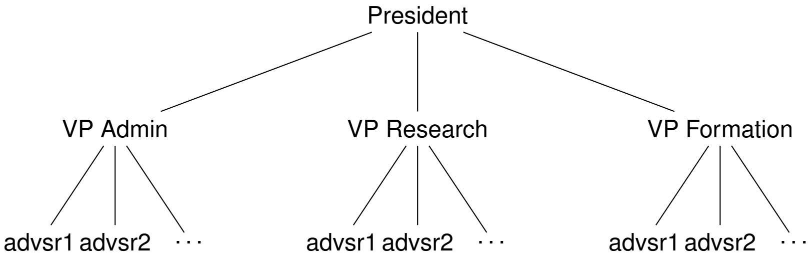
INF231:
Functional Algorithmic and Programming
Lecture 7: Tree-based structures

Academic Year 2023 - 2024

About Trees

Some motivation and intuition

Presidency of a University:



Remark

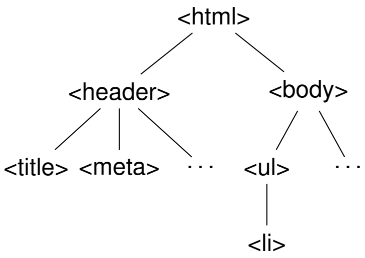
- "root" at the topmost level
- nodes with/without "subtrees"
- Hierarchical structure
- Implicit order or hierarchy. . . or not
- possible repetition

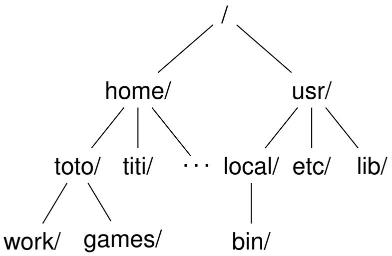
About Trees

Some motivation and intuition

Widely used in computer science mostly because of its notion of hierarchy: (contrarily to lists)

- sorting
- storing "efficiently"
  (e.g., a file system where files are organized in directories)
- compiling: programs are represented with trees
- structured documents, e.g., a web page
- modelling





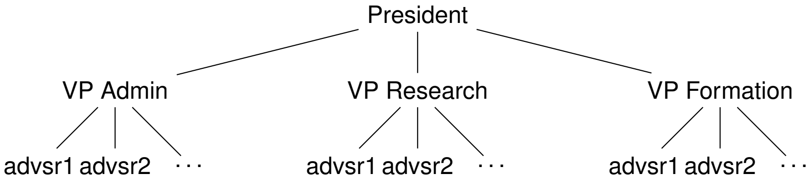
Defining trees
The definition

Definition (Tree)

A *tree* is a *hierarchical recursive data structure* which is either:

- empty
- a node containing a data and (sub) trees

Stores together data *of the same type* (similarly to list)



Vocabulary

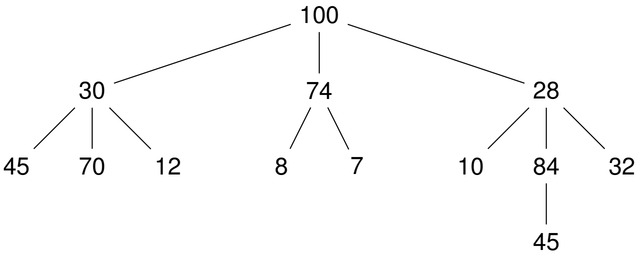
- The topmost node is called the root
- The data associated to a node is called its label / content
- The sub trees of a node are called the children
- The node directly containing subtrees is called the father of the subtrees
- The node containing subtrees is called an ancestor of the subtrees
- A node with an empty tree is called a leaf or a terminal node
- A branch of a tree is the list of nodes corresponding to a path from the root to a leaf
- level of a node: length of the branch to this node
- depth of a tree: the maximal level of the nodes in the tree
- size of a tree: the number of nodes in the tree

Remark Constraints can be put on, e.g.,

- the (maximal) number of children a node can have (e.g., binary trees: 2 children per node)
- how labels are ordered in the tree

An example

Example (A tree)



- root: 100
- labels: 100, 30, 64, 28, 45, 70, 12, 8, 7, 10, 84, 32
- leaves: 45, 70, 12, 8, 7, 10, 84, 32
- children of 30 are 45, 70, 12
- 100 is the direct father of 30
- 100 is at level 0, 7 is at level 2
- the depth of the tree is 3
- [100;30;12] is a path

Outline

Binary trees

Binary Search Trees

Definition (Binary Tree)

