Crash Course on Random Matrix Theory

Part I: Basic notions and applications to wireless communications

Afternoon Session: Wireless Communications

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SUPELEC

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Outline

Outline

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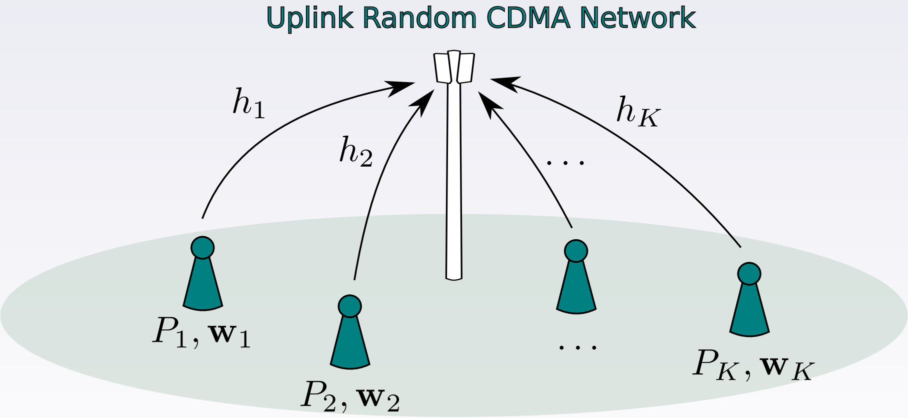
Uplink random CDMA



Capacity of uplink random CDMA

- System model conditions,
  - uplink random CDMA
  - K mobile users, 1 base station
  - N chips per CDMA spreading code.
  - User k, $k \in \{1, \ldots, K\}$ has code $\mathbf{w}_k \sim \mathcal{CN}(0, \mathbf{I}_N)$
  - User k transmits the symbol s_k.
  - User k's channel is $h_k\sqrt{P_k}$, with P_k the power of user k
- The base station receives

$$\mathbf{y} = \sum_{k=1}^{K} h_k \mathbf{w}_k \sqrt{P_k} s_k + \mathbf{n}$$

- This can be written in the more compact form

$$\mathbf{y} = \mathbf{W}\mathbf{H}\mathbf{P}^{\frac{1}{2}}\mathbf{s} + \mathbf{n}$$

with

- $\mathbf{s} = [s_1, \ldots, s_K]^\mathsf{T} \in \mathbb{C}^K$,
- $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_K] \in \mathbb{C}^{N \times K}$,
- $\mathbf{P} = \mathrm{diag}(P_1, \ldots, P_K) \in \mathbb{C}^{K \times K}$,
- $\mathbf{H} = \mathrm{diag}(h_1, \ldots, h_K) \in \mathbb{C}^{K \times K}$.

MMSE decoder

- Consists into decoding symbol of user k as

$$r_k = \mathbf{w}_k^{\mathrm{H}} \left(\mathbf{WHPH}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}} + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{y}.$$

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- The SINR for user's k signal is

$$\begin{aligned}\gamma_k^{(\mathrm{MMSE})} &= P_k |h_k|^2 \mathbf{w}_k^{\mathrm{H}} \Big(\sum_{\substack{1 \leqslant i \leqslant K \\ i \neq k}} P_i |h_i|^2 \mathbf{w}_i \mathbf{w}_i^{\mathrm{H}} + \sigma^2 \mathbf{I}_N \Big)^{-1} \mathbf{w}_k \\ &= P_k |h_k|^2 \mathbf{w}_k^{\mathrm{H}} (\mathbf{WHPH}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}} - P_k |h_k|^2 \mathbf{w}_k \mathbf{w}_k^{\mathrm{H}} + \sigma^2 \mathbf{I}_N)^{-1} \mathbf{w}_k.\end{aligned}$$

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Lemma (Trace Lemma)

If $\mathbf{x} \in \mathbb{C}^N$ has i.i.d. entries of zero mean, variance $1/N$, and $\mathbf{A} \in \mathbb{C}^{N \times N}$ is independent of $\mathbf{x}$ with $\|\mathbf{A}\|$ bounded,

$$\mathbf{x}^{\mathsf{H}} \mathbf{A} \mathbf{x} - \frac{1}{N} \operatorname{tr} \mathbf{A} \xrightarrow{\text{a.s.}} 0.$$

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- Then, for N large,

$$\mathbf{w}_k^{\mathsf{H}} (\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k |h_k|^2 \mathbf{w}_k \mathbf{w}_k^{\mathsf{H}} + \sigma^2 \mathbf{I}_N)^{-1} \mathbf{w}_k - \frac{1}{N} \operatorname{tr} (\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k |h_k|^2 \mathbf{w}_k \mathbf{w}_k^{\mathsf{H}} + \sigma^2 \mathbf{I}_N)^{-1} \xrightarrow{\text{a.s.}} 0.$$

MMSE decoder

$$\mathbf{w}_k^{\mathsf{H}}(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N)^{-1}\mathbf{w}_k - \frac{1}{N}\mathrm{tr}\,(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N)^{-1} \xrightarrow{\text{a.s.}} 0.$$

- From the rank-1 perturbation lemma,

$$\frac{1}{N}\mathrm{tr}\left(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N\right)^{-1} - \frac{1}{N}\mathrm{tr}\left(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} + \sigma^2\mathbf{I}_N\right)^{-1} \to 0,$$

MMSE decoder

$$\mathbf{w}_k^{\mathsf{H}}\left(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}}-P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}}+\sigma^2\mathbf{I}_N\right)^{-1}\mathbf{w}_k-\frac{1}{N}\operatorname{tr}\left(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}}-P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}}+\sigma^2\mathbf{I}_N\right)^{-1}\xrightarrow{\text{a.s.}}0.$$

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- The RHS is the Stieltjes transform of $\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}}$ in $z=-\sigma^2$!

$$m_{\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}}}(-\sigma^2)$$

MMSE decoder

$$\mathbf{w}_k^{\mathsf{H}}(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N)^{-1}\mathbf{w}_k - \frac{1}{N}\operatorname{tr}(\mathbf{WHPH}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} - P_k|h_k|^2\mathbf{w}_k\mathbf{w}_k^{\mathsf{H}} + \sigma^2\mathbf{I}_N)^{-1} \xrightarrow{\text{a.s.}} 0.$$

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SINR and Stieltjes transform

Physical interpretation of the Stieltjes transform: SINR at the output of MMSE receiver!

MMSE decoder

- From previous result,

$$m_{\mathbf{W}\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}}}(-\sigma^2) - m_N(-\sigma^2) \xrightarrow{\text{a.s.}} 0$$

with $m_N(-\sigma^2)$ the unique positive solution of

$$m = \left[\frac{1}{N}\operatorname{tr}\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{H}}\left(m\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{H}} + \mathbf{I}_K\right)^{-1} + \sigma^2\right]^{-1}$$

independent of k, or equivalently

$$m = \left[\sigma^2 + \frac{1}{N}\sum_{1\leqslant i\leqslant K}\frac{P_i|h_i|^2}{1 + mP_i|h_i|^2}\right]^{-1}$$

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- Finally,

$$\gamma_k^{(\mathrm{MMSE})} - P_k|h_k|^2 m_N(-\sigma^2) \xrightarrow{\text{a.s.}} 0$$

and the mutual information reads

$$C^{(\mathrm{MMSE})}(\sigma^2) - \frac{1}{K}\sum_{k=1}^{K}\log_2(1+P_k|h_k|^2 m_N(-\sigma^2)) \xrightarrow{\text{a.s.}} 0.$$

MMSE decoder

$$C^{(\mathrm{MMSE})}(\sigma^2) - \frac{1}{K}\sum_{k=1}^{K}\log_2(1 + P_k|h_k|^2 m_N(-\sigma^2)) \xrightarrow{\mathrm{a.s.}} 0.$$

- AWGN channel, $P_k = P$, $h_k = 1$,

$$C^{(\mathrm{MMSE})}(\sigma^2) \xrightarrow{\mathrm{a.s.}} c\log_2\left(1 + \frac{-(\sigma^2 + (c-1)P) + \sqrt{(\sigma^2 + (c-1)P)^2 + 4P\sigma^2}}{2\sigma^2}\right)$$

- Rayleigh channel, $P_k = P$, $|h_k|$ Rayleigh,

$$m = \left[\sigma^2 + c\int \frac{Pt}{1 + Ptm}e^{-t}dt\right]^{-1}$$

and

$$C_{\mathrm{MMSE}}(\sigma^2) \xrightarrow{\mathrm{a.s.}} c\int \log_2\left(1 + Ptm(-\sigma^2)\right)e^{-t}dt.$$

Matched-Filter and Optimal decoder

- Similarly, we can compute deterministic equivalents for the matched-filter performance,

$$C_{\mathrm{MF}}(\sigma^2) - \frac{1}{N}\sum_{k=1}^{K} \log_2\left(1 + \frac{P_k|h_k|^2}{\frac{1}{N}\sum_{i=1}^{K} P_i|h_i|^2 + \sigma^2}\right) \xrightarrow{\mathrm{a.s.}} 0$$

- AWGN case,

$$C_{\mathrm{MF}}(\sigma^2) \xrightarrow{\mathrm{a.s.}} c\log_2\left(1 + \frac{P}{Pc + \sigma^2}\right)$$

- Rayleigh case,

$$C_{\mathrm{MF}}(\sigma^2) \xrightarrow{\mathrm{a.s.}} -c\log_2(e) e^{\frac{Pc+\sigma^2}{P}} \mathrm{Ei}\left(-\frac{Pc + \sigma^2}{P}\right)$$

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- ... and the optimal joint-decoder performance

$$C_{\mathrm{opt}}(\sigma^2) - \log_2\left(1 + \frac{1}{\sigma^2 N}\sum_{k=1}^{K}\frac{P_k|h_k|^2}{1 + cP_k|h_k|^2 m_N(-\sigma^2)}\right) - \frac{1}{N}\sum_{k=1}^{K}\log_2\left(1 + cP_k|h_k|^2 m_N(-\sigma^2)\right)$$
$$- \log_2(e)\left(\sigma^2 m_N(-\sigma^2) - 1\right) \xrightarrow{\text{a.s.}} 0.$$

with $m_N(-\sigma^2)$ defined as previously.

- Similar expressions are obtained for the AWGN and Rayleigh cases.

Simulation results: AWGN case

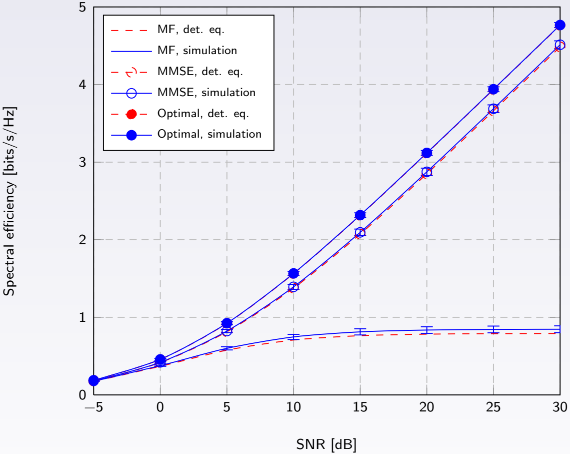

Figure: Spectral efficiency of random CDMA decoders, AWGN channels. Comparison between simulations and deterministic equivalents (det. eq.), for the matched-filter, the MMSE decoder and the optimal decoder, $K = 16$ users, $N = 32$ chips per code. Rayleigh channels. Error bars indicate two standard deviations.

