Future Random Matrix Tools for Large Dimensional Signal Processing

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High-dimensional data

- Consider n observations $\mathbf{x}_1, \cdots, \mathbf{x}_n$of size N, independent and identically distributed with zero-mean and covariance $\mathbf{C}_N$, i.e, $\mathbb{E}\left[\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}}\right] = C_N$,
- Let $\mathbf{X}_N = [\mathbf{x}_1, \cdots, \mathbf{x}_n]$. The sample covariance estimate $\hat{S}_N$ of $\mathbf{C}_N$ is given by: $\hat{S}_N = \frac{1}{n}\mathbf{X}_N\mathbf{X}_N^{\mathrm{H}} = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$,
- From the law of large numbers, as $n \to +\infty$,

$$\hat{S}_N \xrightarrow{\text{a.s.}} \mathbf{C}_N.$$

$\longrightarrow$ Convergence in the operator norm

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$\longrightarrow$ Convergence in the operator norm

- In practice, it might be difficult to afford $n \to +\infty$,
  - if $n \gg N$, $\hat{S}_N$ can be sufficiently accurate,
  - if $N/n = \mathcal{O}(1)$, we model this scenario by the following assumption: $N \to +\infty$ and $n \to +\infty$ with $\frac{N}{n} \to c$,
  - Under this assumption, we have pointwise convergence to each element of C_N, i.e,

$$\left(\hat{S}_N\right)_{i,j} \xrightarrow{\text{a.s.}} (C_N)_{i,j}$$

  but $\|S_N - C_N\|$ does not converge to zero.

  $\longrightarrow$ The convergence in the operator norm does not hold.

Illustration

Consider $C_N = I_N$, the spectrum of $\hat{S}_N$ is different from that of C_N

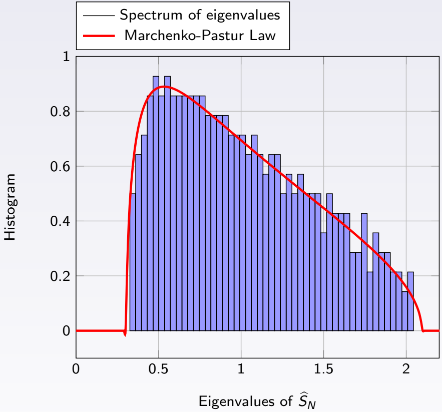

Figure: Spectrum of eigenvalues when $N = 400$ and $n = 2000$

$\longrightarrow$ The asymptotic spectrum can be characterized by the Marchenko-Pastur Law.

Reasons of interest for signal processing

- Scale similarity in array processing applications: large antenna arrays vs limited number of observations,
- Need for detection and estimation based on large dimensional random inputs: subspace methods in array processing.
- The assumption "number of obervations $\gg$ dimension of observation" is no longer valid: large arrays, systems with fast dynamics.

Example

MUSIC with "few" samples (or in large arrays) Call $\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)] \in \mathbb{C}^{N \times K}$, N large, K small, the steering vectors to identify and $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$ the n samples, taken from

$$\mathbf{x}_t = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\sqrt{p}_k s_{k,t} + \sigma w_t.$$

The MUSIC localization function reads $\gamma(\theta) = \mathbf{a}(\theta)^{\mathsf{H}} \hat{\mathbf{U}}_W \hat{\mathbf{U}}_W^{\mathsf{H}} \mathbf{a}(\theta)$ in the "signal vs. noise" spectral decomposition $\mathbf{X}\mathbf{X}^{\mathsf{H}} = \hat{\mathbf{U}}_S \hat{\mathbf{\Lambda}}_S \hat{\mathbf{U}}_S^{\mathsf{H}} + \hat{\mathbf{U}}_W \hat{\mathbf{\Lambda}}_W \hat{\mathbf{U}}_W^{\mathsf{H}}$.

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Writing equivalently $\mathbf{A}(\Theta)\mathbf{P}\mathbf{A}(\Theta)^{\mathsf{H}} + \sigma^2\mathbf{I}_N = \mathbf{U}_S\mathbf{\Lambda}_S\mathbf{U}_S^{\mathsf{H}} + \sigma^2\mathbf{U}_W\mathbf{U}_W^{\mathsf{H}}$, as $n, N \to \infty$, $n/N \to c$, from our previous remarks

$$\hat{\mathbf{U}}_W\hat{\mathbf{U}}_W^{\mathsf{H}} \not\to \mathbf{U}_W\mathbf{U}_W^{\mathsf{H}}$$

$\Rightarrow$ Music is NOT consistent in the large N, n regime! We need improved RMT-based solutions.

Outline

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Stieltjes Transform

Definition

Let F be a real probability distribution function. The Stieltjes transform m_F of F is the function defined, for $z \in \mathbb{C}^+$, as

$$m_F(z) = \int \frac{1}{\lambda - z} dF(\lambda)$$

For $a < b$ continuity points of F, denoting $z = x + iy$, we have the inverse formula

$$F(b) - F(a) = \lim_{y \to 0} \frac{1}{\pi} \int_a^b \Im[m_F(x + iy)] dx$$

If F has a density f at x, then

$$f(x) = \lim_{y \to 0} \frac{1}{\pi} \Im[m_F(x + iy)]$$

The Stieltjes transform is to the Cauchy transform as the characteristic functin is to the Fourier transform.

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Equivalence $F \leftrightarrow m_F$

Similar to the Fourier transform, knowing m_F is the same as knowing F.

Stieltjes transform of a Hermitian matrix

- Let $\mathbf{X}$ be a $N \times N$ random matrix. Denote by dF^X the empirical measure of its eigenvalues $\lambda_1, \cdots, \lambda_N$, i.e, $dF^X = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$. The Stieltjes transform of $\mathbf{X}$ denoted by $m_{\mathbf{X}} = m_F$ is the stieltjes transform of its empirical measure:

$$m_{\mathbf{X}}(z) = \int \frac{1}{\lambda - z} dF(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{N}\operatorname{tr}\left(\mathbf{X} - z\mathbf{I}_N\right)^{-1}.$$

- The Stieltjes transform of a random matrix is the trace of the resolvent matrix $\mathbf{Q}(z) = (\mathbf{X} - z\mathbf{I}_N)^{-1}$. The resolvent matrix plays a key role in the derivation of many of the results of random matrix theory.
- For compactly supported F, $m_F(z)$ is linked to the moments $M_k = \mathbb{E}\frac{1}{N}\operatorname{tr}\mathbf{X}^k$,

$$m_F(z) = -\sum_{k=0}^{+\infty} M_k z^{-k-1}$$

- m_F is defined in general on $\mathbb{C}_+$ but exists everywhere outside the support of F.

Side remark: the "Shannon"-transform

A. M. Tulino, S. Verdù, "Random matrix theory and wireless communications," Now Publishers Inc., 2004.

Definition

Let F be a probability distribution, m_F its Stieltjes transform, then the Shannon-transform $\mathcal{V}_F$ of F is defined as

$$\mathcal{V}_F(x) \triangleq \int_0^\infty \log(1 + x\lambda) dF(\lambda) = \int_x^\infty \left(\frac{1}{t} - m_F(-t) \right) dt$$

- This quantity is fundamental to wireless communication purposes!
- Note that m_F itself is of interest, not F!

Proof of the Marčenko-Pastur law

V. A. Marčenko, L. A. Pastur, "Distributions of eigenvalues for some sets of random matrices", Math USSR-Sbornik, vol. 1, no. 4, pp. 457-483, 1967.

The theorem to be proven is the following

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ have i.i.d. zero mean variance $1/n$ entries with finite eighth order moments. As $n, N \to \infty$ with $\frac{N}{n} \to c \in (0,\infty)$, the e.s.d. of $\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}$ converges almost surely to a nonrandom distribution function F_c with density f_c given by

$$f_c(x) = (1-c^{-1})^+\delta(x) + \frac{1}{2\pi cx}\sqrt{(x-a)^+(b-x)^+}$$

where $a = (1-\sqrt{c})^2$, and $b = (1+\sqrt{c})^2$.

The Marčenko-Pastur density

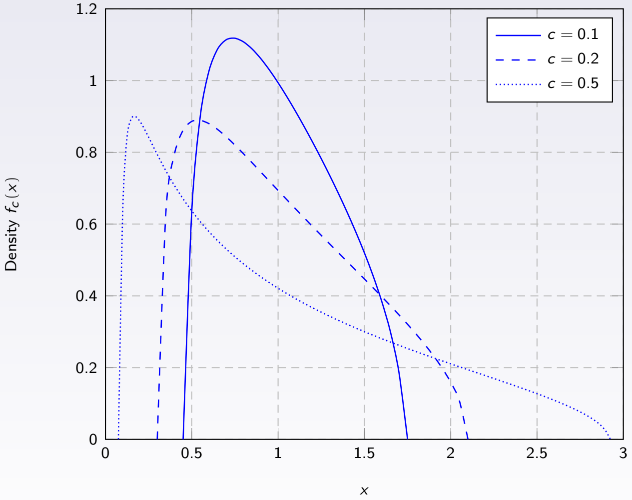

Figure: Marčenko-Pastur law for different limit ratios $c = \lim_{N\to\infty} N/n$.

Diagonal entries of the resolvent

Since we want an expression of m_F, we start by identifying the diagonal entries of the resolvent $(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N)^{-1}$ of $\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}$. Denote

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{y}^{\mathsf{H}} \\ \mathbf{Y} \end{bmatrix}$$

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$$\mathbf{X}_N = \begin{bmatrix} \mathbf{y}^{\mathsf{H}} \\ \mathbf{Y} \end{bmatrix}$$

Now, for $z \in \mathbb{C}^+$, we have

$$\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} = \begin{bmatrix} \mathbf{y}^{\mathsf{H}}\mathbf{y} - z & \mathbf{y}^{\mathsf{H}}\mathbf{Y}^{\mathsf{H}} \\ \mathbf{Y}\mathbf{y} & \mathbf{Y}\mathbf{Y}^{\mathsf{H}} - z\mathbf{I}_{N-1} \end{bmatrix}^{-1}$$

Consider the first diagonal element of $(\mathbf{R}_N - z\mathbf{I}_N)^{-1}$. From the matrix inversion lemma,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}$$

which here gives

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} = \frac{1}{-z - z\mathbf{y}^{\mathsf{H}}(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n)^{-1}\mathbf{y}}$$

Trace Lemma

Z. Bai, J. Silverstein, "Spectral Analysis of Large Dimensional Random Matrices", Springer Series in Statistics, 2009.

To go further, we need the following result,

Theorem

Let $\{\mathbf{A}_N\} \in \mathbb{C}^{N\times N}$ with bounded spectral norm. Let $\{\mathbf{x}_N\} \in \mathbb{C}^N$, be a random vector of i.i.d. entries with zero mean, variance $1/N$ and finite 8^{th} order moment, independent of $\mathbf{A}_N$. Then

$$\mathbf{x}_N^{\mathsf{H}}\mathbf{A}_N\mathbf{x}_N - \frac{1}{N}tr\mathbf{A}_N \xrightarrow{\text{a.s.}} 0.$$

For large N, we therefore have approximately

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} \simeq \frac{1}{-z - z\frac{1}{N}\text{tr}\left(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n\right)^{-1}}$$

Rank-1 perturbation lemma

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

It is somewhat intuitive that adding *a single column* to $\mathbf{Y}$ won't affect the trace in the limit.

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $N \times N$ *with* $\mathbf{B}$ *Hermitian positive definite, and* $\mathbf{v} \in \mathbb{C}^N$*. For* $z \in \mathbb{C} \setminus \mathbb{R}^-$,

$$\left| \frac{1}{N} tr \left((\mathbf{B} - z\mathbf{I}_N)^{-1} - (\mathbf{B} + \mathbf{v}\mathbf{v}^{\mathsf{H}} - z\mathbf{I}_N)^{-1} \right) \mathbf{A} \right| \leqslant \frac{1}{N} \frac{\|\mathbf{A}\|}{\operatorname{dist}(z, \mathbb{R}^+)}$$

with $\|\mathbf{A}\|$ *the spectral norm of* $\mathbf{A}$*, and* $\operatorname{dist}(z, A) = \inf_{y \in A} \|y - z\|$.

Therefore, for large N, we have approximately,

$$\begin{aligned} \left[\left(\mathbf{X}_N \mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N \right)^{-1} \right]_{11} &\simeq \frac{1}{-z - z\frac{1}{N}\operatorname{tr}\left(\mathbf{Y}^{\mathsf{H}}\mathbf{Y} - z\mathbf{I}_n\right)^{-1}} \\ &\simeq \frac{1}{-z - z\frac{1}{N}\operatorname{tr}\left(\mathbf{X}_N^{\mathsf{H}}\mathbf{X}_N - z\mathbf{I}_n\right)^{-1}} \\ &= \frac{1}{-z - z\frac{n}{N} m_{\underline{F}}(z)} \end{aligned}$$

in which we recognize the Stieltjes transform $m_{\underline{F}}$ of the l.s.d. of $\mathbf{X}_N^{\mathsf{H}}\mathbf{X}_N$.

End of the proof

We have again the relation

$$\frac{n}{N} m_{\underline{F}}(z) = m_F(z) + \frac{N-n}{N}\frac{1}{z}$$

hence

$$\left[\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\right]_{11} \simeq \frac{1}{\frac{n}{N} - 1 - z - zm_F(z)}$$

Note that the choice $(1,1)$ is irrelevant here, so the expression is valid for all pair (i,i). Summing over the N terms and averaging, we finally have

$$m_F(z) = \frac{1}{N}\operatorname{tr}\left(\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \simeq \frac{1}{c - 1 - z - zm_F(z)}$$

which solve a polynomial of second order. Finally

$$m_F(z) = \frac{c-1}{2z} - \frac{1}{2} + \frac{\sqrt{(c-1-z)^2 - 4z}}{2z}.$$

From the inverse Stieltjes transform formula, we then verify that m_F is the Stieltjes transform of the Marčenko-Pastur law.

Related bibliography

- V. A. Marčenko, L. A. Pastur, "Distributions of eigenvalues for some sets of random matrices", Math USSR-Sbornik, vol. 1, no. 4, pp. 457-483, 1967.
- J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.
- Z. D. Bai and J. W. Silverstein, "Spectral analysis of large dimensional random matrices, 2nd Edition" Springer Series in Statistics, 2009.
- R. B. Dozier, J. W. Silverstein, "On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices," Journal of Multivariate Analysis, vol. 98, no. 4, pp. 678-694, 2007.
- V. L. Girko, "Theory of Random Determinants," Kluwer, Dordrecht, 1990.
- A. M. Tulino, S. Verdù, "Random matrix theory and wireless communications," Now Publishers Inc., 2004.

Asymptotic results involving Stieltjes transform

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

Theorem

Let $\mathbf{Y}_N = \frac{1}{\sqrt{n}}\mathbf{X}_N\mathbf{C}_N^{\frac{1}{2}}$*, where* $\mathbf{X}_N \in \mathbb{C}^{n\times N}$ *has i.i.d entries of mean* 0 *and variance* 1*. Consider the regime* $n, N \to +\infty$ *with* $\frac{N}{n} \to c$*. Let* $\hat{\underline{m}}_N$ *be the Stieltjes transform associated to* $\mathbf{X}_N\mathbf{X}_N^*$*. Then,* $\hat{\underline{m}}_N - \underline{m}_N \to 0$ *almost surely for all* $z \in \mathbb{C}\backslash\mathbb{R}_+$*, where* $\underline{m}_N(z)$ *is the unique solution in the set* $\{z \in \mathbb{C}_+, \underline{m}_N(z) \in \mathbb{C}_+\}$ *to:*

$$\underline{m}_N(z) = \left(\int \frac{ctdF^{\mathbf{C}_N}}{1 + t\underline{m}_N(z)} - z\right)^{-1}$$

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$$\underline{m}_N(z) = \left(\int \frac{ctdF^{\mathbf{C}_N}}{1 + t\underline{m}_N(z)} - z\right)^{-1}$$

- in general, no explicit expression for $\underline{F}_N$, the distribution whose Stietljes transform is $\underline{m}_N(z)$.
- The theorem above characterizes also the Stieltjes transform of $\mathbf{B}_N = \mathbf{X}_N^{\mathrm{H}}\mathbf{X}_N$ denoted by m_N,

$$m_N = c\underline{m}_N + (c-1)\frac{1}{z}$$

This gives access to the spectrum of the sample covariance matrix model of $\mathbf{x}$, when $\mathbf{y}_i = \mathbf{C}_N^{\frac{1}{2}}\mathbf{x}_i$, $\mathbf{x}_i$ i.i.d., $\mathbf{C}_N = E[\mathbf{y}\mathbf{y}^{\mathrm{H}}]$.

Getting F' from m_F

- Remember that, for $a < b$ real,

$$F'(x) = \lim_{y \to 0} \frac{1}{\pi} \Im[m_F(x + iy)]$$

where m_F is (up to now) only defined on $\mathbb{C}^+$.

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- to plot the density F',
  - *first approach:* span $z = x + iy$ on the line $\{x \in \mathbb{R}, y = \varepsilon\}$ parallel but close to the real axis, solve $m_F(z)$ for each z, and plot $\Im[m_F(z)]$.

