INF231:
Functional Algorithmic and Programming
Lecture 3: Advanced types

Academic Year 2023 - 2024

Modelling information/concepts

What is modelling?

Why modelling?

How to model?

- defining specific data types
- defining functions manipulating these data types

Defining a type

The general form

```
type t = ... (* possibly with constraints *)
```

Now we are going to see how we can define some more complex types using existing types. . .

Outline

Defining a synonym type

Motivations:

- context-specific types
- easier to remember
- re-use

General syntax:

```
type new_type = existing_type
     (* possibly with informative usage constraints *)
```

Example (Soldes)

- `type price = float (* > 0 *)`
- `type rate = int (* 0, ..., 99 *)`
- Defining a function to reduce prices:
  - Description: `reducedPrice`(`p`,`r`) is the price `p` reduced by `r`%
  - Profile: `reducedPrice`: `price` ⋆ `rate` → `price`
  - Examples: `reducedPrice`(100., 25) = 75.

  (note that it is more meaningful than the "anonymous signature" `reducedPrice`: `float` ⋆ `int` → `float`)

Outline

Synonym types

Enumerated types

Product types

Union/Sum types

Case study: Modelling 4 card games

Enumerated types

Motivation: How can we model/define/use:

- the family of a card? $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$
- the color of a card? $\{\text{black}, \text{white}\}$

From a mathematical point of view: sets defined extensively
↪ i.e., by an *explicit enumeration*

Defining an enumerated type in OCaml:

```
type new_type = Value_1 | Value_2 | ... | Value_n
```

Remark

- Capital letters are mandatory
- `new_type` is said to be an *enumerated type*
- `Value_1`, ..., `Value_n` are said to be *symbolic constants*
- `Value_1`, ..., `Value_n` are of type `new_type`
- Implicit order between constants (consequence of the definition)

Enumerated types: Some examples

Painting / Modelling a card game

Example (Some paint colors)

```
type paint =
  | Red
  | Blue
  | Yellow
```

Example (Types of a Card game)

```
type family = Spade | Heart
            | Diamond | Club
type color = White | Black
```

DEMO: types of card game

Example (Color of a family)

Returning the color associated to a family card

- Description: `colorFamily` returns the family of a given card.
  - Heart and Diamond are associated to White
  - Spade and Club are associated to Black
- Signature: `colorFamily`: `family` $\rightarrow$ `color`
- Examples: `colorFamily Spade = Black, ...`

DEMO: Implementation of colorFamily

Back to the language constructs: **pattern-matching**

Your best friend

One of the most powerful features of OCaml (and functional languages)

Pattern-matching: computation by **case analysis**

Specified by the following syntax:

```
match expression with
  | pattern_1 → expression_1
  | pattern_2 → expression_2
    ...
  | pattern_n → expression_n
```

Meaning:

- `expression` is matched against the patterns, i.e., its value is evaluated and then compared to the patterns in order
  ↪ "matching" depends on the type of `expression`!
- the expression associated to the first matching pattern is returned

Remark

- First vertical bar is optional
- may use _ as a *wild-card* (should be the last pattern)

(Pattern) Matching on an example

The card game

Example (`colorFamily` using `if...then...else`)

```
let colorFamily (f:family):color =
  if (f=Spade || f = Club) then Black
  else (* necessarily f = Heart || f = Diamond *)
   White
```

Example (`colorFamily` using pattern-matching)

```
let colorFamily (f:family):color =
   match f with
     | Spade → Black
     | Club → Black
     | Heart → White
     | Diamond → White
```

(Pattern) Matching on an example

The card game with more concise pattern-matching

Example (`colorFamily` using a more concise pattern-matching)

```
let colorFamily (f:family):color =
  match f with
    Spade | Club → Black
    | Heart | Diamond → White
```

Example (`colorFamily` using an even more concise pattern-matching)

```
let colorFamily (f:family):color =
  match f with
    Spade | Club → Black
     | _ → White
```

Example (`colorFamily` using an even even more concise pattern-matching)

```
let colorFamily = function | Spade | Club → Black
  | _ → White
```