Simulation results: Rayleigh case

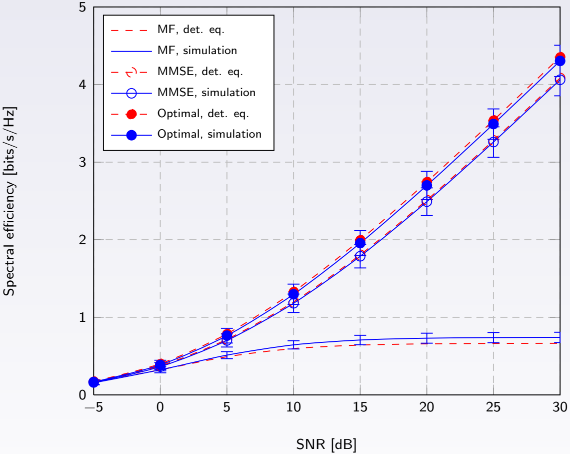

Figure: Spectral efficiency of random CDMA decoders, Rayleigh fading channels. Comparison between simulations and deterministic equivalents (det. eq.), for the matched-filter, the MMSE decoder and the optimal decoder, $K = 16$ users, $N = 32$ chips per code. Rayleigh channels. Error bars indicate two standard deviations.

Simulation results: Performance as a function of K/N

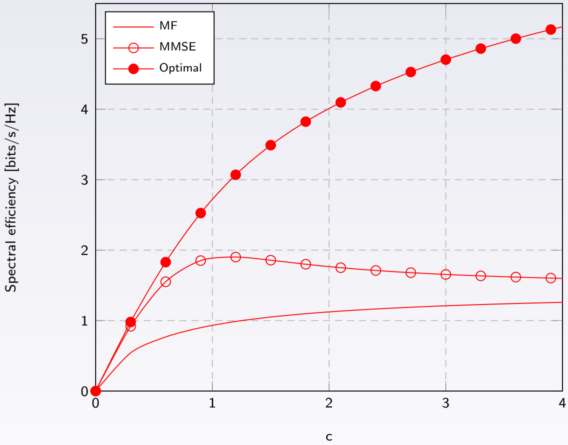

Figure: Spectral efficiency of random CDMA decoders, for different asymptotic ratios $c = K/N$, SNR=10 dB, AWGN channels. Deterministic equivalents for the matched-filter, the MMSE decoder and the optimal decoder. Rayleigh channels.

Related bibliography

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Point-to-point MIMO

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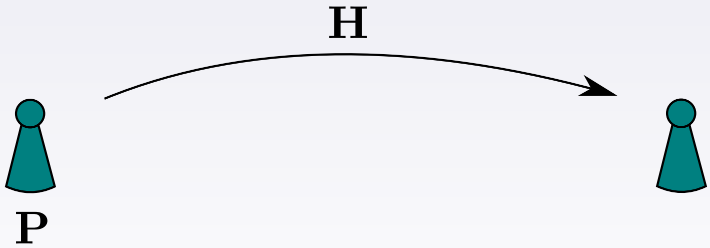

$$\frac{1}{n_t}\mathrm{tr}(\mathbf{P}) = P$$

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- Quasi-static capacity:

$$C^{(n_r,n_t)}(\sigma^2) = \max_{\substack{\mathbf{P} \\ \frac{1}{n_t}\operatorname{tr}\mathbf{P} \leqslant P}} \mathcal{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P})$$

with $\mathcal{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P})$ the mutual information

$$\mathcal{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}) \triangleq \log_2 \det\left(\mathbf{I}_{n_r} + \frac{1}{\sigma^2}\mathbf{HPH}^{\mathsf{H}}\right)$$

and $\mathbf{P}$ the Gaussian signal covariance.

- Capacity achieved by water-filling algorithm for all finite n_r, n_t.
- For $\mathbf{H}$ such that $F^{\mathbf{HH}^{\mathsf{H}}} \Rightarrow F$,

$$\frac{1}{n_r} C^{(n_r,n_t)}(\sigma^2) \xrightarrow{\text{a.s.}} \int \log\left(1 + \frac{1}{\sigma^2}\lambda\left[\mu - \frac{\sigma^2}{\lambda}\right]^+\right) dF(\lambda)$$

with μ such that

$$\int \left[\mu - \frac{\sigma^2}{\lambda}\right]^+ dF(\lambda) = P.$$

Ergodic capacity

- Ergodic mutual information and capacity:

$$C_{\text{ergodic}}^{(n_r,n_t)}(\sigma^2) = \max_{\substack{\mathbf{P} \\ \frac{1}{n_t}\operatorname{tr}\mathbf{P} \leqslant P}} \mathrm{E}\left[\mathcal{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P})\right].$$

- Capacity unknown in closed form! (solution given by convex optimization algo)

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- Capacity unknown in closed form! (solution given by convex optimization algo)
- For classical channel models (Gaussian i.i.d., Kronecker, Rice, etc.), deterministic equivalent optimization can be solved!
- *Example*: Kronecker channel model $\mathbf{H} = \mathbf{R}^{\frac{1}{2}}\mathbf{X}\mathbf{T}^{\frac{1}{2}}$, $\mathbf{X} \in \mathbb{C}^{n_r\times n_t}$ i.i.d. Gaussian, entries $(0, \frac{1}{n_t})$.
- For all covariance $\mathbf{P}$ bounded,

$$\frac{1}{n_r}\mathrm{E}\left[I^{(n_r,n_t)}(\sigma^2;\mathbf{P})\right] - \frac{1}{n_r}\bar{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}) \to 0$$

with

$$\frac{1}{n_r}\bar{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}) \triangleq \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_r} + \bar{e}\mathbf{R}\right) + \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_t} + ce\mathbf{T}\mathbf{P}\right) - \sigma^2\bar{e}e$$

where $c = n_r/n_t$ and

$$e = \frac{1}{\sigma^2 n_r}\operatorname{tr}\mathbf{R}\left(\mathbf{I}_{n_r} + \bar{e}\right)^{-1}, \quad \bar{e} = \frac{1}{\sigma^2 n_t}\operatorname{tr}\mathbf{T}\mathbf{P}\left(\mathbf{I}_{n_t} + ce\mathbf{T}\mathbf{P}\right)^{-1}.$$

Ergodic capacity: Capacity achieving covariance matrix

- We wish to determine $\mathbf{P}^\circ$ such that

$$\mathbf{P}^\circ = \arg\max_{\mathbf{P}} \bar{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}).$$

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- Under some conditions, and with convexity arguments,

$$\mathrm{E}\left[I^{(n_r,n_t)}(\sigma^2;\mathbf{P}^\star)\right] - \mathrm{E}\left[I^{(n_r,n_t)}(\sigma^2;\mathbf{P}^\circ)\right] \to 0$$

with $\mathbf{P}^\star$ the capacity maximizing precoder.

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  with $\mathbf{P}^{\star}$ the capacity maximizing precoder.
- To determine $\mathbf{P}^{\circ}$, we use the differentiation chain rule

$$\frac{d}{d\mathbf{P}}\bar{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}) = \left[\frac{\partial V}{\partial \mathbf{P}} + \frac{\partial V}{\partial f}\frac{\partial f}{\partial \mathbf{P}} + \frac{\partial V}{\partial \bar{f}}\frac{\partial \bar{f}}{\partial \mathbf{P}}\right](e,\bar{e},\mathbf{P})$$

  with

$$V : (f,\bar{f},\mathbf{P}) \mapsto \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_r} + \bar{f}\mathbf{R}\right) + \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_t} + cf\mathbf{T}\mathbf{P}\right) - \sigma^2\bar{f}f.$$

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- Under some conditions, and with convexity arguments,

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with

$$V : (f,\bar{f},\mathbf{P}) \mapsto \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_r} + \bar{f}\mathbf{R}\right) + \frac{1}{n_r}\log\det\left(\mathbf{I}_{n_t} + cf\mathbf{T}\mathbf{P}\right) - \sigma^2\bar{f}f.$$

- We now observe that

$$\frac{\partial V}{\partial f}(e,\bar{e},\mathbf{P}) = \frac{1}{n_t}\operatorname{tr}\mathbf{T}\mathbf{P}\left(\mathbf{I}_{n_t} + ce\mathbf{T}\mathbf{P}\right) - \sigma^2\bar{e} = 0$$

$$\frac{\partial V}{\partial \bar{f}}(e,\bar{e},\mathbf{P}) = \frac{1}{n_r}\operatorname{tr}\mathbf{R}\left(\mathbf{I}_{n_r} + \bar{e}\mathbf{R}\right) - \sigma^2 e = 0.$$

Ergodic capacity: Capacity achieving covariance matrix (2)

- Therefore

$$\frac{d}{d\mathbf{P}}\bar{I}^{(n_r,n_t)}(\sigma^2;\mathbf{P}) = \frac{\partial V}{\partial \mathbf{P}}(e,\bar{e},\mathbf{P}) = \frac{\partial}{\partial \mathbf{P}}\left[\frac{1}{n_r}\log\det\left(\mathbf{I}_{n_t} + ce\mathbf{TP}\right)\right]$$

whose zero corresponds to the water-filling solution when $e = e^\circ$ ($e^\circ = e(\mathbf{P}^\circ)$).

- Therefore, for $\mathbf{T} = \mathbf{U}_T\mathbf{\Lambda}_T\mathbf{U}_T^{\mathsf{H}}$,

$$\mathbf{P}^\circ = \mathbf{U}_T\mathbf{Q}\mathbf{U}_T^{\mathsf{H}}, \quad Q_{ij} = \delta_i^j\left(\mu - \frac{1}{ce^\circ T_{ii}}\right)^+$$

with μ such that $\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\mu - \frac{1}{ce^\circ T_{ii}}\right)^+ = P$.

- Solution can be found by the iterative water-filling algorithm.
- It is known that, upon convergence, the algorithm convergences to the correct solution.