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Example (Sample covariance matrix)

For N multiple of 3, let $F^C(x) = \frac{1}{3}\mathbf{1}_{x \leqslant 1} + \frac{1}{3}\mathbf{1}_{x \leqslant 3} + \frac{1}{3}\mathbf{1}_{x \leqslant K}$ and let $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{Z}_N^{\mathrm{H}}\mathbf{Z}_N\mathbf{C}_N^{\frac{1}{2}}$ with $F^{B_N} \to F$, then

$$m_F = c m_{\underline{F}} + (c-1)\frac{1}{z}$$

$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^C(t) - z \right)^{-1}$$

We take $c = 1/10$ and alternatively $K = 7$ and $K = 4$.

Spectrum of the sample covariance matrix

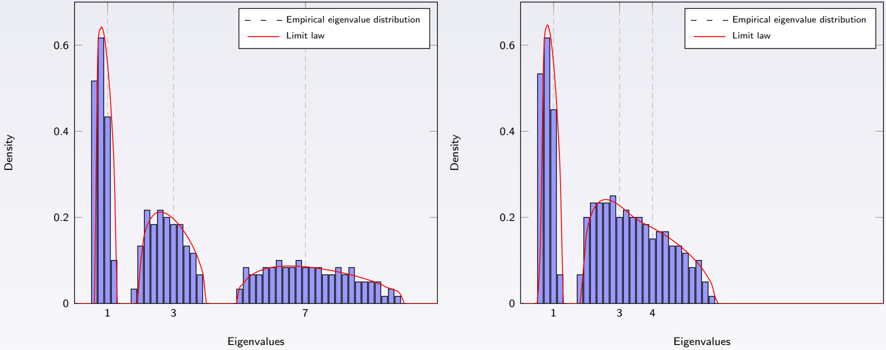

Figure: Histogram of the eigenvalues of $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{Z}_N^{\mathsf{H}}\mathbf{Z}_N\mathbf{C}_N^{\frac{1}{2}}$, $N = 3000$, $n = 300$, with $\mathbf{C}_N$ diagonal composed of three evenly weighted masses in (i) 1, 3 and 7 on top, (ii) 1, 3 and 4 at bottom.

Outline

Support of a distribution

The support of a density f is the closure of the set $\{x, f(x) \neq 0\}$.
For instance the support of the marčenko-Pastur law is $\left[(1-\sqrt{c})^2, (1+\sqrt{c})^2\right]$.

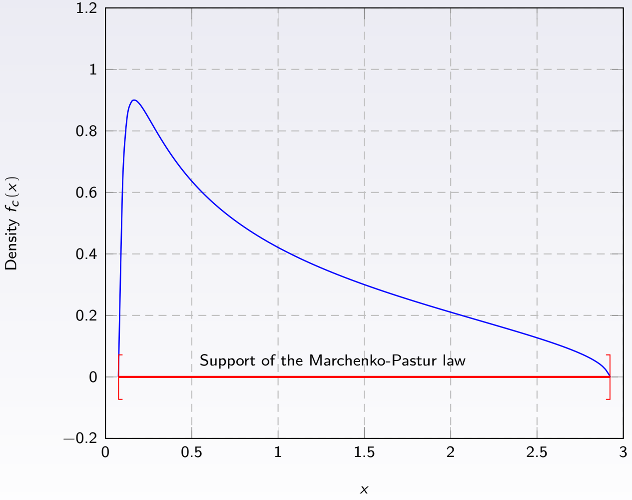

Figure: Marčenko-Pastur law for different limit ratios $c = 0.5$.

Extreme eigenvalues

- Limiting spectral results are insufficient to infer about the location of extreme eigenvalues.
- Example: Consider $dF_N(x) = \frac{1}{N}\sum_{k=1}^{N} \delta_{a_k}$. Then, $dF_N^0 = \frac{N-1}{N} dF_N + \frac{1}{N}\delta_{A_N}(x)$ and dF_N with $A_N \geqslant a_N$ satisfy:

$$dF_N - dF_N^0 \Rightarrow 0.$$

- However, the supports of F_N and F_{N_0} differ by the mass A_N.

Question: How is the behaviour of the extreme eigenvalues of random covariance matrices?

No eigenvalue outside the support of sample covariance matrices

Z. D. Bai, J. W. Silverstein, "No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices," The Annals of Probability, vol. 26, no.1 pp. 316-345, 1998.

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N \times n}$ with i.i.d. entries with zero mean, unit variance and infinite fourth order. Let $\mathbf{C}_N \in \mathbb{C}^{N \times N}$ be nonrandom and bounded in norm. Let $\underline{m}_N$ be the unique solution in $\mathbb{C}_+$ of

$$\underline{m}_N = -\left(z - \frac{N}{n}\int \frac{\tau}{1+\tau \underline{m}_N} dF^{\mathbf{C}_N}(\tau)\right)^{-1}, \quad \underline{m}_N(z) = \frac{N}{n} m_N(z) + \frac{N-n}{n}\frac{1}{z}, z \in \mathbb{C}_+,$$

Let F_N be the distribution associated to the Stieltjes transform $m_N(z)$. Consider $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathrm{H}}\mathbf{C}_N^{\frac{1}{2}}$. We know that $F^{\mathbf{B}_N} - F_N$ converge weakly to zero. Choose $N_0 \in \mathbb{N}$ and $[a, b]$, $a > 0$, outside the support of F_N for all $N \geqslant N_0$. Denote $\mathcal{L}_N$ the set of eigenvalues of $\mathbf{B}_N$. Then,

$$P(\mathcal{L}_N \cap [a, b] \neq \emptyset \text{ i.o.}) = 0.$$

No eigenvalue outside the support: which models?

J. W. Silverstein, P. Debashis, "No eigenvalues outside the support of the limiting empirical spectral distribution of a separable covariance matrix," J. of Multivariate Analysis vol. 100, no. 1, pp. 37-57, 2009.

- It has already been shown that (for all large N) there is no eigenvalues outside the support of
  - *Marčenko-Pastur law*: $\mathbf{X}\mathbf{X}^{\mathsf{H}}$, $\mathbf{X}$ i.i.d. with zero mean, variance $1/N$, finite 4^{th} order moment.
  - *Sample covariance matrix*: $\mathbf{C}^{\frac{1}{2}}\mathbf{X}\mathbf{X}^{\mathsf{H}}\mathbf{C}^{\frac{1}{2}}$ and $\mathbf{X}^{\mathsf{H}}\mathbf{C}\mathbf{X}$, $\mathbf{X}$ i.i.d. with zero mean, variance $1/N$, finite 4^{th} order moment.
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J. Silverstein, Z. Bai, "No eigenvalues outside the support of the limiting spectral distribution of information-plus-noise type matrices" to appear in Random Matrices: Theory and Applications.

- Only recently, information plus noise models, $\mathbf{X}$ with i.i.d. zero mean, variance $1/N$, finite 4^{th} order moment

$$(\mathbf{X}+\mathbf{A})(\mathbf{X}+\mathbf{A})^{\mathsf{H}},$$

  and the generally correlation model where each column of $\mathbf{X}$ has correlation $\mathbf{R}_i$.

Extreme eigenvalues: Deeper into the spectrum

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Extreme eigenvalues: Deeper into the spectrum

- In order to derive statistical detection tests, we need more information on the extreme eigenvalues.
- We will study the fluctuations of the extreme eigenvalues (second order statistics)
- However, the Stieltjes transform method is not adapted here!

Distribution of the largest eigenvalues of $\mathbf{X}\mathbf{X}^{\mathsf{H}}$

C. A. Tracy, H. Widom, "On orthogonal and symplectic matrix ensembles," Communications in Mathematical Physics, vol. 177, no. 3, pp. 727-754, 1996.
K. Johansson, "Shape Fluctuations and Random Matrices," Comm. Math. Phys. vol. 209, pp. 437-476, 2000.

Theorem

Let $\mathbf{X} \in \mathbb{C}^{N \times n}$ have i.i.d. Gaussian entries of zero mean and variance $1/n$. Denoting λ_N^+ the largest eigenvalue of $\mathbf{X}\mathbf{X}^{\mathsf{H}}$, then

$$N^{\frac{2}{3}} \frac{\lambda_N^+ - (1+\sqrt{c})^2}{(1+\sqrt{c})^{\frac{4}{3}} c^{\frac{1}{2}}} \Rightarrow X^+ \sim F^+$$

with $c = \lim_N N/n$ and F^+ the Tracy-Widom distribution given by

$$F^+(t) = \exp\left(- \int_t^{\infty} (x-t)^2 q^2(x) dx \right)$$

with q the Painlevé II function that solves the differential equation

$$q''(x) = xq(x) + 2q^3(x)$$
$$q(x) \sim_{x \to \infty} \mathrm{Ai}(x)$$

in which $\mathrm{Ai}(x)$ is the Airy function.

The law of Tracy-Widom

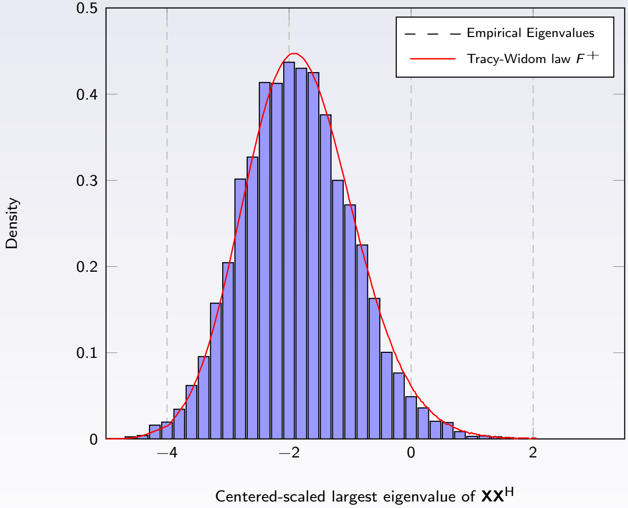

Figure: Distribution of $N^{\frac{2}{3}} c^{-\frac{1}{2}} (1+\sqrt{c})^{-\frac{4}{3}} \left[\lambda_N^+ - (1+\sqrt{c})^2\right]$ against the distribution of X^+ (distributed as Tracy-Widom law) for $N = 500$, $n = 1500$, $c = 1/3$, for the covariance matrix model $\mathbf{XX}^{\mathsf{H}}$. Empirical distribution taken over 10,000 Monte-Carlo simulations.

Techniques of proof

Method of proof requires very different tools:

- *orthogonal (Laguerre) polynomials*: to write joint *unordered* eigenvalue distribution as a kernel determinant.

$$\rho_N(\lambda_1, \ldots, \lambda_p) = \det_{i,j=1}^{p} K_N(\lambda_i, \lambda_j)$$

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- *Fredholm determinants*: we can write hole probability as a Fredholm determinant.

$$P\left(N^{2/3}\left(\lambda_i - (1+\sqrt{c})^2\right) \in A, i = 1, \ldots, N\right) = 1 + \sum_{k\geqslant 1} \frac{(-1)^k}{k!} \int_{A^c} \cdots \int_{A^c} \det_{i,j=1}^{k} K_N(x_i, x_j) \prod dx_i$$

$$\triangleq \det(\mathbf{I}_N - \mathcal{K}_N).$$

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- *kernel theory*: show that K_N converges to a Airy kernel.

$$K_N(x,y) \to K_{\mathrm{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

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- *differential equation tricks*: hole probability in $[t,\infty)$ gives right-most eigenvalue distribution, which is simplified as solution of a Painelvé differential equation: the Tracy-Widom distribution.

$$F^+(t) = e^{-\int_t^\infty (x-t)q(x)^2dx}, \quad q'' = tq+2q^3,\ q(x)\sim_{x\to\infty}\mathrm{Ai}(x).$$

Comments on the Tracy-Widom law

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- gives a hint on convergence speed
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- deeper result than limit eigenvalue result
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- fairly biased on the left: even fewer eigenvalues outside the support.
- can be shown to hold for other distributions than Gaussian under mild assumptions

Outline

Spiked models

- We consider n independent observations $\mathbf{x}_1, \cdots, \mathbf{x}_n$ of size N,
- The correlation structure is in general "white + low rank",

$$\mathbb{E}\left[\mathbf{x}_1 \mathbf{x}_1^{\mathrm{H}}\right] = \mathbf{I} + \mathbf{P}$$

where $\mathbf{P}$ is of low rank,
- Objective: to infer the eigenvalues and/or the eigenvectors of $\mathbf{P}$

The first result

J. Baik, J. W. Silverstein, "Eigenvalues of large sample covariance matrices of spiked population models," Journal of Multivariate Analysis, vol. 97, no. 6, pp. 1382-1408, 2006.

Theorem

Let $\mathbf{B}_N = \frac{1}{n}(\mathbf{I}+\mathbf{P})^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}(\mathbf{I}+\mathbf{P})^{\frac{1}{2}}$*, where* $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ *has i.i.d., zero mean and unit variance entries, and* $\mathbf{P}_N \in \mathbb{R}^{N\times N}$ *with eigenvalues given by:*

$$\mathrm{eig}(\mathbf{P}) = \mathit{diag}(\omega_1,\ldots,\omega_K,\underbrace{0,\ldots,\ldots,0}_{N-K})$$

with $\omega_1 > \ldots > \omega_K > -1$*,* $c = \lim_N N/n$*. Let* $\lambda_1,\cdots,\lambda_N$ *be the eigenvalues of* $\mathbf{B}_N$*. We then have*

- *if* $\omega_j > \sqrt{c}$*,* $\lambda_j \xrightarrow{\text{a.s.}} 1+\omega_j + c\frac{1+\omega_j}{\omega_j}$ *(i.e. beyond the Marčenko–Pastur bulk!)*
- *if* $\omega_j \in (0,\sqrt{c}]$*,* $\lambda_j \xrightarrow{\text{a.s.}} (1+\sqrt{c})^2$ *(i.e. right-edge of the Marčenko–Pastur bulk!)*
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- *for the other eigenvalues, we discriminate over* c*:*
  - *if* $\omega_j < -\sqrt{c}$, $c<1$, $\lambda_j \xrightarrow{\text{a.s.}} 1+\omega_j+c\frac{1+\omega_j}{\omega_j}$ *(i.e. beyond the Marčenko–Pastur bulk!)*
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Illustration of spiked models

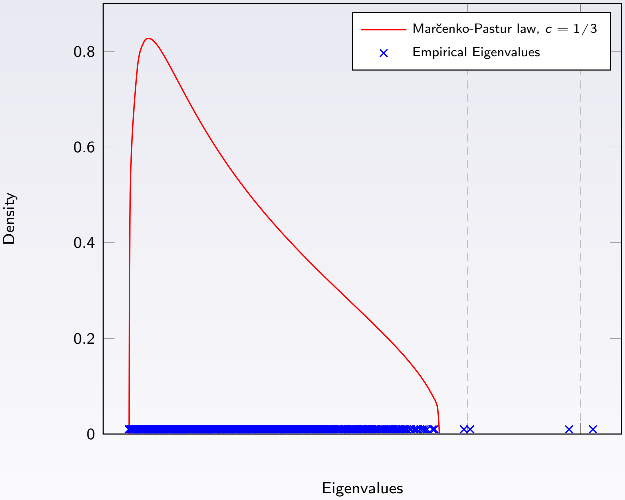

Figure: Eigenvalues of $\mathbf{B}_N = \frac{1}{n}(\mathbf{P}+\mathbf{I})^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N{}^{\mathsf{H}}(\mathbf{P}+\mathbf{I})^{\frac{1}{2}}$, where $\omega_1 = \omega_2 = 1$ and $\omega_3 = \omega_4 = 2$ Dimensions: $N = 500$, $n = 1500$.

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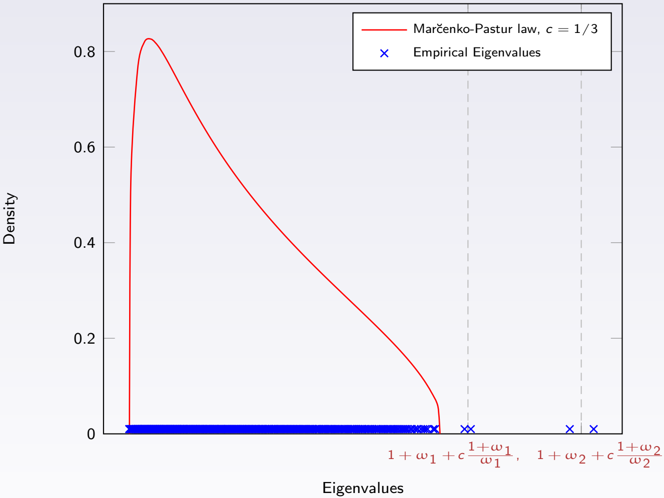

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- if so, no way to decide on the existence of the spikes *from looking at the largest eigenvalues*
- in signal processing words, signals might be missed using largest eigenvalues methods.
- as a consequence,
  - the more the sensors (N),
  - the larger $c = \lim N/n$,
  - the more probable we miss a spike

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$$\begin{aligned}\det(\mathbf{B}_N - x\mathbf{I}_N) &= \det(\mathbf{I}_N + \mathbf{P})\det(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N + x[\mathbf{I}_N - (\mathbf{I}_N + \mathbf{P})^{-1}]) \\ &= \det(\mathbf{I}_N + \mathbf{P})\det(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}\det(\mathbf{I}_N + x\mathbf{P}(\mathbf{I}_N + \mathbf{P})^{-1}(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}).\end{aligned}$$

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- due to unitary invariance of $\mathbf{X}$,

  $$\mathbf{U}^{\mathsf{H}}(\mathbf{X}\mathbf{X}^{\mathsf{H}} - x\mathbf{I}_N)^{-1}\mathbf{U} \xrightarrow{\text{a.s.}} \int (t-x)^{-1} dF^{MP}(t)\mathbf{I}_r \triangleq m(x)\mathbf{I}_r$$

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  with F^{MP} the MP law, and $m(x)$ the Stieltjes transform of the MP law (often known for $r = 1$ as trace lemma).
- finally, we have that the *limiting* solutions x_k satisfy $x_k m(x_k) + (1+\omega_k)\omega_k^{-1} = 0$.
- replacing $m(x)$, this is finally:

$$\lambda_k \xrightarrow{\text{a.s.}} x_k \triangleq 1 + \omega_k + c(1+\omega_k)\omega_k^{-1}, \textit{ if } \omega_k > \sqrt{c}$$

Comments on the result

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- there exists a "phase transition" when the largest population eigenvalues move from inside to outside $(0, 1 + \sqrt{c})$.
- more importantly, for $t_1 < 1 + \sqrt{c}$, we still have the same Tracy-Widom,
  - no way to see the spike even when zooming in
  - in fact, simulation suggests that convergence rate to the Tracy-Widom is slower with spikes.