Pattern-matching for enumerated types

To the enumerated type

```
type newtype = Value_1 | Value_2 | ... | Value_n
```

is associated the pattern matching

```
match expression with (* expression is of type newtype *)
  | Value_1→ expression_1
  | Value_2 → expression_2
  ...
  | Value_n → expression_n
```

Rules

- Pattern-matching "follows" the definition of the type (not necessarily with the same order)
- `expression_i` for $i \in \{1, \ldots, n\}$ should be of the same type
- Should be exhaustive (or use the wild-card symbol _)

```
match expression with
  | Value_1→ expression_1
  ...
  | _ → expression
```

Let's practice enumerated types

Exercise

- Define the enumerated type `month` which represents the twelve months of the year
- Define the function `nb_of_days`: `month` $\rightarrow$ `int` which associates to each month its number of days

Matching (also) works (more or less) with (some) predefined types

Pattern-matching is a generalization of the `if`...`then`...`else`...
↪ works with existing/predefined types: int, bool, float, char, string

Example (Is an integer an even number?)

```
let is_even (n : int) : bool =
 match n with
  | 0 → true
  | 1 → false
  | 2 → true
  | n → if n mod 2 = 0 then true else false
```

Example (Is a character in upper case?)

```
let is_uppercase (c:char) = match c with
   'A' → true
  | 'B' → true
  | ... (* 23 conditions *)
  | 'Z' → true
  | c → false
```

Example (Matching with floats is dangerous)

```
match 4.3 −. 1.2 with
 3.1 → true
 _  → false
```

⇝ returns `false`

Some shortcuts with pattern-matching

For enumerated types

"Disjuncting equivalent patterns":

```
match something with
 ...
 | p1 → v
 | p2 → v
 ...
 | pm → v
 ....
```

can be shortened into

```
match something with
 ...
 | p1 | p2 | pm → v
 ....
```

Example ("Disjuncting equivalent patterns")

```
let is_uppercase (c:char) = match c with
   'A' | 'B' | 'C' | 'D' | 'E' | 'F' | 'G' | 'H' | 'I' | 'J' | 'K' | 'L' | 'M'
   | 'N' | 'O' | 'P' | 'R' | 'S' | 'T' | 'U' | 'V' | 'W' | 'X' | 'Y' | 'Z' → true
   | c → false
```

Some shortcuts with pattern-matching - ctd

For characters

"Leveraging the order between characters":

```
match something with
 ...
 | p1 → v
 | p2 → v
 ...
 | pm → v
 ....
```

⇝

```
match something with
 ...
 | p1 .. pm → v
 ....
```

or

```
match something with
 ...
 | pm .. p1 → v
 ....
```

where `p1`, .., `pm` are *consecutive* characters and `p1` and `pm` are the minimal and the maximal characters (not necessarily in this order)

Example ("Leveraging the order between the elements of characters")

```
let is_uppercase (c:char)
    = match c with
  'A' .. 'Z' → true
  | c → false
```

or

```
let is_uppercase (c:char)
    = match c with
  'Z' .. 'A' → true
  | c → false
```

Outline

Product type: motivating example(s) and connection with maths

Example (Some complex numbers)

How can we model complex numbers?
In maths, we define:

$$\mathbb{C} \quad = \{a + ib \mid a \in \mathbb{R}, b \in \mathbb{R}\}$$

z	**a**	**b**
$3.0 + i * 2.5$	3.0	2.5
$12.0 + i * 1.5$	12.0	1.5
$(1.0 + i) * (1.0 - i)$		

Actually, we could also define:

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}$$

The operation $\times$ is the Cartesian product of sets

Example (Defining card)

Same reasoning can be followed if we want to define the type of a card. . .