Mutual information vs. outage

- We wish to evaluate, for given R,

$$\mathbb{P}\left(I^{(n_r,n_t)}(\sigma^2;\mathbf{P}) < R\right).$$

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$$\frac{1}{\theta_{n_r}}\left(I^{(n_r,n_t)}(\sigma^2;\mathbf{P}) - \mathbb{E}\left[I^{(n_r,n_t)}(\sigma^2;\mathbf{P})\right]\right) \Rightarrow \mathcal{N}(0,1).$$

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- In the case of i.i.d. Gaussian channel,

$$\theta_{n_r}^2 = \theta^2 = \log\left(1 - \frac{\sigma^4}{16c}\left(\sqrt{\frac{(1+\sqrt{c})^2}{\sigma^2}+1} - \sqrt{\frac{(1-\sqrt{c})^2}{\sigma^2}+1}\right)^4\right).$$

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- Many results exist for the CLT of more generic models.
- *This is outside the scope of this lecture.*

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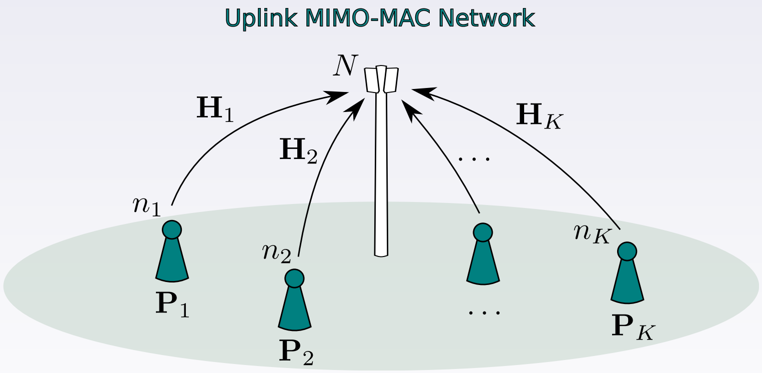
Outline

Outline

MIMO-MAC



MIMO-MAC, SINR of the MMSE receiver

R. Couillet, M. Debbah, J. W. Silverstein, "A Deterministic Equivalent for the Analysis of Correlated MIMO Multiple Access Channels", IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3493-3514, 2011.

$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{H}_k^{\mathsf{H}}, \text{ with } \mathbf{H}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{X}_k \mathbf{T}_k^{\frac{1}{2}}$$

with $\mathbf{X}_k \in \mathbb{C}^{N \times n_k}$ with i.i.d. entries of zero mean, variance $1/n_k$, $\mathbf{R}_k$ Hermitian nonnegative definite, $\mathbf{T}_k$ diagonal. Denote $c_k = N/n_k$. Then, as all N and n_k grow large, with ratio c_k,

$$\frac{1}{N}\operatorname{tr}\left(\mathbf{B}_N + \sigma^2 \mathbf{I}_N\right)^{-1} - \frac{1}{\sigma^2}\frac{1}{N}\operatorname{tr}\left(\mathbf{I}_N + \sum_{k=1}^{K} \bar{e}_k \mathbf{R}_k\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

where the set of $\{e_i\}$ form the unique positive solution to the K equations

$$e_i = \frac{1}{\sigma^2}\frac{1}{N}\operatorname{tr} \mathbf{R}_i \left(\mathbf{I}_N + \sum_{k=1}^{K} \bar{e}_k \mathbf{R}_k\right)^{-1}, \quad \bar{e}_i = \frac{1}{\sigma^2}\frac{1}{n_i}\operatorname{tr} \mathbf{T}_i \left(\mathbf{I}_{n_i} + c_i e_i(z) \mathbf{T}_i\right)^{-1}.$$

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Hence, the SINR at the output of the MMSE receiver for user stream i of user k, γ_{ik}, satisfies

$$\gamma_{ik} = \mathbf{h}_{k,i}^{\mathrm{H}}\left(\mathbf{B}_N - \mathbf{h}_{k,i}\mathbf{h}_{k,i}^{\mathrm{H}} + \sigma^2 I_N\right)^{-1} - t_{k,i} e_i \xrightarrow{\text{a.s.}} 0.$$

MIMO-MAC, sum-rate

- Under the previous model for $\mathbf{B}_N$, as N, n_k grow large,

$$E\left[\frac{1}{N}\log\det\left(\mathbf{I}_N+\frac{1}{\sigma^2}\mathbf{B}_N\right)\right]$$
$$-\left[\frac{1}{N}\log\det\left(\mathbf{I}_N+\sum_{k=1}^{K}\bar{e}_k\mathbf{R}_k\right)+\sum_{k=1}^{K}\frac{1}{N}\log\det\left(\mathbf{I}_{n_k}+c_k e_k\mathbf{T}_k\mathbf{P}_k\right)-\sigma^2\sum_{k=1}^{K}\bar{e}_k e_k\right]\to 0.$$

- The deterministic-equivalent maximizing precoders $\mathbf{P}_1^\circ,\ldots,\mathbf{P}_K^\circ$ satisfy

$$\mathbf{P}_k^\circ=\mathbf{U}_k\mathrm{diag}(p_{k,1}^\circ,\ldots,p_{k,n_k}^\circ)\mathbf{U}_k^H,\quad \text{where } \mathbf{T}_k=\mathbf{U}_k\mathrm{diag}(t_{k,1},\ldots,t_{k,n_k})\mathbf{U}_k^H$$

and $p_{k,i}$ defined by iterative water-filling as

$$p_{k,i}^\circ=\left(\mu_k-\frac{1}{c_k e_k^\circ t_{k,i}}\right)^+$$

with μ_k such that $\frac{1}{n_k}\mathrm{tr}\,\mathbf{P}_k^\circ=P_k$, and e_k° defined as e_k for $(\mathbf{P}_1,\ldots,\mathbf{P}_K)=(\mathbf{P}_1^\circ,\ldots,\mathbf{P}_K^\circ)$.

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- Moreover, under some conditions,

$$\|\mathbf{P}_k^\star - \mathbf{P}_k^\circ\| \to 0.$$

Outline

Linearly precoded broadcast channels

Deterministic equivalents of sum-rate capacity for linearly precoded broadcast channels,

- K users, N antennas at the base station, $c = N/K$, MISO channels $\mathbf{h}_1, \ldots, \mathbf{h}_K$, $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_K]$.

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- accounting for base station antenna correlation $\mathbf{R}_k$ to user k, user path loss t_k, $\mathbf{h}_k = \sqrt{t_k}\mathbf{R}_k^{\frac{1}{2}}\mathbf{x}_k$, $\mathbf{x}_k \sim \mathcal{CN}(0, \frac{1}{N}\mathbf{I}_N)$.
- assuming imperfect channel state information $\hat{\mathbf{h}}_k$ of $\mathbf{h}_k$

$$\hat{\mathbf{h}}_k = \sqrt{1-\tau_k^2}\mathbf{h}_k + \tau_k \mathbf{w}_k, \quad \mathbf{w}_k \sim \mathcal{CN}(0, \frac{1}{N}\mathbf{I}_N)$$

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- focus on the output SINR γ_k of linear receivers $\hat{\mathbf{G}} = [\hat{\mathbf{g}}_1, \ldots, \hat{\mathbf{g}}_K]$ for each user k, with $\hat{\mathbf{G}} = f(\hat{\mathbf{H}})$.

Linearly precoded broadcast channels, system model

- Signal model for user k,

$$y_k = \mathbf{h}_k^{\mathsf{H}} \left[\hat{\mathbf{G}} \mathbf{x}_t + \sigma \mathbf{w}_t \right] = \mathbf{h}_k^{\mathsf{H}} \left[\hat{\mathbf{g}}_k x_{k,t} + \sum_{j \neq k} \hat{\mathbf{g}}_j x_{j,t} + \sigma \mathbf{w}_t \right].$$

- Output SINR

$$\gamma_k = \frac{|\mathbf{h}_k^{\mathsf{H}} \hat{\mathbf{g}}_k|^2}{\mathbf{h}_k^{\mathsf{H}} \hat{\mathbf{G}} \hat{\mathbf{G}}^{\mathsf{H}} \mathbf{h}_k - |\mathbf{h}_k^{\mathsf{H}} \hat{\mathbf{g}}_k|^2 + \sigma^2}.$$

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- Specific precoders (with ξ power normalization):
  - **Matched-filter:** $\hat{\mathbf{G}}_{mf} = \xi \hat{\mathbf{H}}$
  - **Zero-forcing:** $\hat{\mathbf{G}}_{zf} = \xi \hat{\mathbf{H}} (\hat{\mathbf{H}}^{\mathsf{H}} \hat{\mathbf{H}})^{-1}$
  - **Regularized zero-forcing:** $\hat{\mathbf{G}}_{rzf} = \xi (\hat{\mathbf{H}} \hat{\mathbf{H}}^{\mathsf{H}} + \alpha \mathbf{I}_N)^{-1} \hat{\mathbf{H}}$
  - **Optimal linear precoder:** $\hat{\mathbf{G}}_{opt}$ limiting solution of iterative formulation.

Linearly precoded broadcast channels

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Results:

- Deterministic equivalents for the output SINR, i.e. we find $\bar{\gamma}_k$ such that

$$\gamma_k - \bar{\gamma}_k \xrightarrow{\text{a.s.}} 0.$$

  - for specific precoders (MF, ZF, RZF)
  - requires deterministic equivalents for several terms in expression of γ_k
  - specific problems appear for ZF due to matrix inversion

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  - specific problems appear for ZF due to matrix inversion
- These results allow one to characterize:
  - optimal number of users to serve
  - optimal parameter for specific precoders (e.g. regularizes zero-forcing)
  - optimal feedback time in block-fading channel models

SINR approximation in white setting

- Deterministic equivalents $\bar{\gamma}_k$ for γ_k, with $\tau \geqslant 0$, $\mathbf{R}_i = \mathbf{I}_N$, $t_i = 1$.

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- RZF Precoding ($\alpha > 0$, $c \geqslant 1$): $\mathbf{G}_{rzf}(\alpha) = \xi \hat{\mathbf{H}} (\hat{\mathbf{H}}^{\mathsf{H}} \hat{\mathbf{H}} + \alpha \mathbf{I}_N)^{-1}$

$$\bar{\gamma}_{k,rmf}(\alpha) = \frac{e(1-\tau_k^2)\left[1+\alpha c(1+e)^2\right]}{1-\tau_k^2[1-(1+e)^2] + \sigma^2(1+e)^2}$$

with

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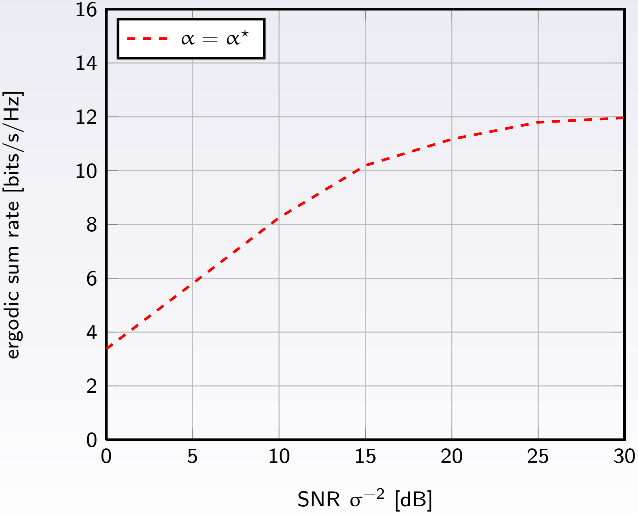
- Optimal linear precoder: difficult to analyze due to recursive definition.