Outline

Stieltjes transform inversion for covariance matrix models

J. W. Silverstein, S. Choi, "Analysis of the limiting spectral distribution of large dimensional random matrices," Journal of Multivariate Analysis, vol. 54, no. 2, pp. 295-309, 1995.

- We know for the model $\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N$, $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ that, if $F^{\mathbf{C}_N} \Rightarrow F^C$, the Stieltjes transform of the e.s.d. of $\underline{\mathbf{B}}_N = \frac{1}{n}\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N\mathbf{X}_N$ satisfies $m_{\underline{\mathbf{B}}_N}(z) \xrightarrow{\text{a.s.}} m_{\underline{F}}(z)$, with

$$m_{\underline{F}}(z) = \left(-z - c\int \frac{t}{1+tm_{\underline{F}}(z)}dF^C(t)\right)^{-1}$$

which is unique on the set $\{z \in \mathbb{C}^+, m_{\underline{F}}(z) \in \mathbb{C}^+\}$.

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  which is unique on the set $\{z \in \mathbb{C}^+, m_{\underline{F}}(z) \in \mathbb{C}^+\}$.
- This can be inverted into

$$z_{\underline{F}}(m) = -\frac{1}{m} - c\int \frac{t}{1 + tm} dF^C(t)$$

  for $m \in \mathbb{C}^+$.

Stieltjes transform inversion and spectrum characterization

Remember that we can evaluate the spectrum density by taking a complex line close to $\mathbb{R}$ and evaluating $\Im[m_{\underline{F}}(z)]$ along this line. Now we can do better.

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It is shown that

$$\lim_{\substack{z \to x \in \mathbb{R}^* \\ z \in \mathbb{C}^+}} m_{\underline{F}}(z) = m_0(x) \quad \text{exists.}$$

We also have,

- for x_0 inside the support, the density $\underline{f}(x)$ of $\underline{F}$ in x_0 is $\frac{1}{\pi}\Im[m_0]$ with m_0 the unique solution $m \in \mathbb{C}^+$ of

$$[z_{\underline{F}}(m) =] \; x_0 = -\frac{1}{m} - c\int \frac{t}{1+tm} dF^C(t)$$

- let $m_0 \in \mathbb{R}^*$ and $x_{\underline{F}}$ the equivalent to $z_{\underline{F}}$ on the real line. Then "x_0 outside the support of $\underline{F}$" is equivalent to "$x'_{\underline{F}}(m_{\underline{F}}(x_0)) > 0$, $m_{\underline{F}}(x_0) \neq 0$, $-1/m_{\underline{F}}(x_0)$ outside the support of F^C".

Stieltjes transform inversion and spectrum characterization

Remember that we can evaluate the spectrum density by taking a complex line close to $\mathbb{R}$ and evaluating $\Im[m_{\underline{F}}(z)]$ along this line. Now we can do better.

It is shown that

$$\lim_{\substack{z \to x \in \mathbb{R}^* \\ z \in \mathbb{C}^+}} m_{\underline{F}}(z) = m_0(x) \quad \text{exists.}$$

We also have,

- for x_0 inside the support, the density $\underline{f}(x)$ of $\underline{F}$ in x_0 is $\frac{1}{\pi}\Im[m_0]$ with m_0 the unique solution $m \in \mathbb{C}^+$ of
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This provides another way to determine the support!. For $m \in (-\infty, 0)$, evaluate $x_{\underline{F}}(m)$. Whenever $x_{\underline{F}}$ decreases, the image is outside the support. The rest is inside.

Another way to determine the spectrum: spectrum to analyze

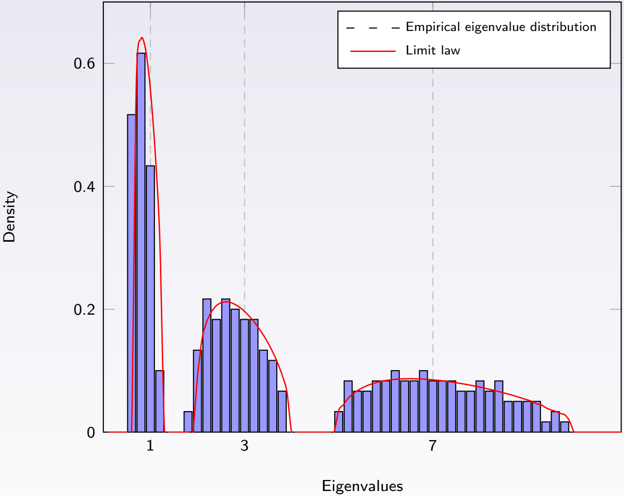

Figure: Histogram of the eigenvalues of $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N^{\frac{1}{2}}$, $N = 300$, $n = 3000$, with $\mathbf{C}_N$ diagonal composed of three evenly weighted masses in 1, 3 and 7.

Another way to determine the spectrum: inverse function method

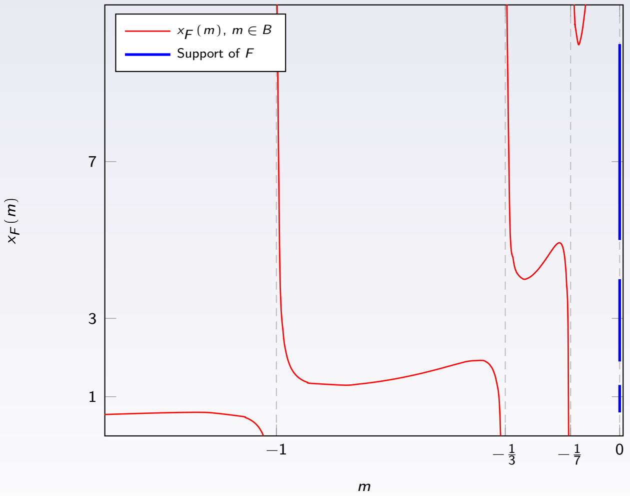

Figure: Stieltjes transform of $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N^{\frac{1}{2}}$, $N = 300$, $n = 3000$, with $\mathbf{C}_N$ diagonal composed of three evenly weighted masses in 1, 3 and 7. The support of F is read on the vertical axis, whenever m_F is decreasing.

Cluster boundaries in sample covariance matrix models

Xavier Mestre, "Improved estimation of eigenvalues of covariance matrices and their associated subspaces using their sample estimates," IEEE Transactions on Information Theory, vol. 54, no. 11, Nov. 2008.

Theorem

Let $\mathbf{X}_N \in \mathbb{C}^{N\times n}$ have i.i.d. entries of zero mean, unit variance, and $\mathbf{C}_N$ be diagonal such that $F^{\mathbf{C}_N} \Rightarrow F^C$, as $n, N \to \infty$, $N/n \to c$, where F^C has K masses in $t_1, \ldots, t_K$ with multiplicity $n_1, \ldots, n_K$ respectively. Then the l.s.d. of $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N^{\frac{1}{2}}$ has support $\mathcal{S}$ given by

$$\mathcal{S} = [x_1^-, x_1^+] \cup [x_2^-, x_2^+] \cup \ldots \cup [x_Q^-, x_Q^+]$$

with $x_q^- = x_F(m_q^-)$, $x_q^+ = x_F(m_q^+)$, and

$$x_F(m) = -\frac{1}{m} - c\frac{1}{n}\sum_{k=1}^{K} n_k \frac{t_k}{1 + t_k m}$$

with $2Q$ the number of real-valued solutions counting multiplicities of $x_F'(m) = 0$ denoted in order $m_1^- < m_1^+ \leqslant m_2^- < m_2^+ \leqslant \ldots \leqslant m_Q^- < m_Q^+$.

Comments on spectrum characterization

Previous results allows to determine

- the spectrum boundaries
- the number Q of clusters
- as a consequence, the total separation $(Q = K)$ or not $(Q < K)$ of the spectrum in K clusters.

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Mestre goes further: to determine local separability of the spectrum,

- identify the K inflexion points, i.e. the K solutions $m_1, \ldots, m_K$ to

$$x_F''(m) = 0$$

- check whether $x_F'(m_i) > 0$ and $x_F'(m_{i+1}) > 0$
- if so, the cluster in between corresponds to a single population eigenvalue.

Exact eigenvalue separation

Z. D. Bai, J. W. Silverstein, "Exact Separation of Eigenvalues of Large Dimensional Sample Covariance Matrices," The Annals of Probability, vol. 27, no. 3, pp. 1536-1555, 1999.

- Recall that the result on "no eigenvalue outside the support"
  - says where eigenvalues are not to be found
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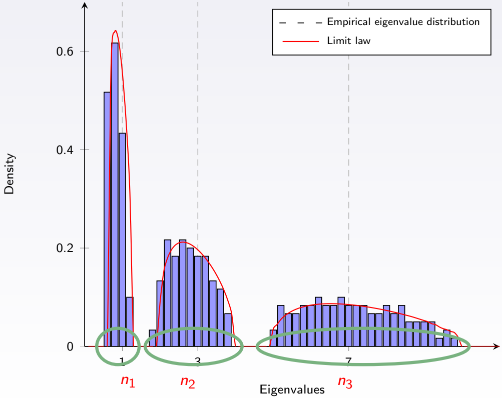
- This is in fact the case,



Eigeninference: Introduction of the problem

- *Reminder:* for a sequence $\mathbf{x}_1, \ldots, x_n \in \mathbb{C}^N$ of independent random variables,

$$\hat{\mathbf{C}}_N = \frac{1}{n} \sum_{k=1}^{n} \mathbf{x}_k \mathbf{x}_k^{\mathsf{H}}$$

is an n-consistent estimator of $\mathbf{C}_N = E[\mathbf{x}_1 \mathbf{x}_1^{\mathsf{H}}]$.

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- If n, N have comparable sizes, this no longer holds.
- Typically, n, N-consistent estimators of the full $\mathbf{C}_N$ matrix perform very badly.
- If only the eigenvalues of $\mathbf{C}_N$ are of interest, things can be done. The process of retrieving information about eigenvalues, eigenspace projections, or functional of these is called eigen-inference.

Girko and the G-estimators

V. Girko, "Ten years of general statistical analysis,"
http://www.general-statistical-analysis.girko.freewebspace.com/chapter14.pdf

- Girko has come up with more than 50 N, n-consistent estimators, called G-estimators (Generalized estimators). Among those, we find
  - *G_1-estimator of generalized variance*. For

$$G_1(\hat{\mathbf{C}}_N) = \alpha_n^{-1} \left[\log \det(\mathbf{C}_N) + \log \frac{n(n-1)^N}{(n-N)\prod_{k=1}^{N}(n-k)} \right]$$

with α_n any sequence such that $\alpha_n^{-2} \log(n/(n-N)) \to 0$, we have

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- However, Girko's proofs are rarely readable, if existent.

A long standing problem

X. Mestre, "Improved estimation of eigenvalues and eigenvectors of covariance matrices using their sample estimates," IEEE trans. on Information Theory, vol. 54, no. 11, pp. 5113-5129, 2008.

- Consider the model $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathrm{H}}\mathbf{C}_N^{\frac{1}{2}}$, where $F^{\mathbf{C}_N}$ is formed of a finite number of masses $t_1, \ldots, t_K$.
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- Only trials were iterative convex optimization methods.
- The problem was partially solved by Mestre in 2008!
- His technique uses elegant complex analysis tools. The description of this technique is the subject of this course.

Reminders

- Consider the sample covariance matrix model $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N^{\frac{1}{2}}$.
- Up to now, we saw:
  - that there is no eigenvalue outside the support with probability 1 for all large N.
  - that for all large N, when the spectrum is divided into clusters, the number of empirical eigenvalues in each cluster is exactly as we expect.

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- Consider the sample covariance matrix model $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathrm{H}}\mathbf{C}_N^{\frac{1}{2}}$.
- Up to now, we saw:
  - that there is no eigenvalue outside the support with probability 1 for all large N.
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- these results are of crucial importance for the following.

Eigen-inference for the sample covariance matrix model

X. Mestre, "Improved estimation of eigenvalues and eigenvectors of covariance matrices using their sample estimates," IEEE trans. on Information Theory, vol. 54, no. 11, pp. 5113-5129, 2008.

Theorem

Consider the model $\mathbf{B}_N = \frac{1}{n}\mathbf{C}_N^{\frac{1}{2}}\mathbf{X}_N\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N^{\frac{1}{2}}$*, with* $\mathbf{X}_N \in \mathbb{C}^{N\times n}$*, i.i.d. with entries of zero mean, unit variance, and* $\mathbf{C}_N \in \mathbb{R}^{N\times N}$ *is diagonal with* K *distinct entries* $t_1,\ldots,t_K$ *of multiplicity* $N_1,\ldots,N_K$ *of same order as* n*. Let* $k \in \{1,\ldots,K\}$*. Then, if the cluster associated to* t_k *is separated from the clusters associated to* $k-1$ *and* $k+1$*, as* $N,n\to\infty$*,* $N/n\to c$*,*

$$\hat{t}_k = \frac{n}{N_k}\sum_{m\in\mathcal{N}_k}(\lambda_m - \mu_m)$$

is an N,n*-consistent estimator of* t_k*, where* $\mathcal{N}_k = \{N-\sum_{i=k}^K N_i + 1,\ldots,N-\sum_{i=k+1}^K N_i\}$*,* $\lambda_1,\ldots,\lambda_N$ *are the eigenvalues of* $\mathbf{B}_N$ *and* $\mu_1,\ldots,\mu_N$ *are the* N *solutions of*

$$\underline{m}_{\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N\mathbf{X}_N}(\mu) = 0$$

or equivalently, $\mu_1,\ldots,\mu_N$ *are the eigenvalues of* $\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$*.*

Remarks on Mestre's result

Assuming cluster separation, the result consists in

- taking the empirical *ordered* λ_i's inside the cluster (note that exact separation ensures there are N_k of these!)
- getting the *ordered* eigenvalues $\mu_1, \ldots, \mu_N$ of

$$\mathrm{diag}(\boldsymbol{\lambda}) - \frac{1}{N}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$$

  with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)^{\mathsf{T}}$. Keep only those of index inside $\mathcal{N}_k$.
- take the difference and scale.

How to obtain this result?

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- Silverstein's Stieltjes transform identity: for the *conjugate* model $\underline{\mathbf{B}}_N = \frac{1}{n}\mathbf{X}_N^{\mathsf{H}}\mathbf{C}_N\mathbf{X}_N$,

$$\underline{m}_N(z) = \left(-z - c\int \frac{t}{1+t\underline{m}_N(z)} dF^{\mathbf{C}_N}(t)\right)^{-1}$$

with $\underline{m}_N$ the deterministic equivalent of $m_{\underline{\mathbf{B}}_N}$. This is the only random matrix result we need.

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  with $\underline{m}_N$ the deterministic equivalent of $m_{\underline{\mathbf{B}}_N}$. This is the only random matrix result we need.
- Before going further, we need some reminders from complex analysis.

Limiting spectrum of the sample covariance matrix

J. W. Silverstein, Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," J. of Multivariate Analysis, vol. 54, no. 2, pp. 175-192, 1995.

Reminder:

- If $F^{\mathbf{C}_N} \Rightarrow F^C$, then $m_{\mathbf{B}_N}(z) \xrightarrow{\text{a.s.}} m_F(z)$ such that

$$m_{\underline{F}}(z) = \left(c \int \frac{t}{1 + t m_{\underline{F}}(z)} dF^C(t) - z \right)^{-1}$$

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  or equivalently

$$m_{F^C}\left(-1/m_{\underline{F}}(z)\right) = -z m_{\underline{F}}(z) m_F(z)$$

  with $m_{\underline{F}}(z) = c m_F(z) + (c-1)\frac{1}{z}$ and $N/n \to c$.

