{-1} z_i$$

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$$+ \frac{1-\rho}{1-(1-\rho)c_N} a^* \hat{C}_N^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \underbrace{\left[\frac{1}{\tilde{d}_i} - \frac{1}{d_i} \right]}_{\text{Term (B)}} z_i z_i^* \right) \hat{S}_N^{-1} b.$$

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- Key lemma for both Terms (A)-(B):

Lemma (Key Lemma, Self-averaging)

$$E\left[\left| \frac{1}{n} \sum_{i=1}^{n} a^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} b \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \gamma_N \right) \right|^{2p} \right] = O\left(N^{-2p}\right)$$

Context (Hypothesis Test)

We observe $x_1, \ldots, x_n$, $x_i = \sqrt{\tau}_i w_i$, $\|w_i\|^2 = N$ isotropic, and receive

$$y = \begin{cases} x & , \mathcal{H}_0 \\ \alpha p + x & , \mathcal{H}_1 \end{cases}$$

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Definition (GLRT Detector)

$$T_N(\rho) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\gamma}{\sqrt{N}}$$

with

$$T_N(\rho) \triangleq \frac{|y^* \hat{C}_N^{-1}(\rho) p|}{\sqrt{y^* \hat{C}_N^{-1}(\rho) y} \sqrt{p^* \hat{C}_N^{-1}(\rho) p}}.$$

Theorem (Asymptotic detector performance [C,Kammoun,Pascal'14])

Under $\mathcal{H}_0$, as $n \to \infty$ with $N/n \to c \in (0,\infty)$,

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| P\left(T_N(\rho) > \frac{\gamma}{\sqrt{N}} \right) - \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho)} \right) \right| \to 0$$

where

$$\sigma_N^2(\rho) \triangleq \frac{1}{2} \frac{p^* C_N Q_N^2(\underline{\rho}) p}{p^* Q_N(\underline{\rho}) p \cdot \frac{1}{N} \operatorname{tr} C_N Q_N(\underline{\rho}) \cdot \left(1 - c(1-\underline{\rho})^2 m(-\underline{\rho})^2 \frac{1}{N} \operatorname{tr} C_N^2 Q_N^2(\underline{\rho}) \right)}$$

with $Q_N(\underline{\rho}) \triangleq (I_N + (1-\underline{\rho}) m(-\underline{\rho}) C_N)^{-1}$ *and* $\underline{\rho} = \rho(\rho + \frac{1}{\gamma_N(\rho)} \frac{1-\rho}{1-(1-\rho)c})^{-1}$*.*

Proposition (Empirical performance optimum)

Let

$$\hat{\sigma}_N^2(\rho) \triangleq \frac{1}{2} \frac{1 - \rho \frac{p^* \hat{C}_N^{-2}(\rho) p}{p^* \hat{C}_N^{-1}(\rho) p}}{(1 - c_N + c_N \rho)(1 - \rho)}.$$

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Besides, let

$$\hat{\rho}_N^* \in \operatorname{argmin}_{\rho \in \mathcal{R}_\kappa} \left\{ \hat{\sigma}_N^2(\rho) \right\}.$$

Then, for every $\gamma > 0$,

$$P\left(\sqrt{N} T_N(\hat{\rho}_N^*) > \gamma \right) - \inf_{\rho \in \mathcal{R}_\kappa} \left\{ P\left(\sqrt{N} T_N(\rho) > \gamma \right) \right\} \to 0.$$

Application to GLRT detection

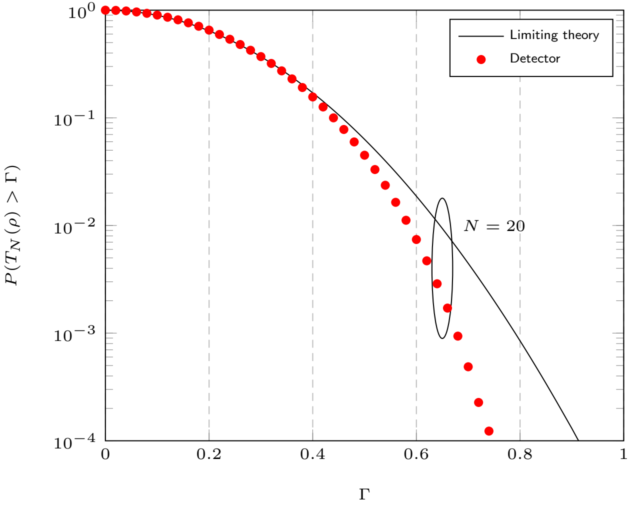
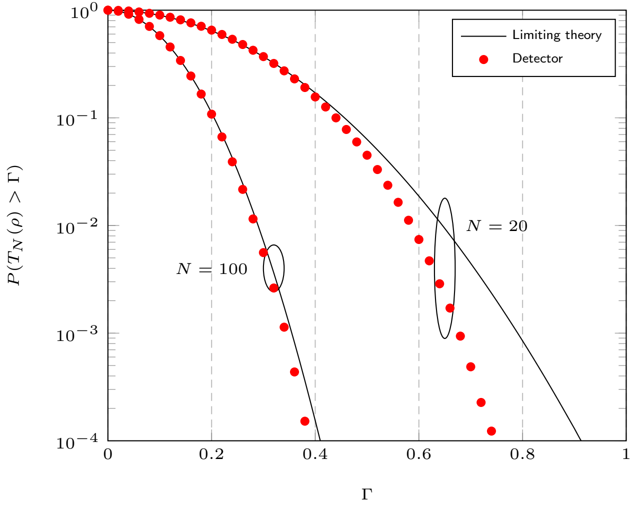


Figure: False alarm rate $P(T_N(\hat{\rho}_N^*) > \Gamma)$, $N = 20$ or $N = 100$, $p = N^{-\frac{1}{2}}[1, \ldots, 1]^\mathsf{T}$, $[C_N]_{ij} = .7^{|i-j|}$, $N/n = 1/2$.



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Perspectives

Takeaway messages

- Robust estimators of scatter: $\left\{ \begin{array}{l} \text{difficult to study for each } N, n \\ \text{become tractable when } N, n \to \infty. \end{array} \right.$
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- In practice, allows for (optimized) robust improvement of classical statistics.

Still on robust statistics

- Further properties of robust estimators of scatter (fluctuations of linear statistics).
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- Sparsity considerations (sparse PCA).

Outline

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The End

Thank you.