Performance of different regularizations

$N = K = 5$, $\mathbf{R}_k = \mathbf{I}_N \ \forall k$, and $\tau_k^2 = 0.1$



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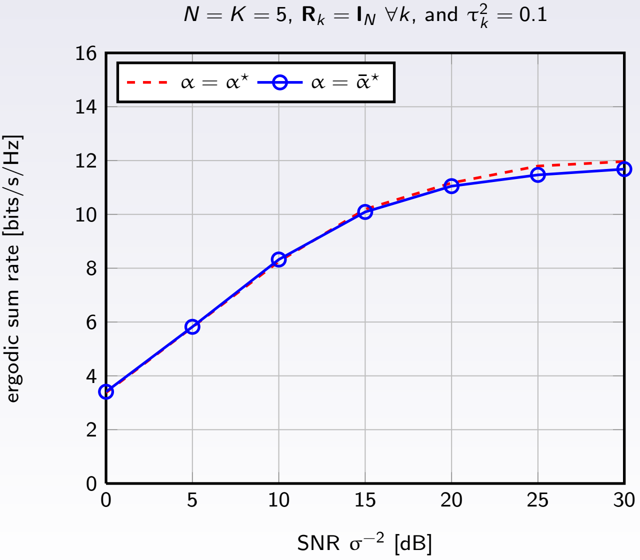

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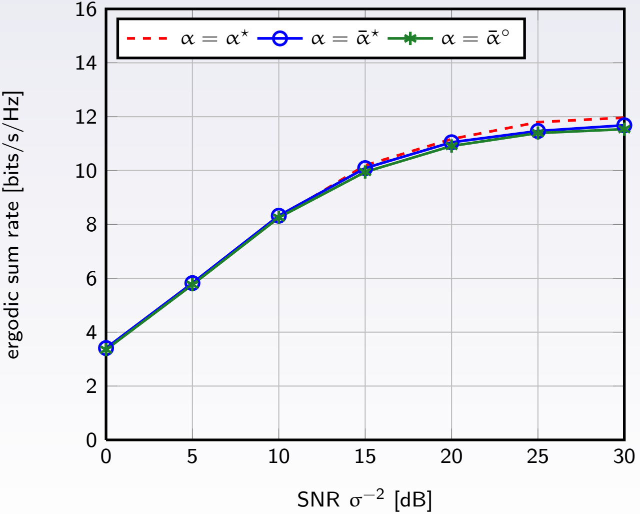

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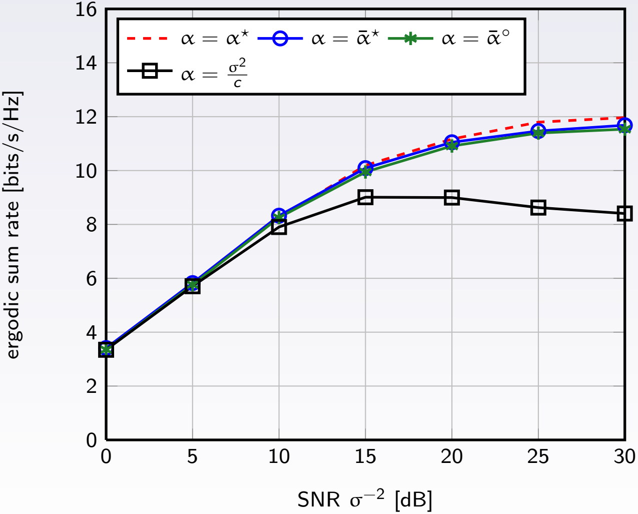

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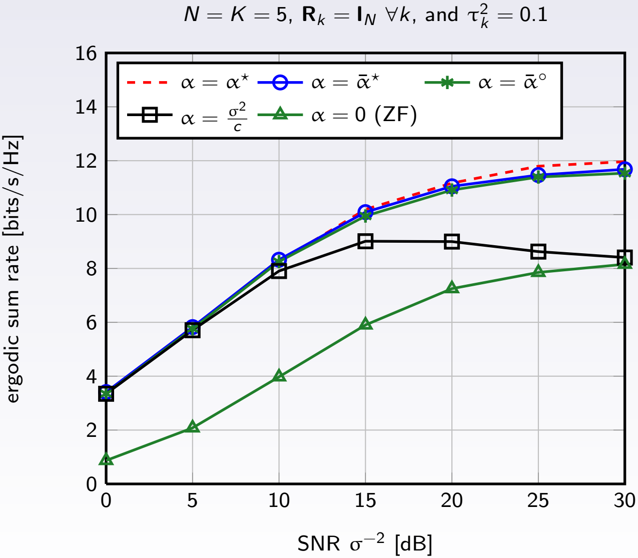
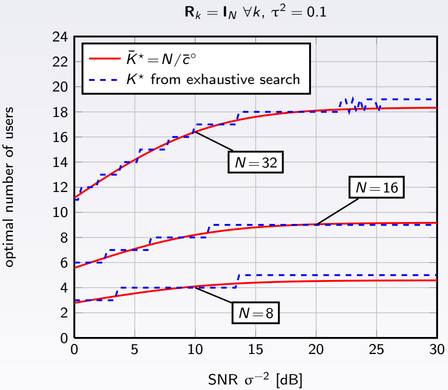


Optimal cell loading $c^\star$



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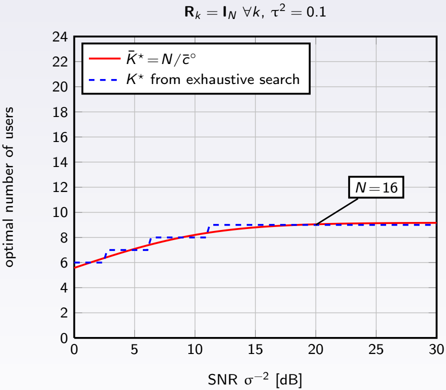

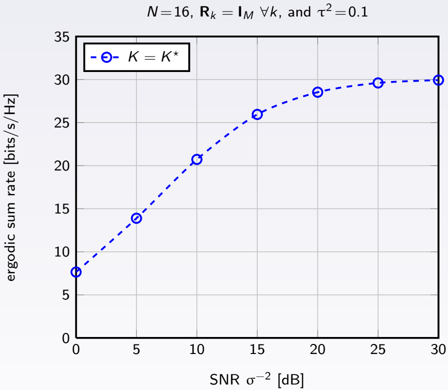

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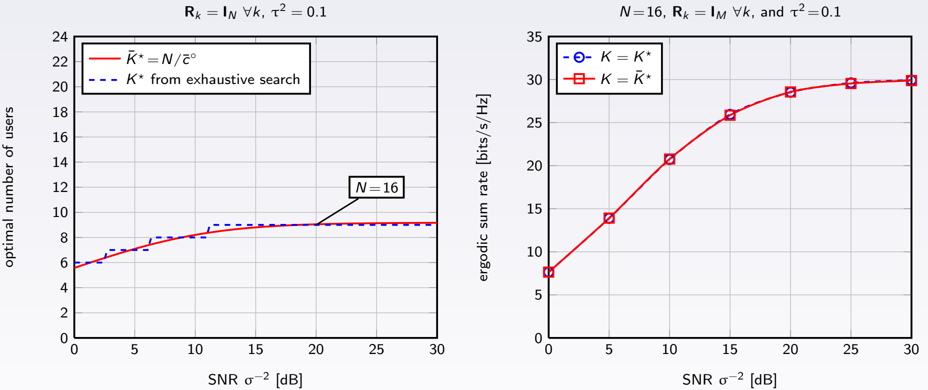

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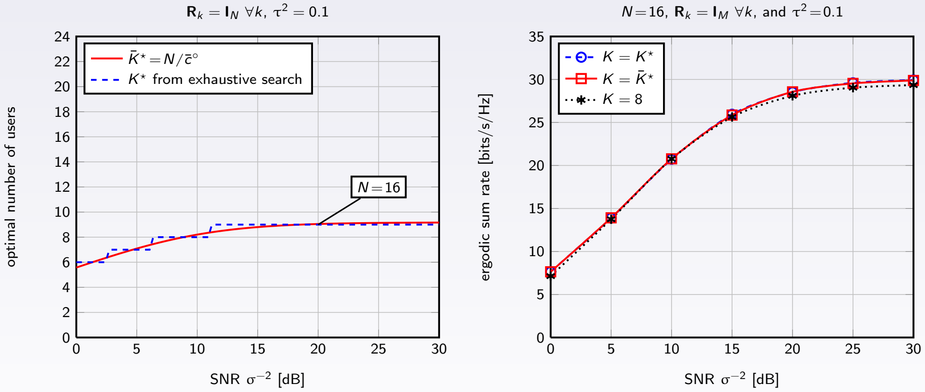

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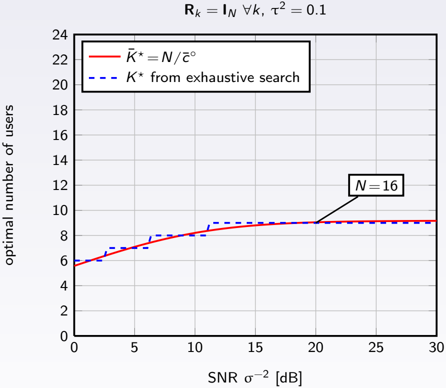
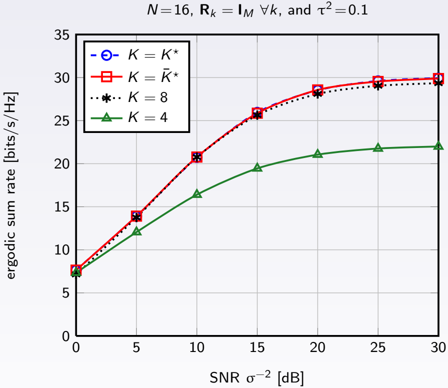
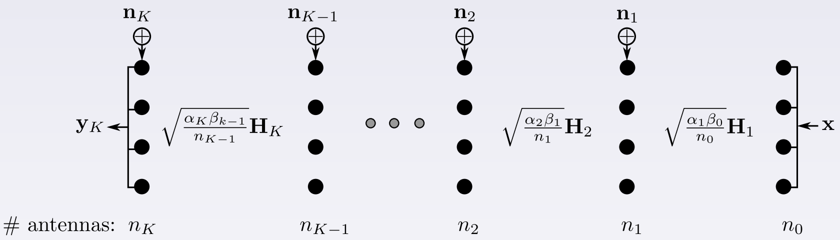




Outline

Amplify-and-forward Multi-hop relay channel



- A source communicates $\mathbf{x}$ to destination via $K-2$ relays.
- Each node receives only from previous node.
- Relays amplify-and-forward to next node.
- We do not account for transmission delay (e.g. $K-1$ TDMA time slots).

Product of random matrices

The model naturally calls for the method of iterative deterministic equivalents.