(Cartesian) Product (of) type

We can build Cartesian product of types, i.e., pairs of object of different types:

Type Constructor	Value Constructors
$\alpha * \beta$	•,•
`int*int`	`1,2`
`int*float`	`1,2.0`

DEMO: A couple of pairs

Defining new product types:

```
type new_type = existing_type1 * existing_type2
```

Two basic operations on pairs:

- `fst` (•$_1$,•$_2$) = •$_1$
- `snd` (•$_1$,•$_2$) = •$_2$

Deconstruction on pairs (hidden pattern matching):

```
let (x1,x2) = (v1,v2) in expression_using_x1_and_x2
```

↪ defines the identifiers `x1` and `x2` locally

DEMO: Product types

General Cartesian product of types

Same principle

Can be generalized to *n*-tuples:

- type definition/constrcution:
  `let my_type = type1 * type2 * ... * typen`
- value *construction*: `v1,v2,...,vn`
- value *deconstruction*:

```
let (x1,...,xn) = (v1,...,vn) in expression
(* expression is depending on x1,...,xn *)
```

DEMO: Generalized Product types

Let's practice product type

Exercise: Getting familiar with tuples

- Define the type `pair_of_int` which implements pairs of integers
- Define the function `swap` which swaps the integers in a `pair_of_int`
- Implement a function `my_fst` which behaves as the predefined function `fst` on `pairs_of_int`

Exercise on Complex numbers

- Define the type `complex` which corresponds to complex numbers
- Define function `real_part` of type `complex` $\rightarrow$ `float` which returns the real part of a complex number
- Define function `im_part` of type `complex` $\rightarrow$ `float` which returns the imaginary part of a complex number
- Define function `conjugation`: `complex` $\rightarrow$ `complex`
  Remainder: the conjugation of $a + b.i$ is $a - b.i$

Exercise on vectors

- Define the type `vect` which corresponds to vectors in the plane
- Define the function `sum` : `vect` $\rightarrow$ `vect` $\rightarrow$ `vect` which performs the sum of two vectors
- What is the type of the function which implements the scalar product?
- Implement a function which performs the scalar product of two vectors
  Remainder: scalar product of two vectors $\overrightarrow{u}$, $\overrightarrow{v}$: $||\overrightarrow{u}||.||\overrightarrow{v}||.\cos(\overrightarrow{u}, \overrightarrow{v})$
  with $\cos(\overrightarrow{u}, \overrightarrow{v}) = \frac{u_x.v_x + u_y.v_y}{||\overrightarrow{u}||.||\overrightarrow{v}||}$
- A vector can represent the position of a point in the plane. The rotation of angle θ of a point of coordinates (x, y) around the origin is expressed by the formula:
  $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$
  Implement the function `rotation`: `float` $\rightarrow$ `vect` $\rightarrow$ `vect` such that `rotation angle v` makes the vector designated by v rotating of an angle `angle`

Outline

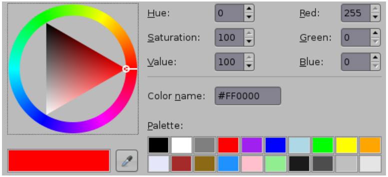
Motivating union types

Mixing carrots and cabbage

. . . in the context of OCaml type system

Some concepts that we cannot model yet:

- How to build a type `figure` which can represent circles, triangles, quadrilaterals?
- How to build a type which allows to represent a full color palette ?



- How to build a card game which can represent various games?

Back to the paint

Introducing Union types through an example

Type definition	Filtering

`type paint =`
`  Blue`
`  | Yellow`
`  | Red`
	`let is_blue (p : paint) : bool =`
	`  match p with`
	`    | Blue → true`
	`    | Yellow → false`
	`    | Red → false`

Remark The type `paint` contains *three constant constructors* □

How can we add to the set of paints, some new paints that do not have a name, but only reference number?

