Local failure localization in large sensor networks

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Track D. Signal Processing and Adaptive Systems

***Abstract*—In this article, the joint fluctuations of the extreme eigenvalues and eigenvectors of a large dimensional sample covariance matrix are analyzed when the associated population covariance is a finite-rank perturbation of the identity matrix. It is shown that these fluctuations are asymptotically normal with zero mean and a variance which is derived explicitly. This result is used in practice to develop an original framework for local failure localization in large sensor networks, among which sudden parameter changes.**

I. Local failure detection: sudden parameter change

In this summary, we concentrate consider on the following sensor network model

$$y = H\theta + \sigma w \tag{1}$$

where $H \in \mathbb{C}^{N\times p}$ is deterministic, $\theta \in \mathbb{C}^p$ and $w \in \mathbb{C}^N$ have independent and identically distributed (i.i.d.) complex standard Gaussian entries and $\sigma > 0$. In a sensor network composed of N nodes, y represents the observation through the channel H of the vector θ, constituted of centered and normalized independent Gaussian system parameters[1], impaired by white Gaussian noise. Therefore, $\mathrm{E}[yy^*] = HH^* + \sigma^2 I_N \triangleq R$.

Suppose now that, $\theta(k)$, the k^{th} entry of θ experiences a sudden change in mean and variance. The resulting observation y' can be modeled as

$$y' = H(I_p + \alpha_k e_k e_k^*)\theta + \mu_k H e_k + \sigma w$$

for some real parameters μ_k, α_k and $e_k \in \mathbb{C}^p$ defined by $e_k(k) = 1$ and $e_k(i) = 0$, $i \neq k$. We suppose here that μ_k and α_k are a priori known to the experimenter (an assumption which will be relaxed in future work). Denoting $R = HH^* + \sigma^2 I_N$ as in the previous example and taking $s = R^{-\frac{1}{2}}y'$, we finally have

$$\mathrm{E}[ss^*] = I_N + (\mu_k^2 + (1+\alpha_k)^2 - 1)R^{-\frac{1}{2}} H e_k e_k^* H^* R^{-\frac{1}{2}}$$

which is here a rank-1 perturbation of the identity matrix by the matrix

$$P_k \triangleq (\mu_k^2 + (1+\alpha_k)^2 - 1)R^{-\frac{1}{2}} H e_k e_k^* H^* R^{-\frac{1}{2}}. \tag{2}$$

This generalizes to sudden changes of multiple parameters. If the means and variances for the sensors $k_1, \ldots, k_M$ are modified simultaneously through the parameters $\mu_{k_1}, \ldots, \mu_{k_M}$ and $\theta(k_1), \ldots, \theta(k_M)$, then

$$\mathrm{E}[ss^*] = I_N + R^{-\frac{1}{2}} H E \Lambda E^* H^* R^{-\frac{1}{2}}$$

with $E = [e_{k_1}, \ldots, e_{k_M}]$ and $\Lambda = \mathrm{diag}(\sigma_{k_1}^2, \ldots, \sigma_{k_M}^2)$, $\sigma_k^2 = \mu_k^2 + (1+\alpha_k)^2 - 1$, which is a rank-$M$ perturbation of the identity matrix.

Similar small rank perturbation models in the context of local failures in large networks are provided in the complete version of this article.

A. Detection and localization

For the model above, let us assume a general scenario with K possible sudden change events, identified by the index $1 \leq k \leq K$ and let now $s_1, \ldots, s_n$ be n successive independent observations of the random variable s. Denote then $\Sigma \triangleq \frac{1}{\sqrt{n}}[s_1, \ldots, s_n] \in \mathbb{C}^{N\times n}$. From the fact that s is Gaussian with zero mean and covariance $(I_N + P_k)$ for a certain k, we can write

$$\Sigma = (I_N + P_k)^{\frac{1}{2}} X \tag{3}$$

where $X \in \mathbb{C}^{N\times n}$ is a given matrix with Gaussian independent entries of zero mean and variance $1/n$. For simplicity here, we assume that for each k the non-zero eigenvalues of P_k are all distinct. We also denote here $P_0 = 0$ for the extra event $k = 0$ corresponding to the no-change scenario. The objective of our study is to provide a detection and localization test based on Σ to decide (i) if $k = 0$ is most likely than $k > 0$ and (ii) if $k > 0$, which model index $k \in \{1, \ldots, K\}$ is the most probable.

II. Main result

The model (3), known as *spike model* has been recently studied in e.g. [1], [2], [3]. These works characterized the limits of the extreme eigenvalues and associated eigenvector projections of $\Sigma\Sigma^*$ leaving the support of the limiting eigenvalue distribution. This happens if and only if $\omega_i > \sqrt{c}$ for some i, an assumption we consider here. The fluctuations of the extreme eigenvalues have also been derived. Our main mathematical result follows this series of works and is concerned with the *joint fluctuations* of the eigenvalue and eigenvector projections of $\Sigma\Sigma^*$.