Channel model

Received signal vector $\mathbf{y}_k \in \mathbb{C}^{n_k}$ at node k:

$$\mathbf{y}_k = \sqrt{\alpha_k}\mathbf{H}_k \underbrace{\sqrt{\frac{\beta_{k-1}}{n_{k-1}}}\mathbf{y}_{k-1}}_{\text{Signal from node } k-1} + \mathbf{w}_k$$

- $\mathbf{H}_k \in \mathbb{C}^{n_k \times n_{k-1}}$, standard complex Gaussian
- $\mathbf{y}_0 = \mathbf{x} \sim \mathcal{CN}(0, \mathbf{I}_{n_0})$, input vector
- $\mathbf{w}_k \sim \mathcal{CN}(0, \mathbf{I}_{n_k})$, noise at node k

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- Transmit signal scaled according to power constraint ρ_k:

$$\beta_k = \frac{\rho_k}{\frac{1}{n_k}\mathrm{tr}\left(\mathbb{E}[\mathbf{y}_k\mathbf{y}_k^{\mathrm{H}}]\right)}$$

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- Large system limit: $n_0 \to \infty$, with

$$0 < \liminf_n c_k \triangleq \frac{n_{k-1}}{n_k} \leqslant \limsup c_k < \infty, \quad \forall k.$$

Mutual information

- Normalized mutual information $\mathcal{I}_k(\boldsymbol{\beta}_k)$ between $\mathbf{y}_k$ and $\mathbf{x}$ can be written as:

$$\mathcal{I}_k(\boldsymbol{\beta}_k) = \mathcal{J}_k(1, \boldsymbol{\beta}_k) - \mathcal{J}_k(1, \boldsymbol{\beta}'_k)$$

where

$$\mathcal{J}_k\left(x, \boldsymbol{\beta}_k\right) = \frac{1}{n_k} \log\det\left(\mathbf{I}_{n_k} + x\frac{\alpha_k \beta_{k-1}}{n_{k-1}}\mathbf{H}_k\mathbf{R}_{k-1}\mathbf{H}_k^{\mathrm{H}}\right)$$

with

$$\mathbf{R}_0 = \mathbb{E}\left[\mathbf{x}\mathbf{x}^{\mathrm{H}}\right] = \mathbf{I}_n$$

$$\mathbf{R}_k = \mathbb{E}\left[\mathbf{y}_k\mathbf{y}_k^{\mathrm{H}}\right] = \mathbf{I}_{n_k} + \frac{\alpha_k \beta_{k-1}}{n_{k-1}}\mathbf{H}_k\mathbf{R}_{k-1}\mathbf{H}_k^{\mathrm{H}}, \qquad k = 1, \ldots, K$$

and $\boldsymbol{\beta}_k = [\beta_0, \cdots, \beta_{k-1}]$, $\boldsymbol{\beta}'_k = [0, \beta_1, \cdots, \beta_{k-1}]$.

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and $\boldsymbol{\beta}_k = [\beta_0, \cdots, \beta_{k-1}]$, $\boldsymbol{\beta}'_k = [0, \beta_1, \cdots, \beta_{k-1}]$.

Example

$$\mathcal{I}_2(\boldsymbol{\beta}_2) = \frac{1}{n_2} \log\det\left(\mathbf{I}_{n_2} + \frac{\alpha_2 \beta_1}{n_1} \mathbf{H}_2 \mathbf{R}_1 \mathbf{H}_2^{\mathrm{H}}\right) - \frac{1}{n_2} \log\det\left(\mathbf{I}_{n_2} + \frac{\alpha_2 \beta_1}{n_1} \mathbf{H}_2 \mathbf{H}_2^{\mathrm{H}}\right)$$

Asymptotic power normalization

- Power normalization

$$\beta_k \xrightarrow{\text{a.s.}} \bar{\beta}_k = \frac{\rho_k}{1 + \alpha_k \rho_{k-1}}, \qquad k = 1, \ldots, K-1$$

where $\beta_0 = \bar{\beta}_0 = \rho_0$.

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- Recursive definition of

$$\mathbf{R}_k = \mathbf{I}_{n_k} + \frac{\alpha_k \beta_{k-1}}{n_{k-1}} \mathbf{H}_k \mathbf{R}_{k-1} \mathbf{H}_k^{\mathrm{H}}$$

allow us to find iterative deterministic equivalents for

$$\mathcal{J}_k(x, \boldsymbol{\beta}_k) = \frac{1}{n_k} \log\det\left(\mathbf{I}_{n_k} + x \frac{\alpha_k \beta_{k-1}}{n_{k-1}} \mathbf{H}_k \mathbf{R}_{k-1} \mathbf{H}_k^{\mathrm{H}}\right).$$

Deterministic equivalent for the capacity

J. Hoydis, R. Couillet, M. Debbah, "Iterative Deterministic Equivalents for the Capacity Analysis of Communication Systems", (submitted to) IEEE Transactions on Information Theory.

Theorem (Deterministic equivalent of $\mathcal{I}_k(x, \boldsymbol{\beta}_k)$)

$$\mathcal{I}_k(x, \boldsymbol{\beta}_k) - \bar{\mathcal{I}}_k(x, \bar{\boldsymbol{\beta}}_k) \xrightarrow[n\to\infty]{a.s.} 0$$

with $\bar{\mathcal{I}}_k(x, \bar{\boldsymbol{\beta}}_k)$ *recursively defined for* $k \geqslant 2$ *as*

$$\begin{aligned}\bar{\mathcal{I}}_k(x, \bar{\boldsymbol{\beta}}_k) = {} & c_k \bar{\mathcal{I}}_{k-1}\left(\frac{x\alpha_k \bar{\beta}_{k-1}}{c_k + x\alpha_k \bar{\beta}_{k-1} + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}, \bar{\boldsymbol{\beta}}_{k-1}\right) \\ & + c_k \log\left(1 + \frac{x\alpha_k \beta_{k-1}}{c_k + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}\right) \\ & + \log\left(1 + \frac{\bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}{c_k}\right) - \frac{\bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}{1 + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}\end{aligned}$$

and $\bar{e}_k(x, \bar{\boldsymbol{\beta}}_{k-1})$ *for* $k \geqslant 0$ *given on next slide.*

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$$\begin{aligned}\bar{\mathcal{J}}_k(x, \bar{\boldsymbol{\beta}}_k) &= c_k \bar{\mathcal{J}}_{k-1}\left(\frac{x\alpha_k \bar{\beta}_{k-1}}{c_k + x\alpha_k \bar{\beta}_{k-1} + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}, \bar{\boldsymbol{\beta}}_{k-1}\right)\\ &+ c_k \log\left(1 + \frac{x\alpha_k \beta_{k-1}}{c_k + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}\right)\\ &+ \log\left(1 + \frac{\bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}{c_k}\right) - \frac{\bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}{1 + \bar{e}_{k-1}(x, \bar{\boldsymbol{\beta}}_{k-1})}\end{aligned}$$

and $\bar{e}_k(x, \bar{\boldsymbol{\beta}}_{k-1})$ *for* $k \geqslant 0$ *given on next slide. Moreover,*

$$\bar{\mathcal{J}}_1(x, \bar{\beta}_0) = c_1 \log\left(1 + \frac{x\alpha_1 \bar{\beta}_0}{c_1 + \bar{e}_0(x, \bar{\beta}_0)}\right) + \log\left(1 + \frac{\bar{e}_0(x, \bar{\beta}_0)}{c_1}\right) - \frac{\bar{e}_0(x, \bar{\beta}_0)}{1 + \bar{e}_0(x, \bar{\beta}_0)}.$$

Deterministic equivalent for the capacity (2)

Theorem (Recursive definition of $\bar{e}_k$)

- $\bar{e}_k\left(x, \bar{\beta}_k\right)$ *is the unique positive solution to*

$$\bar{e}_k\left(x, \bar{\beta}_k\right) = c_{k+1}\left(c_{k+1} + \bar{e}_k\left(x, \bar{\beta}_k\right)\right) - \frac{c_{k+1}\left(c_{k+1} + \bar{e}_k\left(x, \bar{\beta}_k\right)\right)^2}{x\alpha_{k+1}\bar{\beta}_k}\bar{m}_{k-1}\left(\frac{x\alpha_{k+1}\bar{\beta}_k}{c_{k+1} + x\alpha_{k+1}\bar{\beta}_k + \bar{e}_k\left(x, \bar{\beta}_k\right)}, \bar{\beta}_{k-1}\right)$$

where

$$\bar{m}_k\left(x, \bar{\beta}_k\right) = \frac{xc_{k+1}}{c_{k+1} + \bar{e}_k\left(x, \bar{\beta}_k\right)}.$$

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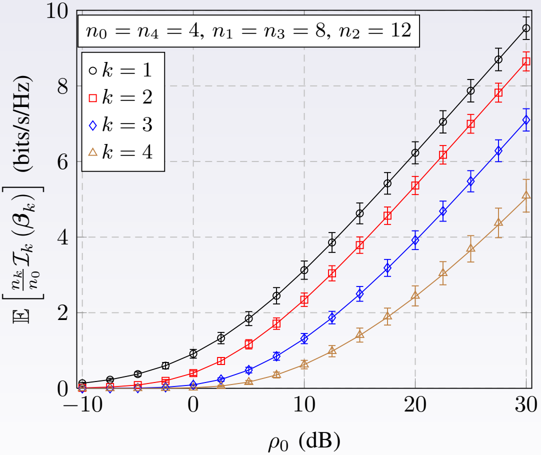
$$\bar{m}_k\left(x, \bar{\beta}_k\right) = \frac{xc_{k+1}}{c_{k+1} + \bar{e}_k\left(x, \bar{\beta}_k\right)}.$$

- *Initial values* $\bar{m}_0(x, \bar{\beta}_0)$ *and* $\bar{e}_0(x, \bar{\beta}_0)$ *given in closed-form:*

$$\bar{m}_0(x, \bar{\beta}_0) = \frac{c_1}{\frac{\alpha_1\bar{\beta}_0}{c_1 + \bar{e}_0(x, \bar{\beta}_0)} + \frac{1}{x}} + (1 - c_1)x$$

$$\bar{e}_0\left(x, \bar{\beta}_0\right) = -\frac{x\alpha_1\bar{\beta}_0(1 - c_1) + c_1}{2} + \frac{\sqrt{\left(x\alpha_1\bar{\beta}_0(1 - c_1) + c_1\right)^2 + 4x\alpha_1\bar{\beta}_0 c_1^2}}{2}.$$

Numerical example: Amplify-and-forward Multi-hop relay channel



$\rho_1 = \rho_3 = 0.7\rho_0,\ \rho_2 = 0.5\rho_0,\ \alpha_1 = 1,\ \alpha_2 = \alpha_4 = 0.7,\ \alpha_3 = 0.5$

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- J. Hoydis, M. Kobayashi and M. Debbah, “Optimal channel training in uplink network MIMO systems”, IEEE Trans. on Signal Processing, vol. 59, no. 6, 2011.
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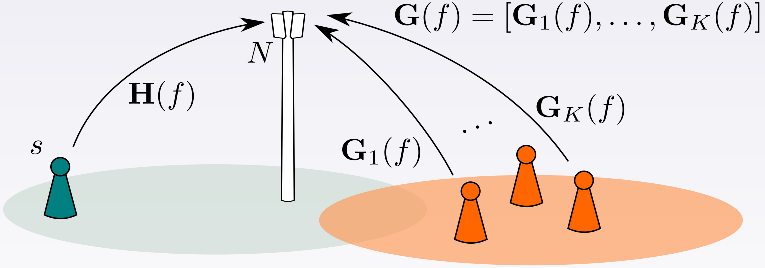
Outline

Outline

Rate inference

Rate inference under interference



Cognitive frequency bands $f \in \{f_1, \ldots, f_B\}$

Rate inference setting

- System setting:
  - Known own-channel $\mathbf{H} \in \mathbb{C}^{N \times s}$
  - Unknown co-channels $\mathbf{G}_1, \ldots, \mathbf{G}_K$, collected as $\mathbf{G} = [\mathbf{G}_1, \ldots, \mathbf{G}_K] \in \mathbb{C}^{N \times n}$, n unknown
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- Received signals

$$\bar{\mathbf{y}}_t = \underbrace{\mathbf{H}\mathbf{s}_t}_{\text{Signal of interest}} + \underbrace{\mathbf{G}\mathbf{x}_t}_{\text{Interference}} + \sigma \mathbf{w}_t$$

collected for $t = 1, \ldots, M$, as

$$\bar{\mathbf{Y}} = \mathbf{H}\mathbf{S} + \mathbf{G}\mathbf{X} + \sigma \mathbf{W}.$$

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- Objective is to infer the rate

$$C \triangleq \frac{1}{N} \log\det\left(\mathbf{H}\mathbf{H}^{\mathsf{H}} + \mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N\right) - \frac{1}{N} \log\det\left(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N\right)$$

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- **Flexible radio setting**: exploration phase after free bandwidth detection, bands $f_1, \ldots, f_B$.
- Choice of preferred bandwidth based on fast rate estimation.