Back to the paint

Introducing Union types through an example

Type definition	Filtering

`type paint =`
`  | Blue`
`  | Yellow`
`  | Red`
`  | Number of int`
	`let is_blue (p : paint) : bool =`
	`  match p with`
	`    | Blue → true`
	`    | Yellow → false`
	`    | Red → false`
	`    | Number i → false`

Remark

- Type `paint` has 3 constant constructors and one non constant constructor.
- `Number` 14 represents the paint numbered 14 (in an imaginary catalogue)

Back to the paint

Introducing Union types through an example

Type definition	Filtering
`type paint =`	`let is_blue (p : paint) : bool =`
`  | Blue`	`  match p with`
`  | Yellow`	`    | Blue → true`
`  | Red`	`    | Yellow → false`
`(* palette RGB *)`	`    | Red → false`
`  | RGB of int * int * int`	`    | RGB (r,g,b) → r = 0 && g = 0 && b = 255`

- Type `paint` has three constant constructors and two non-constant constructors
- `RGB`(255,0,0) corresponds to red
- `RGB`(255,255,0) corresponds to yellow
- ...

Union types (aka union type, tagged union, algebraic data types)

The general form

Syntax of union types:

```
type new_type =
  | Identifier_1 of type_1
  | Identifier_2 of type_2
  ...
  | Identifier_n of type_n
```

Note that:

- `Identifier_i`, $i \in [1, n]$, is an explicit name called a constructor
- the definition "`of type_i`" is optional
- `type_i`, $i \in [1, n]$, can be any (existing) type
- Constructor name must be capitalized

Expression Declaration (of some type `t`):

```
let expression = Identifier v
```

(if `Identifier of tt` is a constructor of type `t` and `v` is a value of type `tt`)

Remark

- Union types are a generalization of enumerated types

An example: Generalization of `int` and `float`

Having two different sets of operations for `int` and `float` is sometimes annoying

Let's define $\text{Numbers} = \mathbb{R} \cup \mathbb{N}$

```
type numbers = INTEGER of int | REAL of float
```

(`INTEGER`, `REAL` sont des contructeurs de type)

Let's define additions on two numbers:

```
let add ((nb1,nb2): number*number) : number= match (nb1,nb2) with
 | (INTEGER(n1), INTEGER(n2)) → INTEGER(n1 + n2)
 | (INTEGER(n) , REAL(r))   → REAL( (float_of_int n) +. r)
 | (REAL(r)  , INTEGER(n)) → REAL( (float_of_int n) +. r)
 | (REAL(r1)  , REAL(r2))  → REAL(r1 +. r2)
```

Remark Has some advantages and disadvantages □

Another example: Geometry

Type definition	Filtering
`type pt = float * float` `type figure =` `	Rectangle of pt * pt` `	Circle of pt * float` `	Triangle of pt * pt * pt`	`let perimeter (f : figure) : float =` `  match f with` `	Rectangle (p1 , p2) → ...` `	Circle (_ , r) → ...` `	Triangle (p1 ,p2 , p3) → ...`

```
let p1 = 1.0, 2.0 and p2 = 3.9, 2.7 in Rectangle (p1,p2)
let p1 = (1.3, 2.9) in Circle (p1,3.6)
```

Exercise

- Define the function `distance`: `pt → pt → float`
- The area of any triangle of edge lengths *a*, *b*, *c* is computed using the Héron's formula:

$$A = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)} \quad \text{with} \quad s = \frac{1}{2} \cdot (a+b+c)$$

Define the function `area`: `figure → float`

Remark: *distinguish* constructors and functions

Constructors and functions take a value of some type and return another value of some other type.

A *function*:

- performs a computation
- cannot be used in pattern matching: the value of all functions is `<fun>`

A *type constructor*:

- constructs a value
- can be used in a pattern-matching

Remark: Difference between union and sum

There is actually a slight difference between union and sum

Consider two sets E and F:

Union	Sum
$E \cup F$	$\{\mathrm{FromE}(x) \mid x \in E\} \cup \{\mathrm{FromF}(x) \mid x \in F\}$
"everything is merged/mixed"	"elements are decorated" and then merged

Second solution is less ambiguous and then preferred by computers

Card Game

Your choice

Playing cards:

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Conclusion

Summary:

- Richer types:

Type	Why?
synonym types	informative type names
enumerated types	Finite set of constants
product types	Cartesian product
sum types	Set Union

- Using filtering and pattern matching to define more complex functions (for each of these types)

Exercise

Find a (personal) example of objects that can be naturally modelled as a union type. Propose/Invent a function using this type.