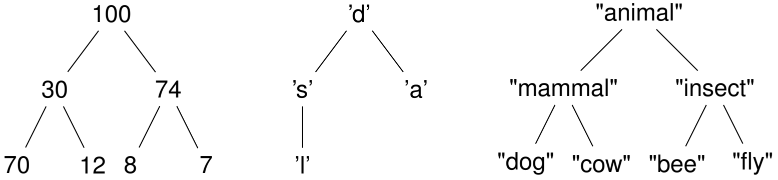
A tree is a binary tree if each node has *at most two children* (possibly empty)
Mathematically:

$$Bt(Elt) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in Elt \wedge tL, tR \in Bt(Elt)\}$$

Example (Binary trees of integers)

$$Bt(\mathbb{N}) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in \mathbb{N} \wedge tL, tR \in Bt(\mathbb{N})\}$$

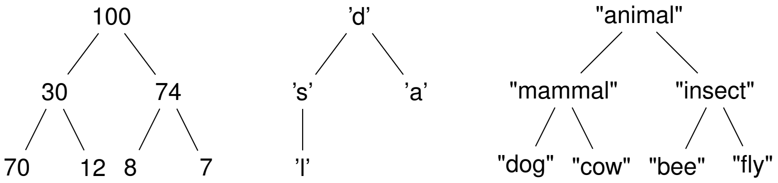
Example (Binary tree)



Vocabulary

- The first (resp. second) child is also called the left-hand (resp. right-hand) subtree
- A binary tree t is complete if $\mathrm{size}(t) = 2^{\mathrm{depth}(t)} - 1$

Example (Binary tree)



Binary trees of integers

In OCaml

```
type binary_tree =
  | Empty
  | Node of int * binary_tree * binary_tree
```

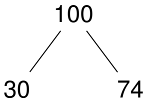
```
type binary_tree =
  | Empty
  | Node of binary_tree * int * binary_tree
```

```
type binary_tree =
  | Empty
  | Node of binary_tree * binary_tree * int
```

Remark

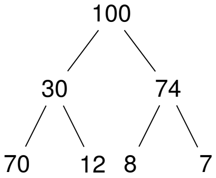
- Three equivalent definitions
- The type `binary_tree` has two constructors
- The constructor `EmptyT` (empty tree) is a constant
- The constructor `Node` is doubly recursive

Example (Defining a tree in OCaml)




```
let bt1 =
  Node (100,
      Node (30,EmptyT,EmptyT),
      Node (74,EmptyT,EmptyT)
   )
```

Example (Another tree)




```
let bt2 =
  Node(
   100,
   Node(30,
      Node(70,EmptyT,EmptyT),
      Node(12,EmptyT,EmptyT)
     ),
   Node(74,
      Node(8,EmptyT,EmptyT),
      Node(7,EmptyT,EmptyT)
     )
  )
```

Some (classical) functions on trees

Example (Depth)

The maximal level of the nodes

```
let rec depth (t:binary_tree):int=
 match t with
  | EmptyT → 0
  | Node (_, t1, t2) → 1+ max (depth t1) (depth t2)
```

Exercise

Define the two following functions

- `sum`: returns the sum of the elements of a tree
- `maximum` returns the maximal integer in the tree. Warning this function should not be called on an empty tree

Binary trees

Let's parameterize binary trees

We can parameterize binary trees by a type (polymorphism)

```
type α binary_tree =
   | EmptyT
   | Node of α * α binary_tree * α binary_tree
```

Many possible sorts of binary trees: `int binary_tree`, `char binary_tree`, `string binary_tree`,...

Remark The element of type α can be placed equivalently in the middle or on the right □

DEMO: Defining some binary trees

Example (Belongs to)

Is an element of type α in an α `binary_tree`?

```
let rec belongsto (elt:α) (t:α bintree):bool =
 match t with
  | Empty → false
  | Node (e,tl,tr) → (e=elt) || belongsto elt tl || belongsto elt tr
```

Example (The list of labels of a tree)

Given an α `binary_tree`, returns the α list of labels

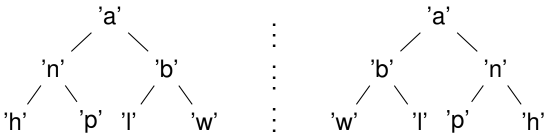
```
let rec labels (t:α bintree):α list=
 match t with
  | Empty → []
  | Node (elt,tl,tr) → (labels tl)@elt::(labels tr)
```

Polymorphic Binary trees

Some functions - let's practice

Exercise: Define the following functions

- `size`: returns the size of the tree (the number of nodes)
- `leaves`: given an α `bintree` returns the α `list` of leaves of this tree
- `maptree`: applies a given function to all elements of an α `bintree`
- `mirror`: returns the mirror image of an α `bintree`