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- Use **G-estimation** to infer on C assuming $M, N, n, s \to \infty$ with nontrivial ratios.

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shows low performance when M, N of similar dimensions.

Bias of the traditional estimator

- $\hat{C}_T$ is biased.
- bias can be determined using deterministic equivalents.

Lemma

Let $\mathbf{T} = \left(\mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{\mathbf{G}\mathbf{G}^{\mathsf{H}}+\sigma^2\mathbf{I}_N}{1+\kappa}\right)^{-1}$ *where* κ *is the unique solution of:*

$$\kappa = \frac{1}{M}\operatorname{tr}\left(\left(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N\right)\left(\mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N)}{1+\kappa}\right)^{-1}\right),$$

then

$$\hat{C}_T - \frac{1}{N}V_T \xrightarrow{\text{a.s.}} 0$$

with V_T *given by*

$$V_T = -\log\det(\mathbf{T}) + M\log(1+\kappa) - M\frac{\kappa}{1+\kappa} - \log\det(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N) + (M-N)\log(1-\frac{N}{M}) + N$$

G-estimator

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- We then prove

$$\hat{C}_T(y) - \frac{1}{N}V_T(y) \xrightarrow{\text{a.s.}} 0$$

where $V_T(y)$ is defined by:

$$V_T(y) = -\log\det(\mathbf{T}(y)) + M\log(1+\kappa(y)) - M\frac{\kappa(y)}{1+\kappa(y)}$$
$$-\log\det(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N) + (M-N)\log\left(\frac{M-N}{M}\right) + N$$

with

$$\mathbf{T}(y) = \left(y\mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N}{1+\kappa(y)}\right)^{-1}$$

and $\kappa(y)$ solution of

$$\kappa(y) = \frac{1}{M}\operatorname{tr}\left((\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N)\left(y\mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{(\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2\mathbf{I}_N)}{1+\kappa(y)}\right)^{-1}\right).$$

G-estimator (2)

- Key idea is to choose y so that C appears in the deterministic equivalent.

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- In our case, y is set to

$$y = \frac{1}{1 + \kappa(y)}.$$

- We have moreover the uniqueness result

Lemma

There exists a unique $y_N^\star$ verifying:

$$y_N^\star = \frac{1}{1 + \kappa(y_N^\star)}$$

Moreover $y_N^\star$ is given by:

$$y_N^\star = 1 - \frac{1}{M}\mathrm{tr}\left((\mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N) \left(\mathbf{H}\mathbf{H}^{\mathsf{H}} + \mathbf{G}\mathbf{G}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N \right)^{-1} \right).$$

G-estimator (3)

- Unfortunately $y_N^\star$ depends on the unknown $\mathbf{G}\mathbf{G}^\mathsf{H} + \sigma^2 \mathbf{I}_N$.

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- We need to estimate it consistently.

Lemma

Let $\hat{y}_N$ the unique solution of

$$y = \frac{1}{M} \operatorname{tr} y \mathbf{H}\mathbf{H}^{\mathsf{H}} \left(y \mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{1}{M} \mathbf{Y}\mathbf{Y}^{\mathsf{H}} \right)^{-1} + \frac{M-N}{M}.$$

Then

$$\hat{y}_N - y_N^\star \xrightarrow{\text{a.s.}} 0.$$

G-estimator (4)

A. Kammoun, R. Couillet, J. Najim, M. Debbah, "Performance of Capacity Inference Methods under Colored Interference", (submitted to) IEEE Transactions on Information Theory, arXiv Preprint 1105.5305.

Using previous results, a consistent estimator of C is then given by the following result.

Theorem

As $M, N \to \infty$ with nontrivial ratio

$$C - \hat{C} \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{C} = \frac{1}{N} \log\det\left(\mathbf{I}_N + \hat{y}_N \mathbf{H}\mathbf{H}^{\mathsf{H}} \left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\right)^{-1}\right) + \frac{(M-N)}{N}\left[\log\left(\frac{M}{M-N}\hat{y}_N\right) + 1\right] - \frac{M}{N}\hat{y}_N$$

and $\hat{y}_N$ is the unique positive solution of

$$y = \frac{1}{M}\operatorname{tr} y\mathbf{H}\mathbf{H}^{\mathsf{H}}\left(y\mathbf{H}\mathbf{H}^{\mathsf{H}} + \frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\right)^{-1} + \frac{M-N}{M}.$$

Fluctuations

We moreover have the following fluctuations of the estimator $\hat{C}$.

Theorem

The G-estimator $\hat{C}$ satisfies:

$$\frac{N}{\theta_N}(\hat{C} - C) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

where θ_N is given by:

$$\theta_N = \log\left(\frac{M^2 y_N^*}{M-N}\right) - \log\left[M - \operatorname{tr}\left(\mathbf{I}_N + \mathbf{H}\mathbf{H}^\mathsf{H}\left(\mathbf{G}\mathbf{G}^\mathsf{H} + \sigma^2\mathbf{I}_N\right)^{-1}\right)^{-2}\right].$$

G-estimator, simulation

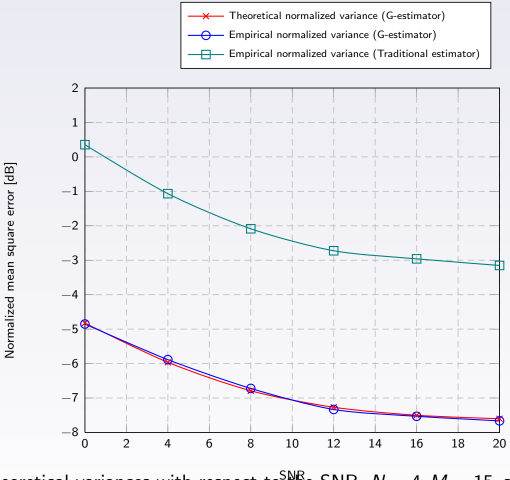

Figure: Empirical and theoretical variances with respect to the SNR, $N = 4$, $M = 15$, $s = 4$, $n = 8$, i.i.d. Gaussian generated channels.

Outline

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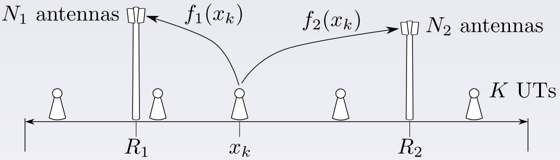
Deterministic equivalents: Cooperation with fixed user terminals



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_B \end{pmatrix} = \mathbf{Hs} + \mathbf{n} = \begin{pmatrix} \mathbf{G}_1 \mathbf{T}_1^{\frac{1}{2}} \\ \vdots \\ \mathbf{G}_B \mathbf{T}_B^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix} + \begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_B \end{pmatrix}$$

- $s_k \sim \mathcal{NC}(0, \rho)$: transmit symbol of UT K
- $\mathbf{n}_b \sim \mathcal{NC}(0, \mathbf{I}_{N_b})$: noise at BS b
- $\mathbf{G}_b \in \mathbb{C}^{N_b \times K}$, $[\mathbf{G}_b]_{i,j} \sim \mathcal{NC}\left(0, \frac{1}{K}\right)$: fast fading
- $\mathbf{T}_b = \text{diag}\left(f_b(x_k)\right)_{k=1}^{K}$ where $f_b(x)$ is a path loss function, e.g.

$$f_b(x) = \frac{1}{(1 + |R_b - x|)^{\beta}}$$

Deterministic equivalents: Mutual information and MMSE sum-rate

Theorem

Denote $c = \frac{N}{K}$, $c_i = \frac{N_i}{K}$ $\forall i$. *For* $N_i, K \to \infty$ *at the same speed,*

$$\frac{1}{N} \log \det \left(\mathbf{I}_N + \rho \mathbf{H}\mathbf{H}^{\mathsf{H}} \right) - \bar{V}_N(\rho) \xrightarrow{\text{a.s.}} 0$$

where

$$\bar{V}_N(\rho) = \sum_{i=1}^{B} c_i \log\left(\frac{\rho}{\Psi_i}\right) + \frac{1}{N}\sum_{k=1}^{K} \log\left(1 + \sum_{i=1}^{B} c_i f_i(x_k)\Psi_i\right) - \frac{1}{N}\sum_{k=1}^{K} \frac{\sum_{i=1}^{B} c_i f_i(x_k)\Psi_i}{1 + \sum_{i=1}^{B} c_i f_i(x_k)\Psi_i}$$

and $\Psi_1, \ldots, \Psi_B$ *are given as the unique positive solution to*

$$\Psi_i = \left(\frac{1}{\rho} + \frac{1}{K}\sum_{k=1}^{K} \frac{f_i(x_k)}{1 + \sum_{i=1}^{B} c_i f_i(x_k)\Psi_i} \right)^{-1}, \qquad i = 1, \ldots, B.$$

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Remark

SINR with MMSE detection: $\gamma_k = \mathbf{h}_k^{\mathrm{H}}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}} - \mathbf{h}_k\mathbf{h}_k^{\mathrm{H}} + \frac{1}{\rho}\mathbf{I}_N\right)^{-1}\mathbf{h}_k \asymp \sum_{i=1}^{B} c_i f_i(x_k)\Psi_i$.

Deterministic equivalents: Mutual information and MMSE sum-rate

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Denote $c = \frac{N}{K}$, $c_i = \frac{N_i}{K}$ $\forall i$. *For* $N_i, K \to \infty$ *at the same speed,*

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$R_{\text{sum}} = \frac{1}{N}\sum_{k=1}^{K} \log(1 + \gamma_k) \asymp \frac{1}{N}\sum_{k=1}^{K} \log\left(1 + \sum_{i=1}^{B} c_i f_i(x_k)\Psi_i\right)$.