Reminders of complex analysis

- Cauchy integration formula

Theorem

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be holomorphic on U. Let $\gamma \subset U$ be a continuous contour (i.e. closed path). Then, for a **inside** *the surface formed by γ, we have*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}dz = f(a)$$

while for a **outside** *the surface formed by γ,*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}dz = 0.$$

Complex integration

- From Cauchy integral formula, denoting $\mathcal{C}_k$ a contour enclosing **only** t_k,

$$t_k = \frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{\omega}{\omega - t_k} d\omega$$

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- After the variable change $\omega = -1/m_{\underline{F}}(z)$,

$$t_k = \frac{N}{N_k}\frac{1}{2\pi i}\oint_{\mathcal{C}_{\underline{F},k}} z m_F(z) \frac{m_{\underline{F}}'(z)}{m_{\underline{F}}^2(z)} dz,$$

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- When the system dimensions are large,

$$m_F(z) \simeq m_{\mathbf{B}_N}(z) \triangleq \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\lambda_k - z}, \quad \text{with} \quad (\lambda_1, \dots, \lambda_N) = \text{eig}(\mathbf{B}_N) = \text{eig}(\mathbf{YY}^{\mathsf{H}}).$$

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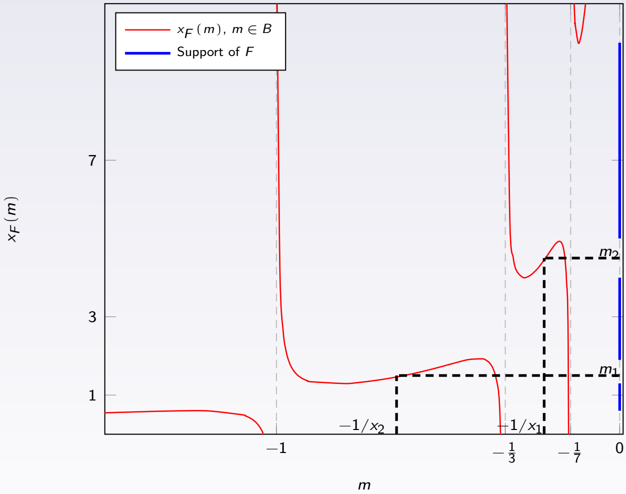
- Dominated convergence arguments then show

$$t_k - \hat{t}_k \xrightarrow{\text{a.s.}} 0 \quad \text{with} \quad \hat{t}_k = \frac{N}{N_k}\frac{1}{2\pi i}\oint_{\mathcal{C}_{\underline{F},k}} z m_{\mathbf{B}_N}(z) \frac{m'_{\underline{\mathbf{B}}_N}(z)}{m^2_{\underline{\mathbf{B}}_N}(z)} dz$$

Understanding the contour change



- **IF** $\mathcal{C}_{\underline{F},k}$ encloses cluster k with real points $m_1 < m_2$
- **THEN** $-1/m_1 = x_1 < t_k < x_2 = -1/m_2$ and $\mathcal{C}_k$ encloses t_k.

Poles and residues

- we find two sets of poles (outside zeros):
  - $\lambda_1, \ldots, \lambda_N$, the eigenvalues of $\mathbf{B}_N$.
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- residue calculus, denote $f(w) = \left(\frac{n}{N} w m_{\underline{\mathbf{B}}_N}(w) + \frac{n-N}{N}\right) \frac{m'_{\underline{\mathbf{B}}_N}(w)}{m_{\underline{\mathbf{B}}_N}(w)^2}$,
  - the λ_k's are poles of order 1 and

$$\lim_{z \to \lambda_k} (z - \lambda_k) f(z) = -\frac{n}{N}\lambda_k$$

  - the μ_k's are also poles of order 1 and by L'Hospital's rule

$$\lim_{z \to \mu_k} (z - \lambda_k) f(z) = \lim_{z \to \mu_k} \frac{n}{N} \frac{(z - \mu_k) z m'_{\underline{\mathbf{B}}_N}(z)}{m_{\underline{\mathbf{B}}_N}(z)} = \frac{n}{N}\mu_k$$

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- remember that

$$m_{\mathbf{B}_N}(w) = \frac{n}{N} m_{\underline{\mathbf{B}}_N}(w) + \frac{n-N}{N}\frac{1}{w}$$

- residue calculus, denote $f(w) = \left(\frac{n}{N} w m_{\underline{\mathbf{B}}_N}(w) + \frac{n-N}{N}\right) \frac{m'_{\underline{\mathbf{B}}_N}(w)}{m_{\underline{\mathbf{B}}_N}(w)^2}$,
  - the λ_k's are poles of order 1 and

$$\lim_{z \to \lambda_k} (z - \lambda_k) f(z) = -\frac{n}{N}\lambda_k$$

  - the μ_k's are also poles of order 1 and by L'Hospital's rule

$$\lim_{z \to \mu_k} (z - \lambda_k) f(z) = \lim_{z \to \mu_k} \frac{n}{N} \frac{(z - \mu_k) z m'_{\underline{\mathbf{B}}_N}(z)}{m_{\underline{\mathbf{B}}_N}(z)} = \frac{n}{N}\mu_k$$

  - So, finally

$$\hat{t}_k = \frac{n}{N_k} \sum_{m \in \text{contour}} (\lambda_m - \mu_m)$$

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- what about μ_1? the trick is to use the fact that

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_k} \frac{1}{z} dz = 0$$

which leads to

$$\frac{1}{2\pi i} \oint_{\partial \Gamma_k} \frac{m'_{\underline{F}}(w)}{m_{\underline{F}}(w)^2} dw = 0$$

the empirical version of which is

$$\#\{i : \lambda_i \in \Gamma_k\} - \#\{i : \mu_i \in \Gamma_k\}$$

Since their difference tends to 0, there are as many λ_k's as μ_k's in the contour, hence μ_1 is asymptotically in the integration contour.

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Outline

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Problem formulation

- We want to test the hypothesis $\mathcal{H}_0$ against $\mathcal{H}_1$,

$$\mathbb{C}^{N\times n} \ni \mathbf{Y} = \begin{cases} \mathbf{h}\mathbf{x}^T + \sigma\mathbf{W} & \text{, information plus noise, hypothesis } \mathcal{H}_1 \\ \sigma\mathbf{W} & \text{, pure noise, hpothesis } \mathcal{H}_0 \end{cases}$$

with $\mathbf{h} \in \mathbb{C}^N$, $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{W} \in \mathbb{C}^{N\times n}$.

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  with $\mathbf{h} \in \mathbb{C}^N$, $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{W} \in \mathbb{C}^{N\times n}$.
- We assume no knowledge whatsoever but that $\mathbf{W}$ has i.i.d. (non-necessarily Gaussian) entries.

Exploiting the conditioning number

L. S. Cardoso, M. Debbah, P. Bianchi, J. Najim, "Cooperative spectrum sensing using random matrix theory," International Symposium on Wireless Pervasive Computing, pp. 334-338 , 2008.

- under either hypothesis,
  - if $\mathcal{H}_0$, for N large, we expect $F_{\mathbf{YY}^\mathsf{H}}$ close to the Marčenko-Pastur law, of support $[\sigma^2\left(1-\sqrt{c}\right)^2, \sigma^2\left(1+\sqrt{c}\right)^2]$.
  - if $\mathcal{H}_1$, if population spike more than $1+\sqrt{\frac{N}{n}}$, largest eigenvalue is further away.
- the conditioning number of $\mathbf{YY}^\mathsf{H}$ is therefore asymptotically, as $N, n \to \infty$, $N/n \to c$,
  - if $\mathcal{H}_0$,

$$\mathrm{cond}(\mathbf{Y}) \triangleq \frac{\lambda_{\max}}{\lambda_{\min}} \to \frac{\left(1-\sqrt{c}\right)^2}{\left(1+\sqrt{c}\right)^2}$$

  - if $\mathcal{H}_1$,

$$\mathrm{cond}(\mathbf{Y}) \to t_1 + \frac{ct_1}{t_1-1} > \frac{\left(1-\sqrt{c}\right)^2}{\left(1+\sqrt{c}\right)^2}$$

  with $t_1 = \sum_{k=1}^{N} |h_k|^2 + \sigma^2$

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  with $t_1 = \sum_{k=1}^{N} |h_k|^2 + \sigma^2$
- the conditioning number is independent of σ. We then have the decision criterion, whether or not σ is known,

$$\text{decide} \begin{cases} \mathcal{H}_0 : & \text{if } \mathrm{cond}(\mathbf{YY}^H) \leqslant \frac{\left(1-\sqrt{\frac{N}{n}}\right)^2}{\left(1+\sqrt{\frac{N}{n}}\right)^2} + \varepsilon \\ \mathcal{H}_1 : & \text{otherwise.} \end{cases}$$

for some security margin ε.

Comments on the method

- Advantages:
  - much simpler than finite size analysis
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Comments on the method

- Advantages:
  - much simpler than finite size analysis
  - ratio independent of σ, so σ needs not be known
- Drawbacks:
  - only stands for very large N (dimension N for which asymptotic results arise function of σ!)
  - *ad-hoc* method, does not rely on performance criterion.

Generalized likelihood ratio test

P. Bianchi, M. Debbah, M. Maida, J. Najim, "Performance of Statistical Tests for Source Detection using Random Matrix Theory," IEEE Trans. on Information Theory, vol. 57, no. 4, pp. 2400-2419, 2011.

- Alternative generalized likelihood ratio test (GLRT) decision criterion, i.e.

$$C(\mathbf{Y}) = \frac{\sup_{\sigma^2,\mathbf{h}} P_{\mathbf{Y}|\mathbf{h},\sigma^2}(\mathbf{Y},\mathbf{h},\sigma^2)}{\sup_{\sigma^2} P_{\mathbf{Y}|\sigma^2}(\mathbf{Y}|\sigma^2)}.$$

- Denote

$$T_N = \frac{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^H)}{\frac{1}{N}\operatorname{tr}\mathbf{Y}\mathbf{Y}^H}$$

To guarantee a maximum false alarm ratio of α,

$$\text{decide} \begin{cases} \mathcal{H}_1 : & \text{if } \left(1-\frac{1}{N}\right)^{(1-N)n} T_N^{-n} \left(1-\frac{\mathbf{T}_N}{N}\right)^{(1-N)n} > \xi_N \\ \mathcal{H}_0 : & \text{otherwise.} \end{cases}$$

for some threshold ξ_N that can be explicitly given as a function of α.

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  for some threshold ξ_N that can be explicitly given as a function of α.
- Optimal test with respect to GLR.
- Performs better than conditioning number test.

Performance comparison for unknown σ^2, P

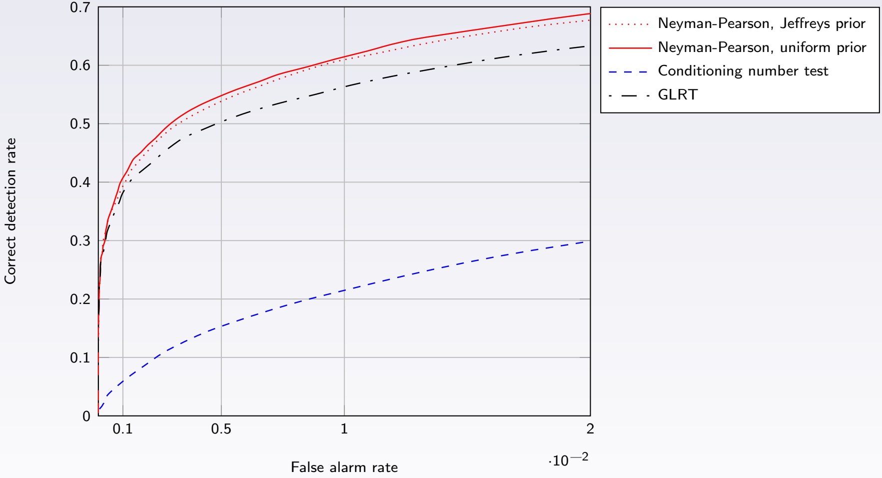

Figure: ROC curve for *a priori* unknown σ^2 of the Neyman-Pearson test, conditioning number method and GLRT, $K = 1$, $N = 4$, $M = 8$, $\mathrm{SNR} = 0 \ \mathrm{dB}$. For the Neyman-Pearson test, both uniform and Jeffreys prior, with exponent $\beta = 1$, are provided.

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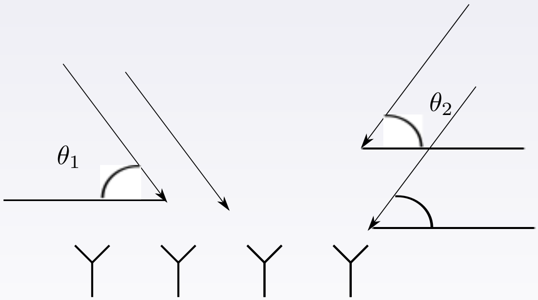
Outline

Source localization

A uniform array of M antennas receives signal from K radio sources during n signal snapshots.
Objective: Estimate the arrival angles $\theta_1, \cdots, \theta_K$.



Source Localization using Music Algorithm

We consider the scenario of K sources and N antenna-array capturing n observations:

$$\mathbf{x}_t = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_{k,t} + \sigma \mathbf{w}_t,\ t = 1, \cdots, n$$

- $\mathbf{A}_N = [\mathbf{a}_N(\theta_1), \cdots, \mathbf{a}_N(\theta_K)]$ with $\mathbf{a}_N(\theta) = \begin{bmatrix} 1 \\ e^{\imath\pi \sin\theta} \\ \cdots \\ e^{\imath(N-1)\pi\sin\theta} \end{bmatrix}$
- σ^2 is the noise variance and is set 1 for simplicity,
- Objective: infer $\theta_1, \cdots, \theta_K$ from the n observations
- Let $\mathbf{X}_N = [\mathbf{x}_1, \cdots, \mathbf{x}_n]$, then,

$$\mathbf{X} = \mathbf{AS} + \mathbf{W} = [\mathbf{A} \quad \mathbf{I}_N] \begin{bmatrix} \mathbf{S} \\ \mathbf{W} \end{bmatrix}$$

- If K is finite while $n, N \to +\infty$, the model correponds to the spiked covariance model.
- MUSIC Algorithm: Let $\mathbf{\Pi}$ be the orthogonal projection matrix on the span of $\mathbf{AA}^*$ and $\mathbf{\Pi}^\perp = \mathbf{I}_N - \mathbf{\Pi}$ (orthogonal projector on the noise subspace). Angles $\theta_1, \cdots, \theta_K$ are the unique ones verifying

$$\eta(\theta) \triangleq \mathbf{a}_N(\theta)^* \mathbf{\Pi} \mathbf{a}_N(\theta) = 0$$

Traditional MUSIC algorithm

- Traditional MUSIC algorithm: Angles are estimated as local minima of:

$$\mathbf{a}_N(\theta)^* \hat{\mathbf{\Pi}} \mathbf{a}_N(\theta)$$

  where $\hat{\mathbf{\Pi}}$ is the orthogonal projection matrix on the eigenspace associated to the K largest eigenvalues of $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$
- It is well-known that this estimator is consistent when $n \to +\infty$ with K, N fixed,
- We consider the case of K finite $\longrightarrow$ spiked covariance model
- What happens when $n, N \to +\infty$?

Asymptotic behaviour of the traditional MUSIC (1)

→ We first need to understand the spectrum of $\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathrm{H}}$

- We know that the weak spectrum is the MP law
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→ Denote $\mathbf{P} = \mathbf{A}\mathbf{A}^{\mathsf{H}} = \mathbf{U}_S \boldsymbol{\Omega} \mathbf{U}_S^{\mathsf{H}}$, $\boldsymbol{\Omega} = \operatorname{diag}(\omega_1, \ldots, \omega_K)$, and $\mathbf{Z} = [\mathbf{S}^{\mathsf{T}}\ \mathbf{W}^{\mathsf{T}}]^{\mathsf{T}}$ to recover (up to one row) the generic spiked model

$$\mathbf{X} = (\mathbf{I}_N + \mathbf{P})^{\frac{1}{2}} \mathbf{Z}.$$

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- Reminder: If x eigenvalue of $\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}}$ with $x > (1 + \sqrt{c})^2$ (edge of MP law), for all large n,

$$x \triangleq \lambda_k \xrightarrow{\text{a.s.}} \rho_k \triangleq 1 + \omega_k + c(1 + \omega_k)\omega_k^{-1}, \ \textit{if} \ \ \omega_k > \sqrt{c}$$

  for some k.

Asymptotic behaviour of the traditional MUSIC (2)

→ Recall the MUSIC approach: we want to estimate

$$\eta(\theta) = \mathbf{a}(\theta)^{\mathsf{H}} \mathbf{U}_W \mathbf{U}_W^{\mathsf{H}} \mathbf{a}(\theta) \quad (\mathbf{U}_W \in \mathbb{C}^{N\times(N-K)} \text{ such that } \mathbf{U}_W^{\mathsf{H}} \mathbf{U}_S = 0)$$

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→ Instead of this quantity, we start with the study of

$$\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a}(\theta),\ k = 1,\ldots,K$$

with $\hat{\mathbf{u}}_1,\ldots,\hat{\mathbf{u}}_N$ the eigenvectors belonging to $\lambda_1 \geqslant \ldots \geqslant \lambda_N$.

→ To fall back on known RMT quantities, we use the Cauchy-integral:

$$\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a}(\theta) = -\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i}\mathbf{a}(\theta)^{\mathsf{H}}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\mathbf{a}(\theta)dz$$

with $\mathcal{C}_i$ a contour enclosing λ_i only.

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with $\mathcal{C}_i$ a contour enclosing λ_i only.