Browsing a binary tree

Given a binary search tree, several functions are defined by "browsing the tree"

When encountering a `Node (elt, lst, rst)`, there are several possibilities according to the "moment" when `elt` is treated:

- treat `elt`, then browse `lst`, then browse `rst`: prefix browsing
- treat `lst`, then browse `elt`, then browse `rst`: infix browsing
- browse `lst`, then browse `rst`, then treat `elt`: suffix browsing

Iterators on binary tree

Iterator on a binary tree: `fold_left_right_root`: applies a function `f`

- to the root, and
- the results of left subtrees and right subtrees

```
let rec fold_lrr (f:α → β → β → β) (acc:β) (t:α bintree):β=
 match t with
   Empty → acc
  | Node (elt, l, r) →
   let rl = fold_lrr f acc l
    and rr = fold_lrr f acc r
        in f elt rl rr
```

Using iterators

Defining functions using iterators

Using the function `fold_lrr`, redefine the following functions:

- `size`
- `depth`
- `mirror`

Pathes in a tree

Exercise: Pathes in a binary tree: function `pathes`

The purpose is to define a function that computes maximal pathes in a tree:

- How can we represent a path and a set of pathes?
- Define a function `add_to_each` that adds an element as the head to each path in a set of pathes
- Using the previously defined function define the function `pathes`

Properties of size and depth

- $depth(t) \leq size(t)$
- $size(t) \leq 2^{depth(t)-1}$

How to prove them?

Structural induction to prove some property P

Consider $Bt(Elt) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in Elt \wedge tL, tR \in Bt(Elt)\}$
To show that $\forall t \in Bt(Elt) : P(t)$

- prove $P(Et)$
- prove

$$\forall tL, tR \in Bt(Elt) : P(tL) \wedge P(tR) \Rightarrow (\forall e \in Elt : P(Node(tL, e, tR))$$

Exercise: some proofs

Prove the above properties using structural induction

Outline

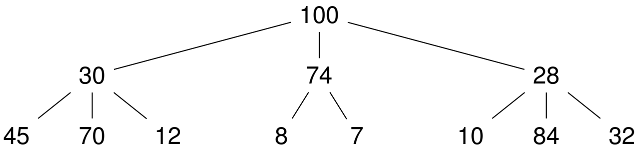
Binary trees

Binary Search Trees

Motivation

Let us come back on the `belongsto` function:

```
let rec belongsto (elt:α) (t:α bintree):bool =
 match t with
  | Empty → false
  | Node (e,tl,tr) → (e=elt) || belongsto elt tl || belongsto elt tr
```



How can we be sure that an element does not belong to the tree?
↪ one has to browse the whole tree
(similarly to what would happen with a list)

Search time depends on the size of the tree

→ solution consists in sorting the elements of the tree

Binary Search Tree: definition

Definition: Binary Search Tree (BST)

A binary search tree is a binary tree s.t. for every node of the tree of the form `Node(elt,lst,rst)`, where `e` is the data carried out by the node, and `lsb` (resp. `rsb`) is the left (resp. right) sub tree of the node, we have:

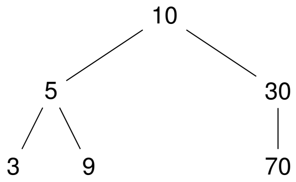
- `lst`, `rst` are binary search trees
- elements of `lst` are all lesser than or equal to `e`
- `e` is (strictly) lesser than all elements in `rst`

Remark

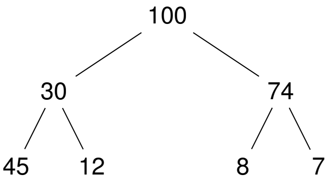
- Binary search trees suppose that the set of the elements of the tree has a total ordering relation
- "lesser than" in the definition is understood w.r.t. this ordering relation
- Elements can be of any type: `int`, `string`, `students`... as long as there is an ordering relation

Binary Search Tree
(counter) Example

Example (A binary search tree)



Example (NOT a binary search tree)



Revisiting the `belongsto` function

We can exploit the property of binary search trees

Example (Does an element belong to a binary search tree?)

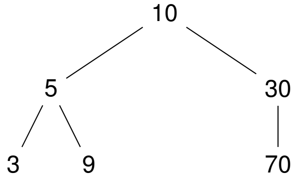
```
let rec belongsto (elt:α) (t:α bst):bool=
  match t with
    | Empty → false
    | Node (e, lbst, rbst) →
      (e=elt)
      || (e>elt) && belongsto elt lbst
      || (e<elt) && belongsto elt rbst
```

One subtree examined at each recursive call

Some sort of "dichotomic" search

An execution of `belongsto`

Let's search 9 in the following tree:



		searching 9 in `Node`(10,...,...)
$9 < 10$	$\Rightarrow$	searching 9 in `Node`(5,...,...)
$9 > 5$	$\Rightarrow$	searching 9 in `Node`(9,...,...)
$9 = 9$	$\Rightarrow$	`true`

DEMO: Tracing `belongsto` 9 ...