Deterministic equivalents: Random user locations

Assume that the positions x_k of the UTs are i.i.d. with distribution F. Then,

$$\frac{1}{N}\sum_{k=1}^{K}\log\left(1+\sum_{i=1}^{B}c_i f_i(x_k)\Psi_i\right)\approx\frac{1}{c}\int\log\left(1+\sum_{i=1}^{B}c_i f_i(x)\Psi_i\right)dF(x).$$

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Similarly,

$$\Psi_i=\left(\frac{1}{\rho}+\frac{1}{K}\sum_{k=1}^{K}\frac{f_i(x_k)}{1+\sum_{i=1}^{B}c_i f_i(x_k)\Psi_i}\right)^{-1}\approx\left(\frac{1}{\rho}+\int\frac{f_i(x)}{1+\sum_{i=1}^{B}c_i f_i(x)\Psi_i}dF(x)\right)^{-1}.$$

Deterministic equivalents: Random user locations

Corollary

Let x_k, $k = 1, \ldots, K$, be i.i.d. with distribution F. Then,

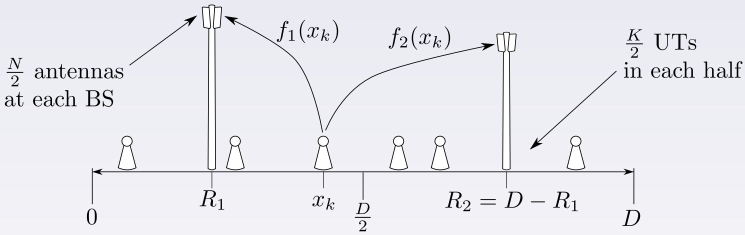
$$\frac{1}{N} \log\det\left(\mathbf{I}_N + \rho \mathbf{H}\mathbf{H}^{\mathrm{H}}\right) - \bar{I}_N(\rho) \xrightarrow{\text{a.s.}} 0$$

$$\bar{I}_N(\rho) = \sum_{i=1}^{B} c_i \log\left(\frac{\rho}{\psi_i}\right) + \frac{1}{c}\int \log\left(1 + \sum_{i=1}^{B} c_i f_i(x)\psi_i\right) dF(x) - \frac{1}{c}\int \frac{\sum_{i=1}^{B} c_i f_i(x)\psi_i}{1 + \sum_{i=1}^{B} c_i f_i(x)\psi_i} dF(x)$$

where $\psi_1, \ldots, \psi_B$ are given as the unique positive solution to

$$\psi_i = \left(\frac{1}{\rho} + \int \frac{f_i(x)}{1 + \sum_{i=1}^{B} c_i f_i(x)\psi_i} dF(x)\right)^{-1}, \qquad i = 1, \ldots, B.$$

Application: Optimal BS-placement

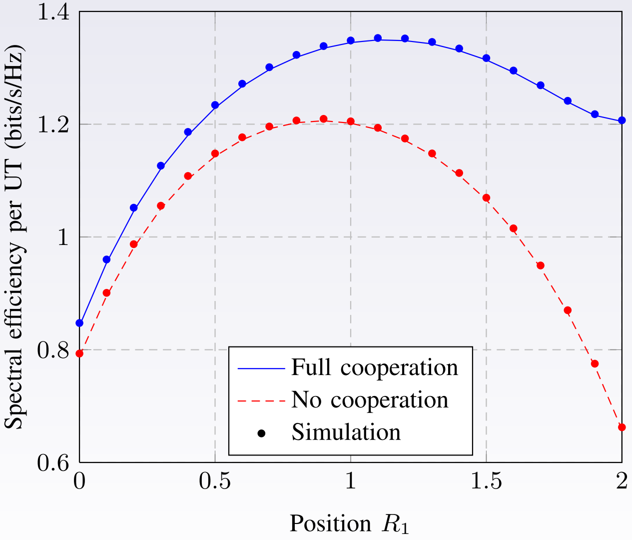


- $\frac{K}{2}$ UTs uniformly distributed on the intervals $[0, \frac{D}{2}]$ and $[\frac{D}{2}, D]$, respectively.
- Path loss functions: $f_i(x) = (1 + |R_i - x||)^{-\beta}, \qquad i = 1, 2.$
- Decompose the channel matrix as $\mathbf{H} = \begin{pmatrix} \mathbf{H}_{1,1} & \mathbf{H}_{1,2} \\ \mathbf{H}_{2,1} & \mathbf{H}_{2,2} \end{pmatrix}$, where $\mathbf{H}_{i,j} \in \mathbb{C}^{N/2 \times K/2}$.
- Mutual information without cooperation:

$$I_N^{\mathrm{nc}}(\rho) = \frac{1}{N} \sum_{i=1}^{2} \log\det\left(\mathbf{I}_{N/2} + \rho\mathbf{H}_{i,i}\mathbf{H}_{i,i}^{\mathsf{H}} + \rho\mathbf{H}_{i,\bar{i}}\mathbf{H}_{i,\bar{i}}^{\mathsf{H}}\right) - \log\det\left(\mathbf{I}_{N/2} + \rho\mathbf{H}_{i,\bar{i}}\mathbf{H}_{i,\bar{i}}^{\mathsf{H}}\right)$$

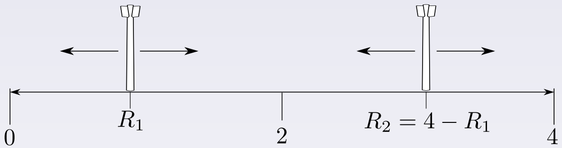
where $\bar{i} = 1 + i \bmod 2$.

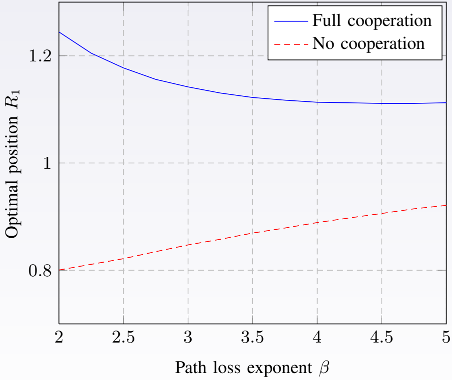
Optimal BS-placement: Numerical results (I)

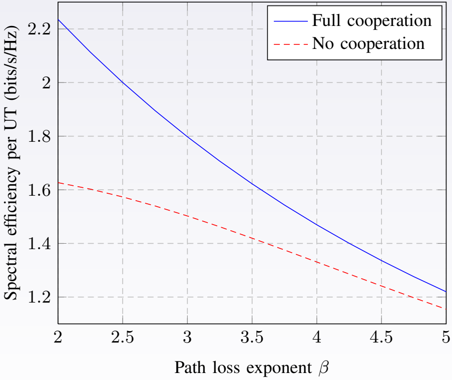


$N = 16,\ K = 12,\ \rho = 10\,\mathrm{dB},\ \beta = 3.7,\ D = 4$

Optimal BS-placement: Numerical results (II)







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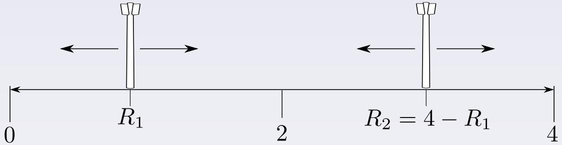

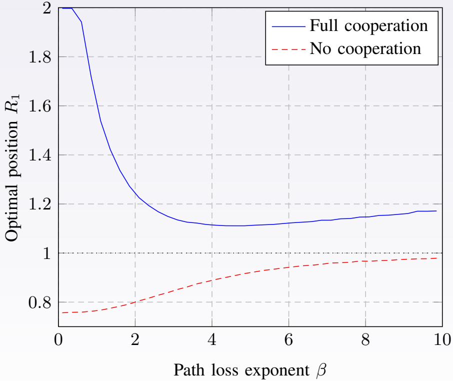

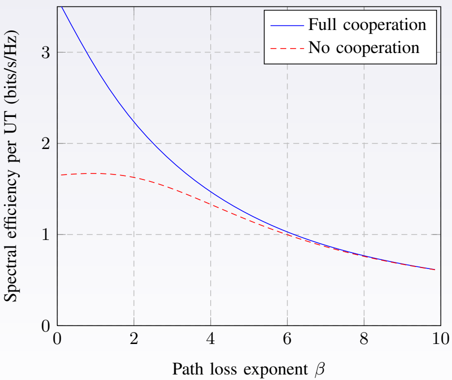

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Some remarks

- We can optimize system parameters with respect to random channel realizations and user distributions, without simulations.
- The same results could be also applied for a variety of other detectors.
- One can also account for imperfect CSI and limited backhaul capacity.
- Extensions to two- or three-dimensional models are possible.

Downlink: Regularized Zero-forcing

Cooperative regularized Zero-forcing: received signal at UT k:

$$y_k = \sqrt{\rho\lambda}\mathbf{h}_k^{\mathrm{H}}\mathbf{w}_k s_k + \sqrt{\rho\lambda}\mathbf{h}_k^{\mathrm{H}}\sum_{j\neq k}\mathbf{w}_j s_j + n_k$$

- Symbol for UT K: $s_k \sim \mathcal{NC}(0,1)$
- Precoding matrix : $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_K] \in \mathbb{C}^{N\times K}$,

$$\mathbf{W} = \left(\mathbf{H}\mathbf{H}^{\mathrm{H}} + \alpha\mathbf{I}_N\right)^{-1}\mathbf{H}$$

- Regularization factor : $\alpha > 0$
- Power normalization: $\lambda = \frac{1}{\operatorname{tr}\mathbf{W}\mathbf{W}^{\mathrm{H}}}$
- Total transmit power $\rho > 0$

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- SINR of UT k :

$$\gamma_k = \frac{\left(\mathbf{h}_k^{\mathsf{H}}\mathbf{Q}_k\mathbf{h}_k\right)^2}{\frac{1}{\rho\lambda}\left(1+\mathbf{h}_k^{\mathsf{H}}\mathbf{Q}_k\mathbf{h}_k\right)^2 + \mathbf{h}_k^{\mathsf{H}}\mathbf{Q}_k\mathbf{h}_k - \alpha\mathbf{h}_k^{\mathsf{H}}\mathbf{Q}_k^2\mathbf{h}_k}$$

where $\mathbf{Q}_k = \left(\mathbf{H}\mathbf{H}^{\mathsf{H}} - \mathbf{h}_k\mathbf{h}_k^{\mathsf{H}} + \alpha\mathbf{I}_N\right)^{-1}$.

Regularized Zero-forcing: Asymptotic analysis

- Using the same approach, we can find asymptotic approximations of

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- We can consider partial and no cooperation: (each BS serves $K/2$ UTs with half the transmit power):

$$\mathbf{W}_{i,\mathrm{pc}} = \left(\mathbf{H}_{i,i}\mathbf{H}_{i,i}^{\mathrm{H}} + \underbrace{\mathbf{H}_{i,\bar{i}}\mathbf{H}_{i,\bar{i}}^{\mathrm{H}}}_{\text{interference to other UTs}} + \alpha_i \mathbf{I}_{N/2}\right)^{-1} \mathbf{H}_{i,i}$$

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- Focus on sum-rate:

$$R_{\mathrm{sum}} = \frac{1}{N}\sum_{k=1}^{K} \log\left(1 + \gamma_k\right)$$

- Goal: Find the asymptotically optimal regularization parameters and BS-positions.