→ Woodbury's identity $(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ gives:

$$\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a} = \frac{-1}{2\pi\imath}\oint_{\mathcal{C}_i}\mathbf{a}^{\mathsf{H}}(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}\left(\frac{\mathbf{Z}\mathbf{Z}^{\mathsf{H}}}{n} - z\mathbf{I}_N\right)^{-1}(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}\mathbf{a}dz + \frac{1}{2\pi\imath}\oint_{\mathcal{C}_i}\hat{\mathbf{a}}_1^{\mathsf{H}}\widehat{\mathbf{H}}^{-1}\hat{\mathbf{a}}_2dz$$

where $\mathbf{P} = \mathbf{U}_S\boldsymbol{\Omega}\mathbf{U}_S^{\mathsf{H}}$, and

$$\begin{cases} \widehat{\mathbf{H}} & = \mathbf{I}_K + z\boldsymbol{\Omega}(\mathbf{I}_K + \boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\frac{1}{n}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{U}_S \\ \hat{\mathbf{a}}_1^{\mathsf{H}} & = z\mathbf{a}(\theta)^{\mathsf{H}}(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}(\frac{1}{n}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{U}_S \\ \hat{\mathbf{a}}_2 & = \boldsymbol{\Omega}(\mathbf{I}_K + \boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\frac{1}{n}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}(\mathbf{I}_N + \mathbf{P})^{-\frac{1}{2}}\mathbf{a}(\theta). \end{cases}$$

Asymptotic behaviour of the traditional MUSIC (3)

- For large n, the first term has no pole, while the second converges to

$$T_i \triangleq \frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \mathbf{a}_1^{\mathsf{H}}\mathbf{H}^{-1}\mathbf{a}_2 dz, \text{ with } \begin{cases} \mathbf{H} & = \mathbf{I}_K + zm(z)\boldsymbol{\Omega}(\mathbf{I}_K+\boldsymbol{\Omega})^{-1} \\ \mathbf{a}_1^{\mathsf{H}} & = zm(z)\mathbf{a}^*(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}\mathbf{U}_S \\ \mathbf{a}_2 & = m(z)\boldsymbol{\Omega}(\mathbf{I}_K+\boldsymbol{\Omega})^{-1}\mathbf{U}_S^{\mathsf{H}}(\mathbf{I}_N+\mathbf{P})^{-\frac{1}{2}}\mathbf{a} \end{cases}$$

which after development is

$$T_i = \sum_{\ell=1}^{K} \frac{1}{1+\omega_\ell}\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \frac{zm^2(z)}{\frac{1+\omega_\ell}{\omega_\ell}+zm(z)}dz.$$

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- Using residue calculus, the sole pole is in ρ_i and we find

$$\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{a}(\theta) \xrightarrow{\text{a.s.}} \frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-1}}\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{u}_i\mathbf{u}_i^{\mathsf{H}}\mathbf{a}(\theta).$$

Therefore,

$$\hat{\eta}(\theta) = \mathbf{a}(\theta)^{\mathsf{H}}\hat{\boldsymbol{\Pi}}\mathbf{a}(\theta) \xrightarrow{\text{a.s.}} \mathbf{a}(\theta)\mathbf{a}(\theta)^{\mathsf{H}} - \sum_{i=1}^{K}\frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-1}}\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{u}_i\mathbf{u}_i^{\mathsf{H}}\mathbf{a}(\theta)$$

Improved G-MUSIC

Recall that:

$$\mathbf{a}(\theta)^{\mathsf{H}}\mathbf{u}_k\mathbf{u}_k^{\mathsf{H}}\mathbf{a}(\theta) - \frac{1+c\omega_k^{-1}}{1-c\omega_k^{-2}}\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_k\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{a}(\theta) \xrightarrow{\text{a.s.}} 0$$

→ The ω_k are however unknown. But they can be estimated from

$$\lambda_k \xrightarrow{\text{a.s.}} \rho_k = 1 + \omega_k + c(1+\omega_k)\omega_k^{-1}$$

→ This gives finally

$$\hat{\eta}_G(\theta) \simeq \mathbf{a}(\theta)^{\mathsf{H}}\mathbf{a}(\theta) - \sum_{k=1}^{K} \frac{1+c\hat{\omega}_k^{-1}}{1-c\hat{\omega}_k^{-2}}\mathbf{a}(\theta)^{\mathsf{H}}\hat{\mathbf{u}}_k\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{a}(\theta)$$

with

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$$\hat{\omega}_k = \frac{\hat{\lambda}_k - (c+1)}{2} + \sqrt{(c+1-\hat{\lambda}_k)^2 - 4c)}$$

→ We then obtain another (N,n)-consistent MUSIC estimator, only valid for K finite!

Simulation results

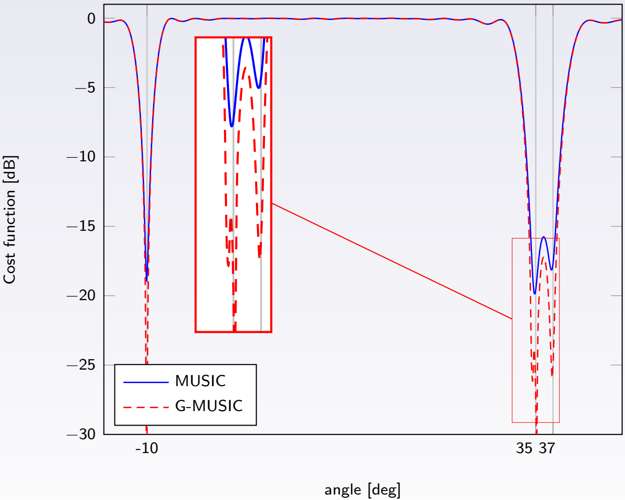

Figure: MUSIC against G-MUSIC for DoA detection of $K = 3$ signal sources, $N = 20$ sensors, $M = 150$ samples, SNR of 10 dB. Angles of arrival of $10°$, $35°$, and $37°$.

Outline of the tutorial

- **Part 1: Basics of Random Matrix Theory for Sample Covariance Matrices**
  - 1.1. Introduction to the Stieltjes transform method, Marčenko–Pastur law, advanced models
  - 1.2. Extreme eigenvalues: no eigenvalue outside the support, exact separation, Tracy–Widom law
  - 1.3. Extreme eigenvalues: the spiked models
  - 1.4. Spectrum analysis and G-estimation

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Covariance estimation and sample covariance matrices

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- The main reasons are:
  - Assuming $E[x] = 0$, $E[xx^*] = C_N$, with $X = [x_1, \ldots, x_n]$, by the LLN

$$\hat{S}_N \triangleq \frac{1}{n} XX^* \xrightarrow{\text{a.s.}} C_N \text{ as } n \to \infty.$$

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  - The SCM $\hat{S}_N$ is the ML estimate of C_N for Gaussian x
    $\to$ One therefore expects $\hat{\theta}$ to closely approximate θ for all finite n.
- This approach however has two limitations:
  - if N, n are of the same order of magnitude,

$$\|\hat{S}_N - C_N\| \not\to 0 \text{ as } N, n \to \infty, \ N/n \to c > 0, \text{ so that in general } |\hat{\theta} - \theta| \not\to 0$$

    $\to$ This motivated the introduction of G-estimators.
  - if x is not Gaussian, but has heavier tails, $\hat{S}_N$ is a poor estimator for C_N.
    $\to$ This motivated the introduction of robust estimators.

Reminders on robust estimation

J. T. Kent, D. E. Tyler, “Redescending M-estimates of multivariate location and scatter”, 1991.
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→ The objectives of robust estimators:

- Replace the SCM $\hat{S}_N$ by another estimate $\hat{C}_N$ of C_N which:
  - rejects (or downscales) observations deterministically
  - or rejects observations inconsistent with the full set of observations

→ **Example**: Huber estimator, $\hat{C}_N$ defined as solution of

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} \beta_i x_i x_i^* \text{ with } \beta_i = \alpha \min\left\{1, \frac{k^2}{\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i}\right\} \text{ for some } \alpha > 1, k^2 \text{ function of } \hat{C}_N.$$

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- Provide scale-free estimators of C_N:
  → **Example**: Tyler's estimator: if one observes $x_i = \tau_i z_i$ for unknown scalars τ_i,

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i} x_i x_i^*$$

  - existence and uniqueness of $\hat{C}_N$ defined up to a constant.
  - few constraints on $x_1, \ldots, x_n$ ($N+1$ of them must be linearly independent)

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  → **Example**: Maronna's estimator for elliptical x

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

with $u(s)$ such that

(i) $u(s)$ is continuous and non-increasing on $[0, \infty)$

(ii) $\phi(s) = su(s)$ is non-decreasing, bounded by $\phi_\infty > 1$. Moreover, $\phi(s)$ increases where $\phi(s) < \phi_\infty$.

(note that Huber's estimator is compliant with Maronna's estimators)

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  - existence is not too demanding
  - uniqueness imposes strictly increasing $u(s)$ (inconsistent with Huber's estimate)
  - consistency result: $\hat{C}_N \to C_N$ **if** $u(s)$ meets the ML estimator for C_N.

Robust Estimation and RMT

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Robust RMT estimation

Can we study the performance of estimators based on the $\hat{C}_N$?

- what are the spectral properties of $\hat{C}_N$?
- can we generate RMT-based estimators relying on $\hat{C}_N$?

Setting and assumptions

- **Assumptions:**
  - Take $x_1, \ldots, x_n \in \mathbb{C}^N$ "elliptical-like" random vectors, i.e. $x_i = \sqrt{\tau_i} C_N^{\frac{1}{2}} w_i$ where
    - $\tau_1, \ldots, \tau_n \in \mathbb{R}^+$ random or deterministic with $\frac{1}{n}\sum_{i=1}^n \tau_i \xrightarrow{\text{a.s.}} 1$
    - $w_1, \ldots, w_n \in \mathbb{C}^N$ random independent with $w_i/\sqrt{N}$ uniformly distributed over the unit-sphere
    - $C_N \in \mathbb{C}^{N\times N}$ deterministic, with $C_N \succ 0$ and $\limsup_N \|C_N\| < \infty$
  - We denote $c_N \triangleq N/n$ and consider the growth regime $c_N \to c \in (0,1)$.

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- **Maronna's estimator of scatter:** (almost sure) unique solution to

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^n u\left(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

where u satisfies

(i) $u : [0,\infty) \to (0,\infty)$ nonnegative continuous and non-increasing

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- **Additional technical assumption:** Let $\nu_n \triangleq \frac{1}{n}\sum_{i=1}^n \delta_{\tau_i}$. For each $a > b > 0$, a.s.

$$\limsup_{t\to\infty} \frac{\limsup_n \nu_n((t,\infty))}{\phi(at) - \phi(bt)} = 0.$$

$\to$ Controls relative speed of the tail of ν_n versus the flattening speed of $\phi(x)$ as $x \to \infty$.
Examples:
  - $\tau_i < M$ for each i. In this case, $\nu_n((t,\infty)) = 0$ a.s. for $t > M$.
  - For $u(t) = (1+\alpha)/(\alpha+t)$, $\alpha > 0$, and τ_i i.i.d., it is sufficient to have $E[\tau_1^{1+\varepsilon}] < \infty$.

Heuristic approach

- **Major issues with $\hat{C}_N$:**
  - Defined implicitly
  - Sum of non-independent rank-one matrices from vectors $\sqrt{u(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i)}x_i$ ($\hat{C}_N$ depends on all x_j's).

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- **But there is some hope:**
  - First remark: we can work with $C_N = I_N$ without generality restriction!
  - Denote

$$\hat{C}_{(j)} = \frac{1}{n}\sum_{i \neq j}^{n} u\left(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i\right) x_i x_i^*$$

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  - We expect in particular (highly non-rigorous but intuitive!!):

$$\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i \simeq \tau_i\frac{1}{N}\operatorname{tr}\hat{C}_{(i)}^{-1} \simeq \tau_i\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}.$$

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- **Our heuristic approach:**
  - Rewrite $\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i$ as $f(\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i)$ for some function f (later called g^{-1})
  - Deduce that

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n}(u\circ f)\left(\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i\right)x_ix_i^*$$

  - Use $\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i \simeq \tau_i\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}$ to get

$$\hat{C}_N \simeq \frac{1}{n}\sum_{i=1}^{n}(u\circ f)\left(\tau_i\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}\right)x_ix_i^*$$

  - Use random matrix results to find a limiting value γ for $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}$, and conclude

$$\hat{C}_N \simeq \frac{1}{n}\sum_{i=1}^{n}(u\circ f)(\tau_i\gamma)x_ix_i^*.$$

Heuristic approach in detail: f and γ

- **Determination of f:** Recall the identity $(A + tvv^*)^{-1}v = A^{-1}/(1 + tv^*A^{-1}v)$. Then

$$\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i = \frac{\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i}{1 + c_N u(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i)\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i}$$

so that

$$\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i = \frac{\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i}{1 - c_N\phi(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i)}.$$

Now the function $g : x \mapsto x/(1 - c_N\phi(x))$ is monotonous increasing (we use the assumption $\phi_\infty < c^{-1}$!), hence, with $f = g^{-1}$,

$$\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i = g^{-1}\left(\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i\right).$$

Heuristic approach in detail: f and γ

- **Determination of γ:** From previous calculus, we expect

$$\hat{C}_N \simeq \frac{1}{n}\sum_{i=1}^n (u\circ g^{-1})\left(\tau_i \frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}\right) x_i x_i^* \simeq \frac{1}{n}\sum_{i=1}^n (u\circ g^{-1})(\tau_i\gamma)\, x_i x_i^*.$$

Hence

$$\gamma \simeq \frac{1}{N}\operatorname{tr}\hat{C}_N^{-1} \simeq \frac{1}{N}\operatorname{tr}\left(\frac{1}{n}\sum_{i=1}^n (u\circ g^{-1})(\tau_i\gamma)\,\tau_i w_i w_i^*\right)^{-1}.$$

Since τ_i are independent of w_i and γ deterministic, this is a Bai-Silverstein model

$$\frac{1}{n}WDW^*,\ W=[w_1,\ldots,w_n],\ D=\operatorname{diag}(D_{ii})=u\circ g^{-1}(\tau_i\gamma).$$

And we have:

$$\gamma \simeq \frac{1}{N}\operatorname{tr}\left(\frac{1}{n}WDW^*\right)^{-1} = m_{\frac{1}{n}WDW^*}(0) \simeq \left(0+\int \frac{t(u\circ g^{-1})(t\gamma)}{1+c(u\circ g^{-1})(t\gamma)m_{\frac{1}{n}WDW^*}(0)}\nu_N(dt)\right)^{-1}$$

$$= \left(\frac{1}{n}\sum_{i=1}^n \frac{\tau_i(u\circ g^{-1})(\tau_i\gamma)}{1+c\tau_i(u\circ g^{-1})(\tau_i\gamma)m_{\frac{1}{n}WDW^*}(0)}\right)^{-1}.$$

Since $\gamma \simeq m_{\frac{1}{n}WDW^*}(0)$, this defines γ as a solution of a fixed-point equation:

$$\gamma = \left(\frac{1}{n}\sum_{i=1}^n \frac{\tau_i(u\circ g^{-1})(\tau_i\gamma)}{1+c\tau_i(u\circ g^{-1})(\tau_i\gamma)\gamma}\right)^{-1}.$$

Main result

R. Couillet, F. Pascal, J. W. Silverstein, “The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples”, (submitted to) Elsevier Journal of Multivariate Analysis.

Theorem (Asymptotic Equivalence)

Under the assumptions defined earlier, we have

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0, \text{ where } \hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^{n} v(\tau_i \gamma) x_i x_i^*$$

$v(x) = (u \circ g^{-1})(x)$, $\psi(x) = xv(x)$, $g(x) = x/(1 - c\phi(x))$ *and* $\gamma > 0$ *unique solution of*

$$1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\tau_i \gamma)}{1 + c\psi(\tau_i \gamma)}.$$

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- **Remarks:**
  - Th. says: first order substitution of $\hat{C}_N$ by $\hat{S}_N$ allowed for large N, n.
  - It turns out that $v \sim u$ and $\psi \sim \phi$ in general behavior.
  - Corollaries:

$$\max_{1 \leqslant i \leqslant n} \left| \lambda_i(\hat{S}_N) - \lambda_i(\hat{C}_N) \right| \xrightarrow{\text{a.s.}} 0$$

$$\frac{1}{N} \operatorname{tr} (\hat{C}_N - zI_N)^{-1} - \frac{1}{N} \operatorname{tr} (\hat{S}_N - zI_N)^{-1} \xrightarrow{\text{a.s.}} 0$$

$\longrightarrow$ Important feature for detection and estimation.

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- **Remarks:**
  - Th. says: first order substitution of $\hat{C}_N$ by $\hat{S}_N$ allowed for large N, n.
  - It turns out that $v \sim u$ and $\psi \sim \phi$ in general behavior.
  - Corollaries:

$$\max_{1 \leqslant i \leqslant n} \left| \lambda_i(\hat{S}_N) - \lambda_i(\hat{C}_N) \right| \xrightarrow{\text{a.s.}} 0$$

$$\frac{1}{N}\operatorname{tr}(\hat{C}_N - zI_N)^{-1} - \frac{1}{N}\operatorname{tr}(\hat{S}_N - zI_N)^{-1} \xrightarrow{\text{a.s.}} 0$$

  $\longrightarrow$ Important feature for detection and estimation.
- **Proof:** So far in the tutorial, we do not have a rigorous proof!

Proof

- **Fundamental idea:** Showing that all $\frac{1}{\tau_i}\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i$ converge to the same limit γ.