Browsing a tree

Given a binary search tree, how to put the elements in order in a list?
↪ browsing the tree

When encountering a `Node (elt, lst, rst)`, there are several possibilities according to the "moment" when `elt` is treated:

- place `elt`, then `lst`, then `rst`: prefix browsing
- place `lst`, then `elt`, then `rst`: infix browsing
- place `lst`, then `rst`, then `elt`: suffix browsing

Following the property of binary search trees, the infix browsing gives us the solution:

```
let rec tolistinorder (t: α bst):α list=
 match t with
   Empty → []
  | Node (elt, lbst, rbst) →
       (tolistinorder lbst)@elt::tolistinorder rbst
```

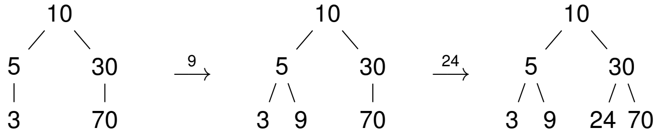
Insertion in a binary search tree

Insert the element as a leaf (simplest method)

Objective: insert an element `elt` in a binary search tree `t`

- preserve the binary search tree property
- insert the element as a leaf of the tree

Example (Inserting two elements)



Idea: recursively distinguish two cases

- `t` is empty, then by inserting `elt` we obtain `Node(elt, Empty, Empty)`
- `t` is not empty, then it is of the form `Node(e,lbst,rbst)`, then
  - if `elt <= e`, then `elt` should be placed in `lbst`
  - if `elt > e`, then `elt` should be placed in `rbst`

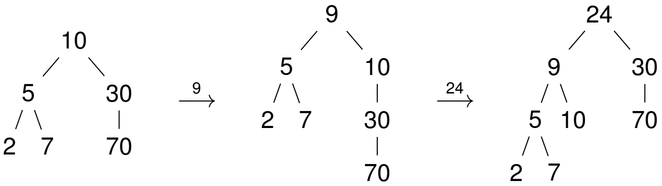
Insertion in a binary search tree

Insert the element as a root

Objective: insert an element `elt` in a binary search tree `t`

- preserve the binary search tree property
- insert the element as the *root* of the tree

Example (Inserting two elements)



Idea: proceed in two steps:

- "cut" the tree into two binary search subtrees `l` and `r` s.t.
  - `l` contains all elements smaller than `elt`
  - `r` contains all elements greater than `elt`
- Build the tree `Node (elt,l,r)`

Exercise: insertion as a leaf

Define the function `insert` that inserts an element in a BST, as a leaf

Exercise: insertion as the root

Define the functions:

- `cut` that cuts a binary search tree as described before
- `insert` that inserts an element in a binary tree as the root, using `cut`

Exercise: Binary Search Tree creation

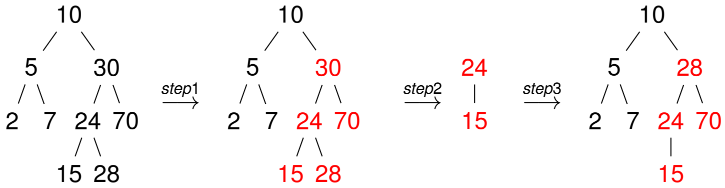
Define two functions `create_bst` that, given a list of elements create a binary search tree of the elements in the list, using the two insertion methods

Suppressing an element in a BST

Suppressing an element in `elt` a BST consists in:

1. Identify the subtree `Node`(`elt`, `lst`, `rst`) (where suppression should occur)
2. Suppress the greatest element `max` of `lst` $\rightarrow$ we obtain a BST `lstprime`
3. Build the tree `Node`(`max`,`lstprime`, `rst`)

Example (Suppressing 30)



Exercise: suppression in a tree

Define the functions:

- `remove_max` that remove the greatest element in a tree
  To ease the definition of the subsequent function, it is better if this function returns both the maximal element and the new tree
- `suppression` that suppresses an element in a BST

Exercise: Is a Binary Tree a Binary Search Tree?

Define the function `is_bst` that checks whether a binary tree is a BST

Conclusion

Summary:

About trees:

- Hierarchical "objects"
- Doubly recursive data type
- Two variants (binary trees and binary search trees) (there exist many others)
- Several functions to manipulate them