Downlink: RZF with full cooperation

Theorem (Deterministic equivalent of the sum-rate with full cooperation)

$$\bar{R}^{\text{full coop., DL}} = \frac{1}{cD} \int_0^D \log\left(1 + \gamma(x)\right) dx$$

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and $\boldsymbol{\psi}' = [\psi_1', \psi_2']\mathrm{T}$ *given as*

$$\boldsymbol{\psi}' = (\mathbf{I}_2 - \mathbf{J})^{-1}\begin{pmatrix}\psi_1^2\\ \psi_2^2\end{pmatrix}, \qquad [\mathbf{J}]_{k,l} = \frac{1}{D}\int_0^D \frac{\frac{c}{2}f_k(x)f_l(x)\psi_k^2}{\left(1+\sum_{b=1}^2 c_b f_b(x)\psi_b\right)^2}dx$$

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$$\bar{R}^{\text{full coop., DL}} = \frac{1}{cD}\int_0^D \log\left(1+\gamma(x)\right)dx$$

where

$$\gamma(x) = \frac{\left(\frac{c}{2}\sum_{i=1}^{2} f_i(x)\psi_i\right)^2}{\frac{1}{\rho\bar{\lambda}}\left(1+\frac{c}{2}\sum_{i=1}^{2} f_i(x)\psi_i\right)^2 + \frac{c}{2}\sum_{i=1}^{2} f_i(x)\psi_i - \alpha\frac{c}{2}\sum_{i=1}^{2} f_i(x)\psi_i'}$$

with $\boldsymbol{\psi} = [\psi_1, \psi_2]\,\mathsf{T}$ *given as the unique solution to*

$$\psi_i = \left(\alpha + \frac{1}{D}\int_0^D \frac{f_i(x)}{1+\sum_{b=1}^{2} c_b f_b(x)\psi_b}dx\right)^{-1}, \qquad i = 1,2$$

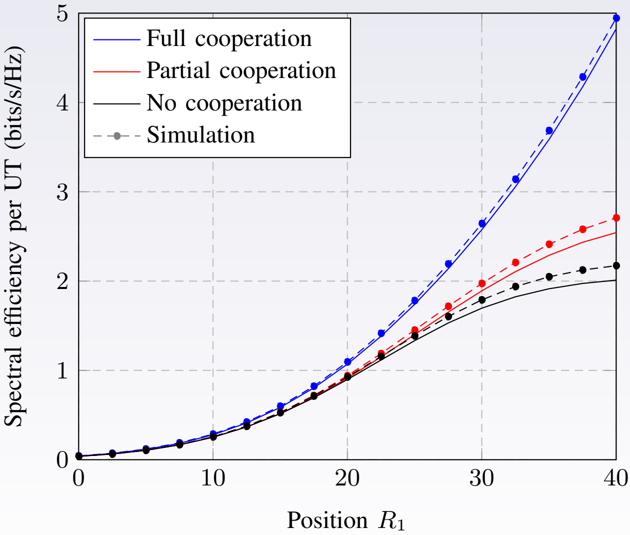
and $\boldsymbol{\psi}' = [\psi_1', \psi_2']\,\mathsf{T}$ *given as*

$$\boldsymbol{\psi}' = (\mathbf{I}_2 - \mathbf{J})^{-1}\begin{pmatrix}\psi_1^2\\ \psi_2^2\end{pmatrix}, \qquad [\mathbf{J}]_{k,l} = \frac{1}{D}\int_0^D \frac{\frac{c}{2} f_k(x) f_l(x)\psi_k^2}{\left(1+\sum_{b=1}^{2} c_b f_b(x)\psi_b\right)^2}dx$$

and

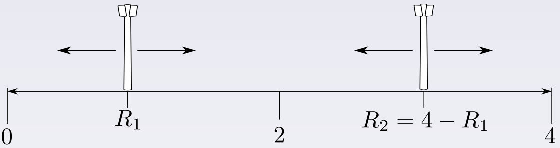
$$\bar{\lambda} = \frac{1}{N\left(\frac{1}{2}(\psi_1+\psi_2) - \alpha\frac{1}{2}(\psi_1'+\psi_2')\right)}.$$

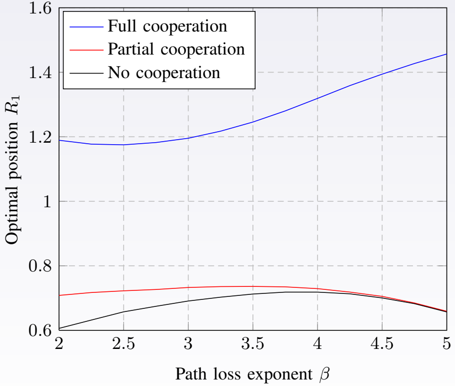
Downlink: Numerical results (I)

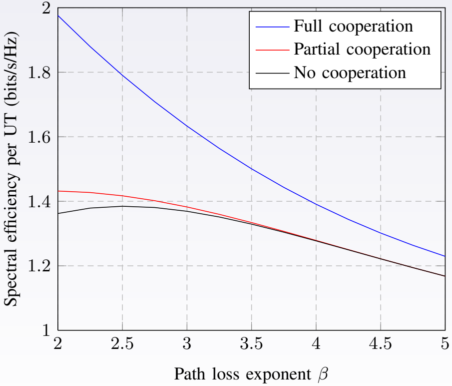


$N = 16$, $K = 12$, $\beta = 3.7$, $D = 4$, $R_1 = 0.35$, $R_2 = 3.65$, optimal regularization

Optimal Downlink BS-placement: Numerical results (II)







$N = 16, \; K = 12, \; \rho = 23\,\text{dB}, \; \beta = 3.7, \; D = 4$

Outline

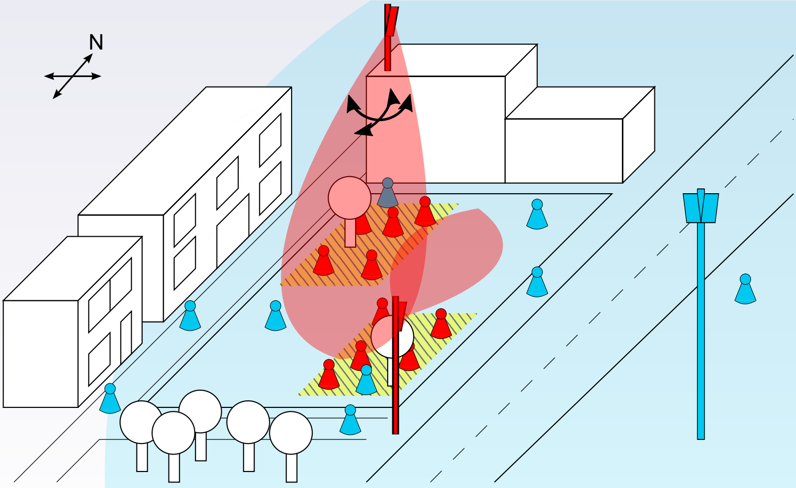
CDMA and point-to-point MIMO capacity
- Performance of CDMA systems
- Point-to-point MIMO performance

Multi-user multi-cell performance
- Sum rate performance and capacity region of MIMO-MAC
- Linearly precoded broadcast channels
- Multi-hop relay channels

Applications to cognitive radios
- Capacity inference methods

Research today: Green self-organizing small cell radios
- Cell planning
- 3D beamforming

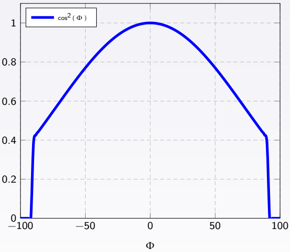
3D analog beamforming application



3D analog beamforming model

Model properties:

- uplink, multi-cell, full base station cooperation
- blue macro and red micro-cells on different bands
- tilting-capable base stations
- users distributed uniformly in patches
- antenna power radiation pattern: truncated $\cos^2$-beam, single beam, no sidelobe (no backside radiation), fixed azimuth angle.



Amplitude pattern (2D)
$\Phi = [-100, 100]^\circ$,

$\Phi_{3dB} = 160^\circ, \Theta = 0^\circ$ (fixed)

Simulation results: Single cell 3D-BF

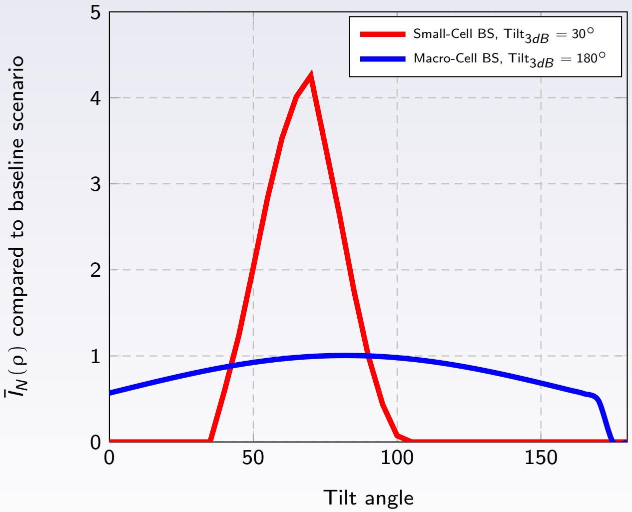


Figure: Sum rate gain w.r.t. 90°-tilt with 3D beamforming for a single base station in macro-cell or micro-cell, constant azimuth angle, uniform user distribution over central square.

Simulation results: Coordinated 3D-BF

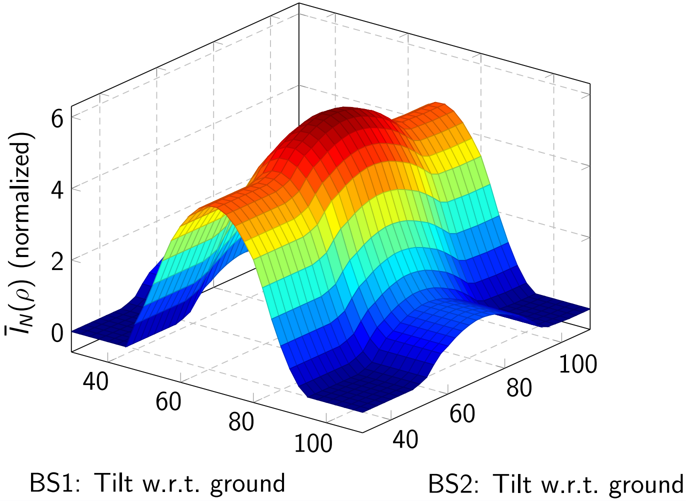


Figure: Achievable sum rate for different tilting scenarios of the small cell BSs.