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and relabel terms such that

$$e_1 \leqslant \ldots \leqslant e_n$$

We shall prove that, for each $\ell > 0$,

$$e_1 > 1 - \ell \text{ i.o. and } e_n < 1 + \ell \text{ i.o.}$$

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- **Some basic inequalities:** Denoting $d_i \triangleq \frac{1}{\tau_i}\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i = \frac{1}{N}w_i^*\hat{C}_{(i)}^{-1}w_i$, we have

$$e_j = \frac{v\left(\tau_j\frac{1}{N}w_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v(\tau_i d_i)w_iw_i^*\right)^{-1}w_j\right)}{v(\tau_j\gamma)} = \frac{v\left(\tau_j\frac{1}{N}w_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v(\tau_i\gamma)e_iw_iw_i^*\right)^{-1}w_j\right)}{v(\tau_j\gamma)}$$

$$\leqslant \frac{v\left(\tau_j\frac{1}{N}w_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v(\tau_i\gamma)e_nw_iw_i^*\right)^{-1}w_j\right)}{v(\tau_j\gamma)} = \frac{v\left(\frac{\tau_j}{e_n}\frac{1}{N}w_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v(\tau_i\gamma)w_iw_i^*\right)^{-1}w_j\right)}{v(\tau_j\gamma)}$$

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- **Specialization to e_n:**

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or equivalently, recalling $\psi(x) = xv(x)$,

$$\frac{\frac{1}{N}w_n^*\left(\frac{1}{n}\sum_{i\neq n}\tau_i v(\tau_i\gamma)w_i w_i^*\right)^{-1}w_n}{\gamma} \leqslant \frac{\psi\left(\frac{\tau_n}{e_n}\frac{1}{N}w_n^*\left(\frac{1}{n}\sum_{i\neq n}\tau_i v(\tau_i\gamma)w_i w_i^*\right)^{-1}w_n\right)}{\psi(\tau_n\gamma)}.$$

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  - By trace lemma, we should have

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  - **Solution**: uniform convergence result.
    By (higher order) moment bounds, Markov inequality, and Borel Cantelli, for all large n a.s.

$$\max_{1\leqslant j\leqslant n}\left|\frac{1}{N}w_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v(\tau_i\gamma)w_iw_i^*\right)^{-1}w_j - \gamma\right| < \varepsilon.$$

Proof

- **Back to original problem**: For all large n a.s., we then have (using growth of ψ)

$$\frac{\gamma - \varepsilon}{\gamma} \leqslant \frac{\psi\left(\frac{\tau_n}{e_n}(\gamma + \varepsilon)\right)}{\psi(\tau_n \gamma)}.$$

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  - **Bounded support for τ_i:** If $0 < \tau_- < \tau_i < \tau_+ < \infty$ for all i, n, then on a subsequence where $\tau_n \to \tau_0$,

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  - **Unbounded support for τ_i:** Importance of relative growth of τ_n versus convergence of ψ to ψ_∞.
Proof consists in dividing $\{\tau_i\}$ in two groups: few large ones versus all others.
Sufficient condition:

$$\limsup_{t\to\infty} \frac{\limsup_n \nu_n((t,\infty))}{\phi(at) - \phi(bt)} = 0.$$

Simulations

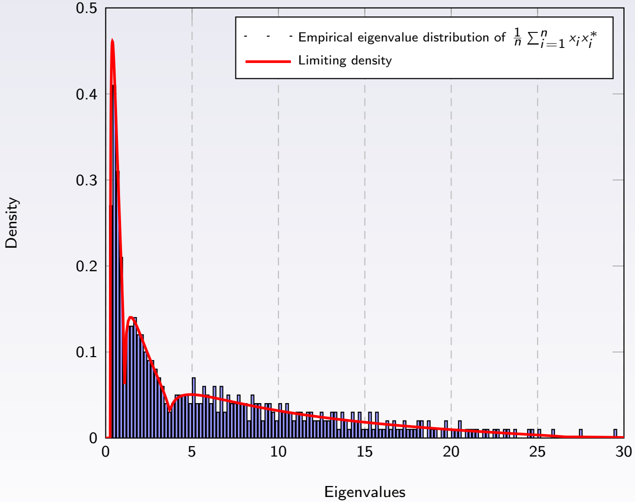

Figure: Histogram of the eigenvalues of $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$ for $n = 2500$, $N = 500$, $C_N = \mathrm{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

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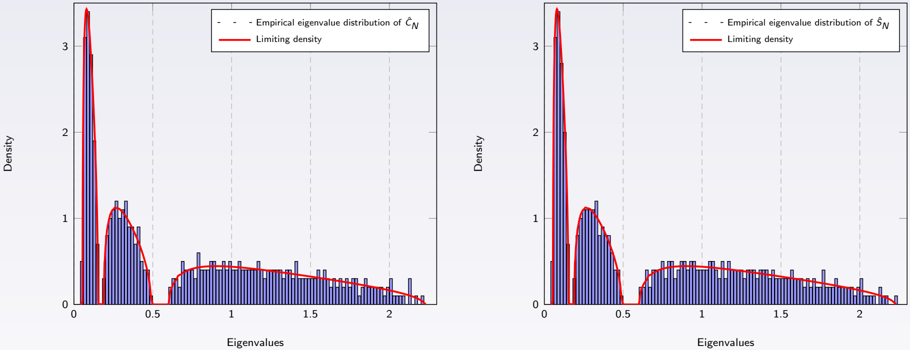

Figure: Histogram of the eigenvalues of $\hat{C}_N$ (left) and $\hat{S}_N$ (right) for $n = 2500$, $N = 500$, $C_N = \mathrm{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

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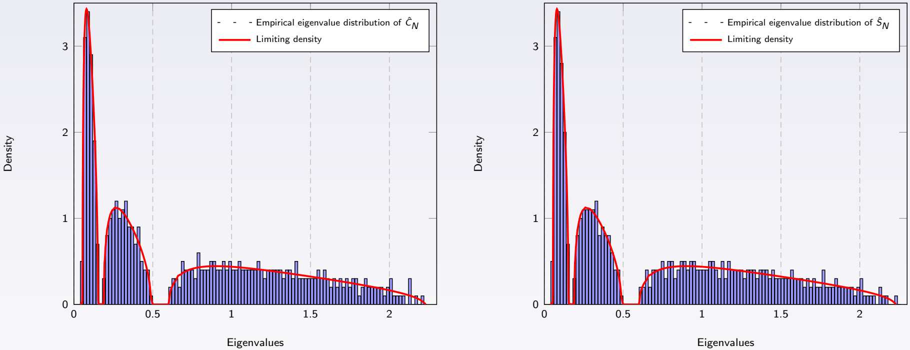

Figure: Histogram of the eigenvalues of $\hat{C}_N$ (left) and $\hat{S}_N$ (right) for $n = 2500$, $N = 500$, $C_N = \mathrm{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

- **Remark/Corollary:** Spectrum of $\hat{C}_N$ a.s. bounded uniformly on n.

Hint on potential applications

- **Spectrum boundedness:** for impulsive noise scenarios,
  - SCM spectrum grows unbounded
  - robust scatter estimator spectrum remains bounded

$\Rightarrow$ Robust estimators improve spectrum separability (important for many statistical inference techniques seen previously)

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- **Application scenarios:**
  - Radar detection in impulsive noise (non-Gaussian noise, possibly clutter)
  - Financial data analytics
  - Any application where Gaussianity is too strong an assumption...

Outline

System Setting

- **Signal model:**

$$y_i = \sum_{l=1}^{L} \sqrt{p_l} a_l s_{li} + \sqrt{\tau_i} w_i = A_i \bar{w}_i$$

$$A_i \triangleq \begin{bmatrix} \sqrt{p_1} a_1 & \ldots & \sqrt{p_L} a_L & \sqrt{\tau_i} I_N \end{bmatrix}, \quad \bar{w}_i \triangleq [s_{1i}, \ldots, s_{Li}, w_i]^{\mathsf{T}}.$$

with $y_1, \ldots, y_n \in \mathbb{C}^N$ satisfying:

1. $\tau_1, \ldots, \tau_n > 0$ random such that $\nu_n \triangleq \frac{1}{n} \sum_{i=1}^{n} \delta_{\tau_i} \to \nu$ weakly and $\int t\nu(dt) = 1$;
2. $w_1, \ldots, w_n \in \mathbb{C}^N$ random independent unitarily invariant $\sqrt{N}$-norm;
3. $L \in \mathbb{N}$, $p_1 \geqslant \ldots \geqslant p_L \geqslant 0$ deterministic;
4. $a_1, \ldots, a_L \in \mathbb{C}^N$ deterministic or random with $A^* A \xrightarrow{\text{a.s.}} \operatorname{diag}(p_1, \ldots, p_L)$ as $N \to \infty$, with $A \triangleq [\sqrt{p_1} a_1, \ldots, \sqrt{p_L} a_L] \in \mathbb{C}^{N \times L}$.
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  $\Rightarrow$ Elliptical model with covariance a low-rank (L) perturbation of I_N.
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- **Application contexts:**
  - *wireless communications:* signals s_{li} from L transmitters, N-antenna receiver; a_l random i.i.d. channels ($a_l^* a_{l'} \to \delta_{l-l'}$, e.g. $a_l \sim \mathcal{CN}(0, I_N/N)$);
  - *array processing:* L sources emit signals s_{li} at steering angle $a_l = a(\theta_l)$. For ULA,

$$[a(\theta)]_j = N^{-\frac{1}{2}} \exp(2\pi\imath dj \sin(\theta)).$$

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- **Signal detection/estimation in impulsive environments:** Two scenarios
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    $\Rightarrow$ False alarms induced by noise impulses!
- **Our results:** In a spiked model with noise impulsions,
  - whatever noise impulsion type, spectrum of $\hat{C}_N$ remains bounded
  - isolated largest eigenvalues may appear, two classes:
    - isolated eigenvalues due to noise impulses CANNOT exceed a threshold!
    - all isolated eigenvalues beyond this threshold are due to signal

    $\Rightarrow$ Detection criterion: everything above threshold is signal.

Theoretical results

Theorem (Extension to spiked robust model)

Under the same assumptions as in previous section,

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n}\sum_{i=1}^{n} v(\tau_i\gamma)A_i\bar{w}_i\bar{w}_i^* A_i^*$$

with γ the unique solution to

$$1 = \int \frac{\psi(t\gamma)}{1 + c\psi(t\gamma)}\nu(dt)$$

and we recall

$$A_i \triangleq \begin{bmatrix} \sqrt{p_1}a_1 & \dots & \sqrt{p_L}a_L & \sqrt{\tau_i}I_N \end{bmatrix}$$
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- **Remark:** For $L = 0$, $A_i = [0, \dots, 0, I_N]$.
  $\Rightarrow$ Recover previous result $A_i\bar{w}_i$ becomes w_i.

Localization of eigenvalues

Theorem (Eigenvalue localization)

Denote

- u_k *eigenvector of k-th largest eigenvalue of* $AA^* = \sum_{i=1}^{L} p_i a_i a_i^*$
- $\hat{u}_k$ *eigenvector of k-th largest eigenvalue of* $\hat{C}_N$

Also define $\delta(x)$ *unique positive solution to*

$$\delta(x) = c\left(-x + \int \frac{tv_c(t\gamma)}{1+\delta(x)tv_c(t\gamma)}\nu(dt)\right)^{-1}.$$

Further denote

$$p_- \triangleq \lim_{x \downarrow S^+} -c\left(\int \frac{\delta(x)v_c(t\gamma)}{1+\delta(x)tv_c(t\gamma)}\nu(dt)\right)^{-1}, \quad S^+ \triangleq \frac{\phi_\infty(1+\sqrt{c})^2}{\gamma(1-c\phi_\infty)}.$$

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Then, if $p_j > p_-$, $\hat{\lambda}_j \xrightarrow{\text{a.s.}} \Lambda_j > S^+$, *otherwise* $\limsup_n \hat{\lambda}_j \leqslant S^+$ *a.s., with* Λ_j *unique positive solution to*

$$-c\left(\delta(\Lambda_j) \int \frac{v_c(\tau\gamma)}{1+\delta(\Lambda_j)\tau v_c(\tau\gamma)} \nu(d\tau)\right)^{-1} = p_j.$$

Simulation

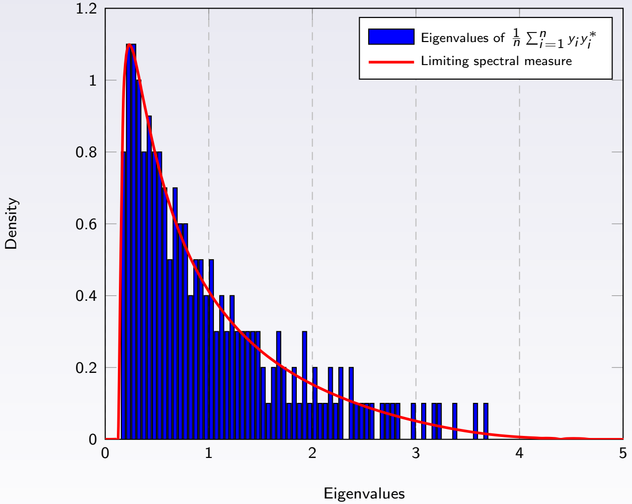

Figure: Histogram of the eigenvalues of $\frac{1}{n}\sum_i y_i y_i^*$ against the limiting spectral measure, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Sudent-t impulsions.

Simulation

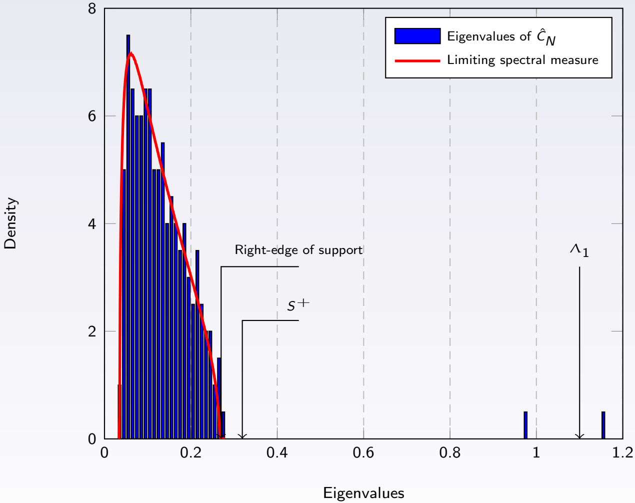

Figure: Histogram of the eigenvalues of $\hat{C}_N$ against the limiting spectral measure, for $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Student-t impulsions.

Comments

- **SCM vs robust:** Spikes invisible in SCM in impulsive noise, reborn in robust estimate of scatter.

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- **SCM vs robust:** Spikes invisible in SCM in impulsive noise, reborn in robust estimate of scatter.
- **Largest eigenvalues:**
  - $\lambda_i(\hat{C}_N) > S^+ \Rightarrow$ Presence of a source!
  - $\lambda_i(\hat{C}_N) \in (\sup(\text{Support}), S^+) \Rightarrow$ May be due to a source or to a noise impulse.
  - $\lambda_i(\hat{C}_N) < \sup(\text{Support}) \Rightarrow$ As usual, nothing can be said.

$\Rightarrow$ Induces a natural source detection algorithm.

Eigenvalue and eigenvector projection estimates

- **Two scenarios:**
  - known $\nu = \lim_n \frac{1}{n} \sum_{i=1}^{n} \delta_{\tau_i}$
  - unknown ν

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- **Two scenarios:**
  - known $\nu = \lim_n \frac{1}{n} \sum_{i=1}^{n} \delta_{\tau_i}$
  - unknown ν

Theorem (Estimation under known ν)

1. Power estimation. *For each $p_j > p_-$,*

$$-c\left(\delta(\hat{\lambda}_j)\int \frac{v_c(\tau\gamma)}{1+\delta(\hat{\lambda}_j)\tau v_c(\tau\gamma)}\nu(d\tau)\right)^{-1} \xrightarrow{\text{a.s.}} p_j.$$

2. Bilinear form estimation. *For each $a,b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$, and $p_j > p_-$*

$$\sum_{k,p_k=p_j} a^* u_k u_k^* b - \sum_{k,p_k=p_j} w_k a^* \hat{u}_k \hat{u}_k^* b \xrightarrow{\text{a.s.}} 0$$

where

$$w_k = \frac{\displaystyle\int \frac{v_c(t\gamma)}{\left(1+\delta(\hat{\lambda}_k)tv_c(t\gamma)\right)^2}\nu(dt)}{\displaystyle\int \frac{v_c(t\gamma)}{1+\delta(\hat{\lambda}_k)tv_c(t\gamma)}\nu(dt)\left(1-\frac{1}{c}\int \frac{\delta(\hat{\lambda}_k)^2 t^2 v_c(t\gamma)^2}{\left(1+\delta(\hat{\lambda}_k)tv_c(t\gamma)\right)^2}\nu(dt)\right)}.$$

Eigenvalue and eigenvector projection estimates

Theorem (Estimation under unknown ν)

1. Purely empirical power estimation. *For each* $p_j > p_-$,

$$-\left(\hat{\delta}(\hat{\lambda}_j)\frac{1}{N}\sum_{i=1}^{n}\frac{v(\hat{\tau}_i\hat{\gamma}_n)}{1+\hat{\delta}(\hat{\lambda}_j)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma}_n)}\right)^{-1} \xrightarrow{\text{a.s.}} p_j.$$

2. Purely empirical bilinear form estimation. *For each* $a,b\in\mathbb{C}^N$ *with* $\|a\|=\|b\|=1$, *and each* $p_j > p_-$,

$$\sum_{k,p_k=p_j} a^* u_k u_k^* b - \sum_{k,p_k=p_j} \hat{w}_k a^* \hat{u}_k \hat{u}_k^* b \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{w}_k = \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{v(\hat{\tau}_i\hat{\gamma})}{\left(1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma})\right)^2}}{\frac{1}{n}\sum_{i=1}^{n}\frac{v(\hat{\tau}_i\hat{\gamma})}{1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma})}\left(1-\frac{1}{N}\sum_{i=1}^{n}\frac{\hat{\delta}(\hat{\lambda}_k)^2\hat{\tau}_i^2 v(\hat{\tau}_i\hat{\gamma})^2}{\left(1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma})\right)^2}\right)}$$

$$\hat{\gamma} \triangleq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i, \quad \hat{\tau}_i \triangleq \frac{1}{\hat{\gamma}}\frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i, \quad \hat{\delta}(x) \text{ as } \delta(x) \text{ but for } (\tau_i,\gamma)\to(\hat{\tau}_i,\hat{\gamma}).$$

Application to G-MUSIC

- Assume the model $a_i = a(\theta_i)$ with

$$a(\theta) = N^{-\frac{1}{2}} [\exp(2\pi \imath d j \sin(\theta))]_{j=0}^{N-1}.$$

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Corollary (Robust G-MUSIC)

Define $\hat{\eta}_{\rm RG}(\theta)$ *and* $\hat{\eta}_{\rm RG}^{\rm emp}(\theta)$ *as*

$$\hat{\eta}_{\rm RG}(\theta) = 1 - \sum_{k=1}^{|\{j, p_j > p_-\}|} w_k a(\theta)^* \hat{u}_k \hat{u}_k a(\theta)$$

$$\hat{\eta}_{\rm RG}^{\rm emp}(\theta) = 1 - \sum_{k=1}^{|\{j, p_j > p_-\}|} \hat{w}_k a(\theta)^* \hat{u}_k \hat{u}_k a(\theta).$$

Then, for each $p_j > p_-$,

$$\hat{\theta}_j \xrightarrow{\rm a.s.} \theta_j$$

$$\hat{\theta}_j^{\rm emp} \xrightarrow{\rm a.s.} \theta_j$$

where

$$\hat{\theta}_j \triangleq \operatorname{argmin}_{\theta \in \mathcal{R}_j^\kappa} \{\hat{\eta}_{\rm RG}(\theta)\}$$

$$\hat{\theta}_j^{\rm emp} \triangleq \operatorname{argmin}_{\theta \in \mathcal{R}_j^\kappa} \{\hat{\eta}_{\rm RG}^{\rm emp}(\theta)\}.$$

Simulations: Single-shot in elliptical noise

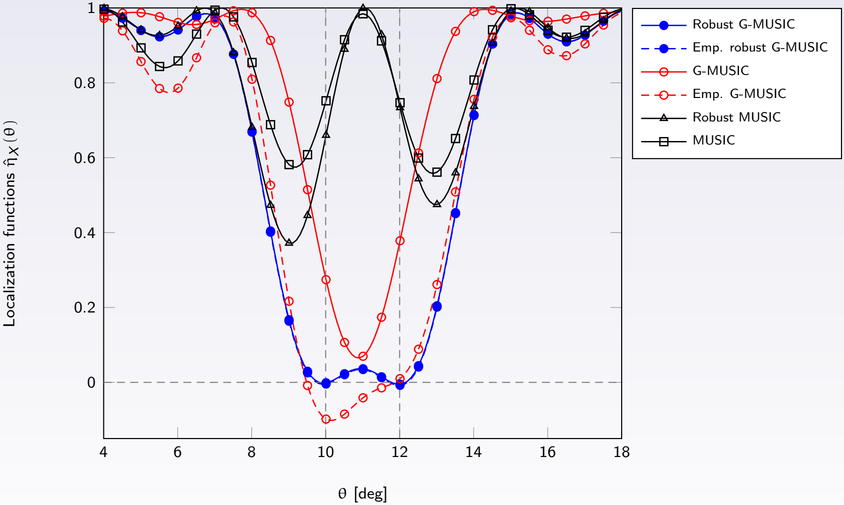

Figure: Random realization of the localization functions for the various MUSIC estimators, with $N = 20$, $n = 100$, two sources at 10° and 12°, Student-t impulsions with parameter $\beta = 100$, $u(x) = (1+\alpha)/(\alpha + x)$ with $\alpha = 0.2$. Powers $p_1 = p_2 = 10^{0.5} = 5$ dB.

Simulations: Elliptical noise

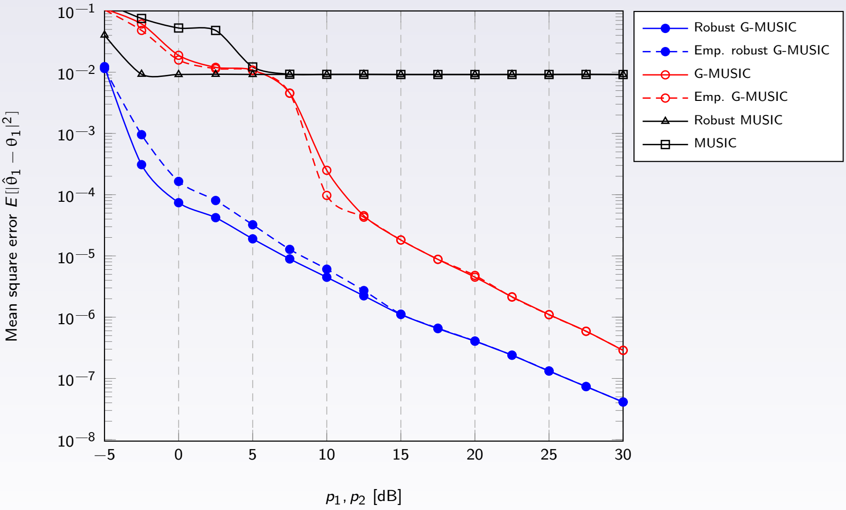

Figure: Means square error performance of the estimation of $\theta_1 = 10^\circ$, with $N = 20$, $n = 100$, two sources at 10° and 12°, Student-t impulsions with parameter $\beta = 10$, $u(x) = (1+\alpha)/(\alpha + x)$ with $\alpha = 0.2$, $p_1 = p_2$.

Simulations: Spurious impulses

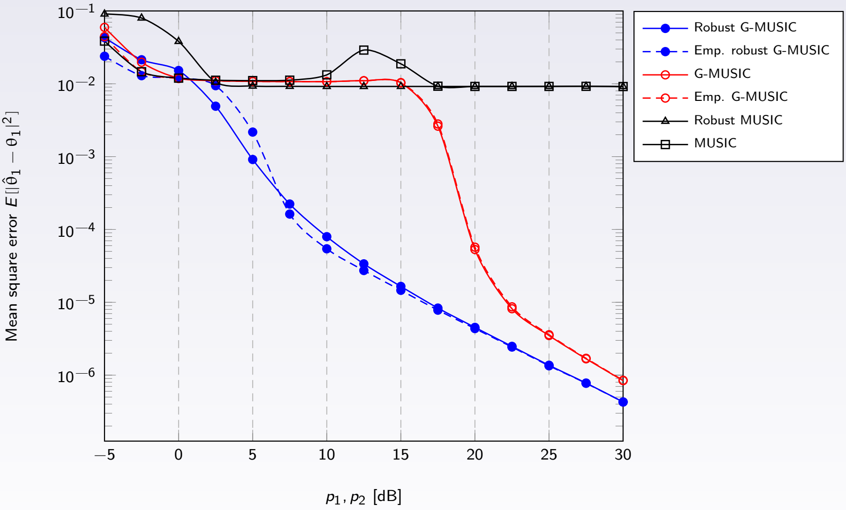

Figure: Means square error performance of the estimation of $\theta_1 = 10^\circ$, with $N = 20$, $n = 100$, two sources at 10° and 12°, sample outlier scenario $\tau_i = 1$, $i < n$, $\tau_n = 100$, $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $p_1 = p_2$.

Outline

Context

Ledoit and Wolf, 2004. A well-conditioned estimator for large-dimensional covariance matrices.
Pascal, Chitour, Quek, 2013. Generalized robust shrinkage estimator – Application to STAP data.
Chen, Wiesel, Hero, 2011. Robust shrinkage estimation of high-dimensional covariance matrices.

- **Shrinkage covariance estimation:** For $N > n$ or $N \simeq n$, shrinkage estimator

$$(1-\rho)\frac{1}{n}\sum_{i=1}^{n} x_i x_i^* + \rho I_N, \text{ for some } \rho \in [0,1].$$

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- **Limitation of Maronna's estimator:**
  - Maronna and Tyler estimators limited to $N < n$, otherwise do not exist
  - introducing shrinkage in robust estimator cannot do much harm anyhow...
- **Introducing the robust-shrinkage estimator:** The literature proposes two such estimators

$$\hat{C}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N^{-1}(\rho) x_i} + \rho I_N, \quad \rho \in \left(\max\{0, \frac{N-n}{N}\}, 1\right] \qquad \text{(Pascal)}$$

$$\check{C}_N(\rho) = \frac{\check{B}_N(\rho)}{\frac{1}{N}\operatorname{tr}\check{B}_N(\rho)}, \quad \check{B}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N^{-1}(\rho) x_i} + \rho I_N, \quad \rho \in (0,1] \qquad \text{(Chen)}$$

Main theoretical result

- **Which estimator is better?**

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Main theoretical result

- **Which estimator is better?**

  Having asked to authors of both papers, their estimator was much better than the other, but the arguments we received were quite vague...

- **Our result:** In the random matrix regime, both estimators tend to be one and the same!
- **Assumptions:** As before, “elliptical-like” model

$$x_i = \tau_i C_N^{\frac{1}{2}} w_i$$

$\longrightarrow$ This time, C_N cannot be taken I_N (due to $+\rho I_N$)!
$\longrightarrow$ Maronna-based shrinkage is possible but more involved...

Pascal's estimator

Theorem (Pascal's estimator)

For $\varepsilon \in (0, \min\{1, c^{-1}\})$*, define* $\hat{\mathcal{R}}_\varepsilon = [\varepsilon + \max\{0, 1 - c^{-1}\}, 1]$*. Then, as* $N, n \to \infty$*,* $N/n \to c \in (0, \infty)$*,*

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} \left\| \hat{C}_N(\rho) - \hat{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{C}_N(\rho) = (1 - \rho)\frac{1}{n}\sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N(\rho)^{-1} x_i} + \rho I_N$$

$$\hat{S}_N(\rho) = \frac{1}{\hat{\gamma}(\rho)} \frac{1 - \rho}{1 - (1 - \rho)c} \frac{1}{n} \sum_{i=1}^{n} C_N^{\frac{1}{2}} w_i w_i^* C_N^{\frac{1}{2}} + \rho I_N$$

and $\hat{\gamma}(\rho)$ *is the unique positive solution to the equation in* $\hat{\gamma}$

$$1 = \frac{1}{N}\sum_{i=1}^{N} \frac{\lambda_i(C_N)}{\hat{\gamma}\rho + (1 - \rho)\lambda_i(C_N)}.$$

Moreover, $\rho \mapsto \hat{\gamma}(\rho)$ *is continuous on* $(0, 1]$*.*

Chen's estimator

Theorem (Chen's estimator)

For $\varepsilon \in (0,1)$, *define* $\check{\mathcal{R}}_\varepsilon = [\varepsilon, 1]$. *Then, as* $N, n \to \infty$, $N/n \to c \in (0, \infty)$,

$$\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \left\| \check{C}_N(\rho) - \check{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\check{C}_N(\rho) = \frac{\check{B}_N(\rho)}{\frac{1}{N} tr \check{B}_N(\rho)}, \quad \check{B}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N} x_i^* \check{C}_N(\rho)^{-1} x_i} + \rho I_N$$

$$\check{S}_N(\rho) = \frac{1-\rho}{1-\rho+T_\rho}\frac{1}{n}\sum_{i=1}^n C_N^{\frac{1}{2}} w_i w_i^* C_N^{\frac{1}{2}} + \frac{T_\rho}{1-\rho+T_\rho} I_N$$

in which $T_\rho = \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)$ *with, for all* $x > 0$,

$$F(x;\rho) = \frac{1}{2}(\rho - c(1-\rho)) + \sqrt{\frac{1}{4}(\rho - c(1-\rho))^2 + (1-\rho)\frac{1}{x}}$$

and $\check{\gamma}(\rho)$ *is the unique positive solution to the equation in* $\check{\gamma}$

$$1 = \frac{1}{N}\sum_{i=1}^N \frac{\lambda_i(C_N)}{\check{\gamma}\rho + \frac{1-\rho}{(1-\rho)c + F(\check{\gamma};\rho)}\lambda_i(C_N)}.$$

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Asymptotic Model Equivalence

Theorem (Model Equivalence)

For each $\rho \in (0,1]$, *there exist unique* $\hat{\rho} \in (\max\{0, 1-c^{-1}\}, 1]$ *and* $\check{\rho} \in (0,1]$ *such that*

$$\frac{\hat{S}_N(\hat{\rho})}{\frac{1}{\hat{\gamma}(\hat{\rho})}\frac{1-\hat{\rho}}{1-(1-\hat{\rho})c} + \hat{\rho}} = \check{S}_N(\check{\rho}) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n} C_N^{\frac{1}{2}} w_i w_i^* C_N^{\frac{1}{2}} + \rho I_N.$$

Besides, $(0,1] \to (\max\{0, 1-c^{-1}\}, 1]$, $\rho \mapsto \hat{\rho}$ *and* $(0,1] \to (0,1]$, $\rho \mapsto \check{\rho}$ *are increasing and onto.*

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- Up to normalization, both estimators behave the same!
- Both estimators behave the same as an impulsion-free Ledoit-Wolf estimator
- **About uniformity:** Uniformity over ρ in the theorems is essential to find optimal values of ρ.

Optimal Shrinkage parameter

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Theorem (Optimal Shrinkage)

For each $\rho \in (0,1]$, *define*

$$\hat{D}_N(\rho) = \frac{1}{N} tr\left(\left(\frac{\hat{C}_N(\rho)}{\frac{1}{N} tr\, \hat{C}_N(\rho)} - C_N\right)^2\right), \quad \check{D}_N(\rho) = \frac{1}{N} tr\left(\left(\check{C}_N(\rho) - C_N\right)^2\right).$$

Denote $D^\star = c\frac{M_2-1}{c+M_2-1}$, $\rho^\star = \frac{c}{c+M_2-1}$, $M_2 = \lim_N \frac{1}{N}\sum_{i=1}^N \lambda_i^2(C_N)$ *and* $\hat{\rho}^\star$, $\check{\rho}^\star$ *unique solutions to*

$$\frac{\hat{\rho}^\star}{\frac{1}{\hat{\gamma}(\hat{\rho}^\star)}\frac{1-\hat{\rho}^\star}{1-(1-\hat{\rho}^\star)c} + \hat{\rho}^\star} = \frac{T_{\check{\rho}^\star}}{1-\check{\rho}^\star + T_{\check{\rho}^\star}} = \rho^\star.$$

Then, letting ε *small enough,*

$$\inf_{\rho \in \hat{\mathcal{R}}_\varepsilon} \hat{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star, \qquad \inf_{\rho \in \check{\mathcal{R}}_\varepsilon} \check{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star$$

$$\hat{D}_N(\hat{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star, \qquad \check{D}_N(\check{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star.$$

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- Careful control of the proofs provide many ways to estimate these.
- Proposition below provides one example.

Optimal Shrinkage Estimate

Let $\hat{\rho}_N \in (\max\{0, 1 - c^{-1}\}, 1]$ and $\check{\rho}_N \in (0, 1]$ be solutions (not necessarily unique) to

$$\frac{\hat{\rho}_N}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho}_N)} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}$$

$$\frac{\check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}}{1 - \check{\rho}_N + \check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}$$

defined arbitrarily when no such solutions exist. Then

$$\hat{\rho}_N \xrightarrow{\text{a.s.}} \hat{\rho}^\star, \quad \check{\rho}_N \xrightarrow{\text{a.s.}} \check{\rho}^\star$$

$$\hat{D}_N(\hat{\rho}_N) \xrightarrow{\text{a.s.}} D^\star, \quad \check{D}_N(\check{\rho}_N) \xrightarrow{\text{a.s.}} D^\star.$$

Simulations

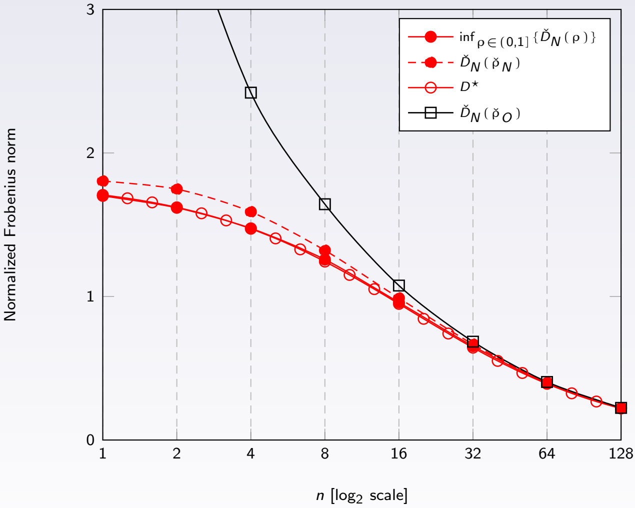

Figure: Performance of optimal shrinkage averaged over 10 000 Monte Carlo simulations, for $N = 32$, various values of n, $[C_N]_{ij} = r^{|i-j|}$ with $r = 0.7$; $\check{\rho}_N$ as above; $\check{\rho}_O$ the clairvoyant estimator proposed in (Chen'11).

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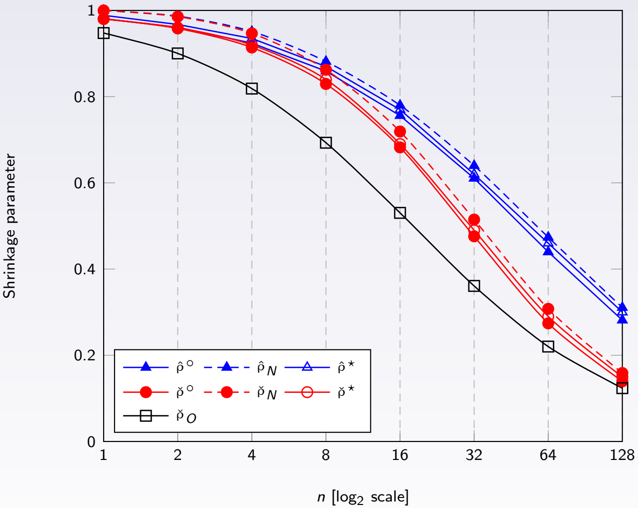

Figure: Shrinkage parameter ρ averaged over 10 000 Monte Carlo simulations, for $N = 32$, various values of n, $[C_N]_{ij} = r^{|i-j|}$ with $r = 0.7$; $\hat{\rho}_N$ and $\check{\rho}_N$ as above; $\check{\rho}_O$ the clairvoyant estimator proposed in (Chen'11); $\hat{\rho}^\circ = \operatorname{argmin}_{\{\rho \in (\max\{0, 1 - c_N^{-1}\}, 1]\}} \{\hat{D}_N(\rho)\}$ and $\check{\rho}^\circ = \operatorname{argmin}_{\{\rho \in (0,1]\}} \{\check{D}_N(\rho)\}$.

Outline

Context

- **Hypothesis testing problem:** Two sets of data
  - Initial pure-noise data: $x_1, \ldots, x_n$, $x_i = \sqrt{\tau_i} C_N^{\frac{1}{2}} w_i$ as before.
  - New incoming data y given by:

$$y = \begin{cases} x & , \mathcal{H}_0 \\ \alpha p + x & , \mathcal{H}_1 \end{cases}$$

with $x = \sqrt{\tau} C_N^{\frac{1}{2}} w$, $p \in \mathbb{C}^N$ deterministic known, α unknown.

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- **GLRT detection test:**

$$T_N(\rho) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \Gamma$$

for some detection threshold Γ where

$$T_N(\rho) \triangleq \frac{|y^* \hat{C}_N^{-1}(\rho) p|}{\sqrt{y^* \hat{C}_N^{-1}(\rho) y} \sqrt{p^* \hat{C}_N^{-1}(\rho) p}}.$$

and $\hat{C}_N(\rho)$ defined in previous section.

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and $\hat{C}_N(\rho)$ defined in previous section.

$\longrightarrow$ In fact, originally found to be $\hat{C}_N(0)$ but
  - only valid for $N < n$
  - introducing ρ may bring improved for arbitrary N/n ratios.

Objectives and main results

- **Initial observations:**
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- **Objectives:**
  - for each ρ, develop central limit theorem to evaluate

$$\lim_{\substack{N,n\to\infty \\ N/n\to c}} P\left(\sqrt{N} T_N(\rho) > \gamma\right)$$

  - determine limiting minimizing ρ
  - empirically estimate minimizing ρ

What do we need?

CLT over $\hat{C}_N$ statistics

- We know that $\|\hat{C}_N(\rho) - \hat{S}_N(\rho)\| \xrightarrow{\text{a.s.}} 0$
  $\longrightarrow$ Key result so far!
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- This requires much more delicate treatment, not discussed in this tutorial.

Main results

Theorem (Fluctuation of bilinear forms)

Let $a, b \in \mathbb{C}^N$ *with* $\|a\| = \|b\| = 1$. *Then, as* $N, n \to \infty$ *with* $N/n \to c > 0$, *for any* $\varepsilon > 0$ *and every* $k \in \mathbb{Z}$,

$$\sup_{\rho \in \mathcal{R}_\kappa} N^{1-\varepsilon} \left| a^* \hat{C}_N^k(\rho) b - a^* \hat{S}_N^k(\rho) b \right| \xrightarrow{\text{a.s.}} 0$$

where $\mathcal{R}_\kappa = [\kappa + \max\{0, 1 - 1/c\}, 1]$.

False alarm performance

Theorem (Asymptotic detector performance)

As $N, n \to \infty$ with $N/n \to c \in (0, \infty)$,

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| P\left(T_N(\rho) > \frac{\gamma}{\sqrt{N}} \right) - \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\hat{\rho})} \right) \right| \to 0$$

where $\rho \mapsto \hat{\rho}$ is the aforementioned mapping and

$$\sigma_N^2(\hat{\rho}) \triangleq \frac{1}{2} \frac{p^* C_N Q_N^2(\hat{\rho}) p}{p^* Q_N(\hat{\rho}) p \cdot \frac{1}{N} \operatorname{tr} C_N Q_N(\hat{\rho}) \cdot \left(1 - c(1-\rho)^2 m(-\hat{\rho})^2 \frac{1}{N} \operatorname{tr} C_N^2 Q_N^2(\hat{\rho})\right)}$$

with $Q_N(\hat{\rho}) \triangleq (I_N + (1-\hat{\rho}) m(-\hat{\rho}) C_N)^{-1}$.

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- Limiting Rayleigh distribution
  $\Rightarrow$ Weak convergence to Rayleigh variable $R_N(\hat{\rho})$
- **Remark:** σ_N and $\hat{\rho}$ not a function of γ
  $\Rightarrow$ There exists a uniformly optimal ρ!

Simulation

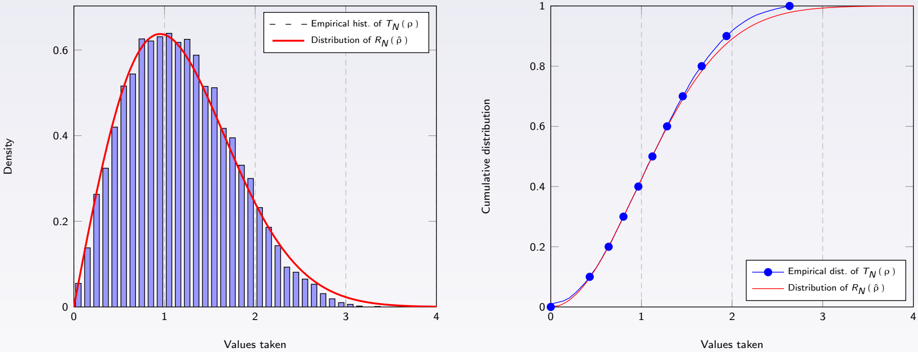

Figure: Histogram distribution function of the $\sqrt{N}T_N(\rho)$ versus $R_N(\hat{\rho})$, $N = 20$, $p = N^{-\frac{1}{2}}[1, \ldots, 1]^{\mathsf{T}}$, C_N Toeplitz from AR of order 0.7, $c_N = 1/2$, $\rho = 0.2$.

Simulation

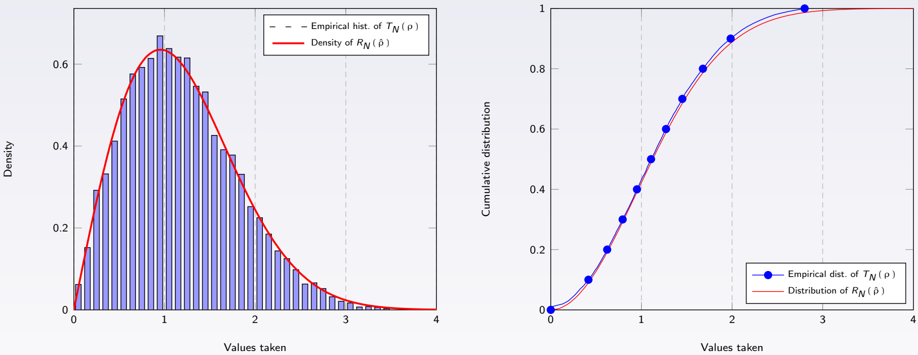

Figure: Histogram distribution function of the $\sqrt{N}T_N(\rho)$ versus $R_N(\hat{\rho})$, $N = 100$, $p = N^{-\frac{1}{2}}[1,\ldots,1]^{\mathsf{T}}$, C_N Toeplitz from AR of order 0.7, $c_N = 1/2$, $\rho = 0.2$.

Empirical estimation of optimal ρ

- Optimal ρ can be found by line search... but C_N unknown!
- We shall successively:
  - empirical estimate $\sigma_N(\hat{\rho})$
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Theorem (Empirical performance estimation)

For $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1)$, *let*

$$\hat{\sigma}_N^2(\hat{\rho}) \triangleq \frac{1}{2} \frac{1 - \hat{\rho} \cdot \frac{p^* \hat{C}_N^{-2}(\rho) p}{p^* \hat{C}_N^{-1}(\rho) p} \cdot \frac{1}{N} tr\, \hat{C}_N(\rho)}{\left(1 - c + c\hat{\rho} \frac{1}{N} tr\, \hat{C}_N^{-1}(\rho) \cdot \frac{1}{N} tr\, \hat{C}_N(\rho)\right)\left(1 - \hat{\rho} \frac{1}{N} tr\, \hat{C}_N^{-1}(\rho) \cdot \frac{1}{N} tr\, \hat{C}_N(\rho)\right)}.$$

Also let $\hat{\sigma}_N^2(1) \triangleq \lim_{\hat{\rho} \uparrow 1} \hat{\sigma}_N^2(\hat{\rho})$. *Then*

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| \sigma_N^2(\hat{\rho}) - \hat{\sigma}_N^2(\hat{\rho}) \right| \xrightarrow{\text{a.s.}} 0.$$

Final result

Theorem (Optimality of empirical estimator)

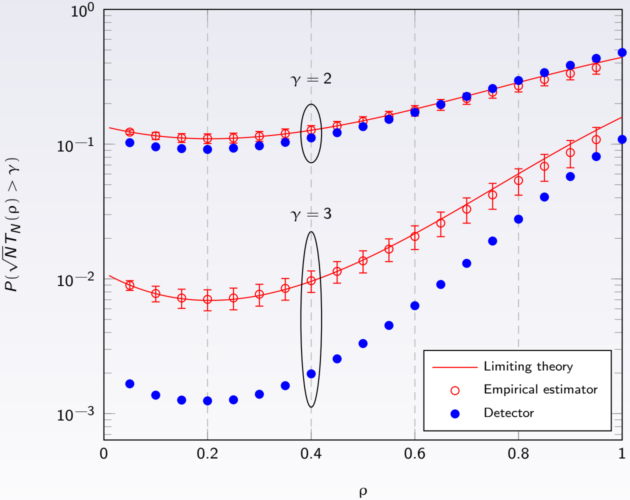
Define

$$\hat{\rho}_N^* = \operatorname{argmin}_{\{\rho \in \mathcal{R}'_\kappa\}} \left\{ \hat{\sigma}_N^2(\hat{\rho}) \right\}.$$

Then, for every $\gamma > 0$,

$$P\left(\sqrt{N} T_N(\hat{\rho}_N^*) > \gamma\right) - \inf_{\rho \in \mathcal{R}_\kappa} \left\{ P\left(\sqrt{N} T_N(\rho) > \gamma\right) \right\} \to 0.$$

Simulations



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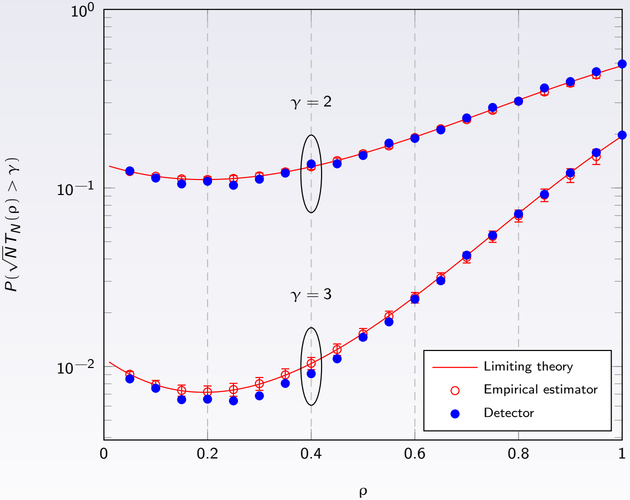

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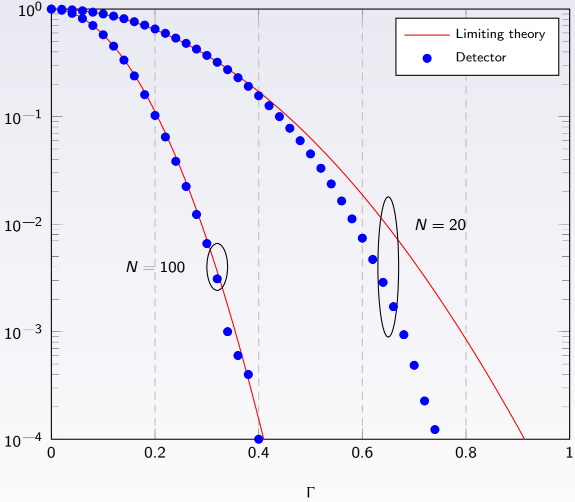

Figure: False alarm rate $P(T_N(\rho) > \Gamma)$ for $N = 20$ and $N = 100$, $p = N^{-\frac{1}{2}}[1, \ldots, 1]^{\mathsf{T}}$, $[C_N]_{ij} = 0.7^{|i-j|}$, $c_N = 1/2$.

Outline

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Motivation: Spectral Clustering

N. El Karoui. The spectrum of kernel random matrices. The Annals of Statistics, 38(1):150, 2010.

- **Objective:** Clustering data $x_1, \ldots, x_n \in \mathbb{C}^N$ in k similarity classes
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- Little is known on such random matrices, but for x_i i.i.d. zero mean and covariance I_N:

$$\left\| W - \left(\alpha 1 1^\mathsf{T} - \beta \frac{1}{n} W W^* \right) \right\| \xrightarrow{\text{a.s.}} 0$$

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for some α, β depending on f and its derivatives.
$\Rightarrow$ Basically, W gets equivalent to a rank-one matrix.

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- Clustering $x_1, \ldots, x_n$ in k often written as:

$$\text{(RatioCut)} \quad \min_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_k \\ \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k = \mathcal{S} \\ \forall i \neq j,\ \mathcal{S}_i \cap \mathcal{S}_j = \emptyset}} \sum_{i=1}^{k} \sum_{j \in \mathcal{S}_i, \bar{j} \in \mathcal{S}_i^c} \frac{f(x_j, x_{\bar{j}})}{|\mathcal{S}_i|}.$$

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- Relaxing M to unitary leads to a simple eigenvalue/eigenvector problem:
$\Rightarrow$ **Spectral clustering**.

Objectives

- Generalization to k distributions for $x_1, \ldots, x_n$ should lead to asymptotically rank-k W matrices.
- If established, specific choices of known "good" kernel better understood.
- Eventually, find optimal choices for kernels.

Outline

Echo-state neural networks

- **Neural network:**
  - Input neuron signal $s_t \in \mathbb{R}$ (could be multivariate)
  - Output neuron signal $y_t \in \mathbb{R}$ (could be multivariate)
  - N neurons with
    - state $x_t \in \mathbb{R}^N$ at time t
    - connectivity matrix $W \in \mathbb{R}^{N \times N}$
    - connectivity vector to input $w_I \in \mathbb{R}^N$
    - connectivity vector to output $w_O \in \mathbb{R}^N$
  - State evolution $x_0 = 0$ (say) and

$$x_{t+1} = S\left(Wx_t + w_I s_t\right)$$

    with S entry-wise sigmoid function.
  - Output observation

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- **Classical neural networks:**
  - Learning phase: input-output data (s_t, y_t) used to learn W, w_O, w_I (via e.g. LS)
  - Interpolation phase: W, w_O, w_I fixed, we observe output y_t from new data s_t.

$\Rightarrow$ Poses overlearning problems, difficult to set up, demands lots of learning data.

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- **Echo-state neural networks:** To solve the problems of neural networks
  - W and w_I set to be a random matrix, no longer learned
  - only w_O is learned

$\Rightarrow$ Reduces amount of data to learn, shows striking performances in some scenarios.

ESN and random matrices

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- **Performance measures:**
  - MSE for training data
  - MSE for interpolated data

  $\Rightarrow$ Optimization to be performed on regression method!, e.g.

$$w_O = (X_{\rm train}X_{\rm train}^T + \gamma I_N)^{-1}X_{\rm train}y_{\rm train}$$

with $X_{\rm train} = [x_1, \ldots, x_T]$, $y_{\rm train} = [y_1, \ldots, y_T]^{\sf T}$, T train period.

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  with $X_{\rm train} = [x_1, \ldots, x_T]$, $y_{\rm train} = [y_1, \ldots, y_T]^{\sf T}$, T train period.
- **In first approximation:** $S = {\rm Id}$.
  $\Rightarrow$ MSE performance with stationary inputs leads to study

$$\sum_{j=1}^{\infty} W^j w_I w_I^{\sf T} (W^T)^j$$

  $\Rightarrow$ New random matrix model, can be analyzed with usual tools though.

Related biography

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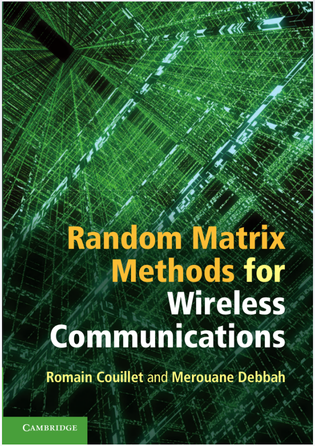
To know more about all this



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