Theorem 1: Let $X \in \mathbb{C}^{N\times n}$ have i.i.d. Gaussian entries of zero mean and variance $1/n$, and $P = \sum_{i=1}^K \omega_i u_i u_i^*$ in its spectral decomposition with $\omega_1 > \ldots > \omega_K > 1$. Denote $\Sigma = (I_N + P)^{\frac{1}{2}} X$ with K largest eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_K$

[1] Up to a right-product of H by a positive diagonal matrix, the variance of the entries of θ can be assumed all equal to one without loss of generality.

and associated eigenvectors $\hat{u}_1, \ldots, \hat{u}_K$. Then, as $N, n \to \infty$ with $N/n \to c$, if $\omega_i > \sqrt{c}$,

$$\sqrt{N}\begin{pmatrix} |u_k^* \hat{u}_k|^2 - \xi(\omega_k) \\ \hat{\lambda}_k - \rho_k \end{pmatrix} \Rightarrow \mathcal{N}(0, C_k)$$

where $\rho_k = 1 + \omega_k + c\frac{1+\omega_k}{\omega_k}$, $\xi(\omega_k) = \frac{1-\omega_k^{-2}}{1+\omega_k^{-1}}$ and

$$C_k = \begin{bmatrix} \frac{c^2(1+\omega_k)^2}{(c+\omega_k)^2(\omega_k^2-c)}\left(c\frac{(1+\omega_k)^2}{(c+\omega_k)^2}+1\right) & \frac{(1+\omega_k)^3 c^2}{(\omega_k+c)^2\omega_k} \\ \frac{(1+\omega_k)^3 c^2}{(\omega_k+c)^2\omega_k} & \frac{c(1+\omega_k)^2(\omega_k^2-c)}{\omega_k^2} \end{bmatrix}.$$

Moreover, for $k \neq k'$ such that $\omega_, \omega_{k'} > \sqrt{c}$, the fluctuations of $\hat{\lambda}_k, \hat{u}_k$ and $\hat{\lambda}_k', \hat{u}_k'$ are asymptotically independent.
The proof of this result relies on a random matrix framework based on the Stieltjes transform method and complex integration. The result provided in the complete version of this article is more general as X is only assumed unitarily invariant but non necessarily Gaussian and the values ω_i may have multiplicities.

Application-wise, the main contribution of this article is the derivation of a method for detecting and localizing sudden parameter changes in the system (1). The detection consists in a known statistical test, already used in the context of signal sensing, to decide whether the largest eigenvalue $\hat{\lambda}_1$ of $\Sigma\Sigma^*$ exceeds a critical threshold (usually around the value $(1+\sqrt{c})^2$, the right-edge of the asymptotic spectrum of $\Sigma\Sigma^*$ under the null hypothesis).

The novelty of the present article lies in the localization framework, upon change detection, which can be performed here by testing the most probable perturbation matrix P_k in the model of Equation (2). The test relies on a "maximum-likelihood"-like procedure based on the asymptotic distribution of the eigenvector projections corresponding to the largest eigenvalues of $\Sigma\Sigma^*$, as well as on the asymptotic distribution of the largest eigenvalues if μ_k, σ_k are a priori known. That is, the test consists in the following estimator $k^\star$ for the change index:

$$k^\star = \arg\max_{1\le k\le K} \prod_{i=1}^{r_k} f\left(\sqrt{N}(\hat{v}_{k,i} - v_{k,i}); C_{k,i}\right)$$

where $f(x; C)$, $C \in \mathbb{C}^{m\times m}$, is the m-variate real normal density of zero mean and variance C at point x, and $C_{k,i}$ is defined as a straightforward application of Theorem 1, where we denoted

$$\hat{v}_{k,i} \triangleq \begin{pmatrix} |u_{k,i}^* \hat{u}_i|^2 \\ \hat{\lambda}_i \end{pmatrix}, \; v_{k,i} \triangleq \begin{pmatrix} \xi(\omega_{k,i}) \\ \rho_{k,i} \end{pmatrix}$$

where we assumed $P_k = \sum_{i=1}^{r_k} \omega_{k,i} u_{k,i} u_{k,i}^*$ and $\rho_{k,i} = 1 + \omega_{k,i} + \frac{N}{n}\frac{1+\omega_{k,i}}{\omega_{k,i}}$.

A. *Simulations*

We apply the localization algorithm above for the network model $y = H\theta + \sigma w$, with $N = 10$, where the entries of HH^* are presented in Figure 1. We assume that the entries of θ are $\mathcal{C}N(0,1)$ are that θ_1 suddenly drops to zero. The minimum observability ratio N/n is 0.8. In Figure 2, we depict Monte

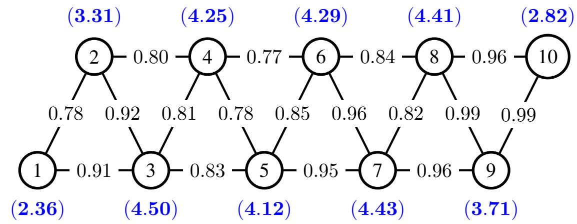

Fig. 1. Network of $N = 10$ sensors. The correlation $\mathrm{E}[y(i)^* y(j)]$ between data on sensors i and j, $i \neq j$, can be read on the link (i, j), while $\mathrm{E}[|y_{(i)}|^2]$ variances are shown in parentheses.

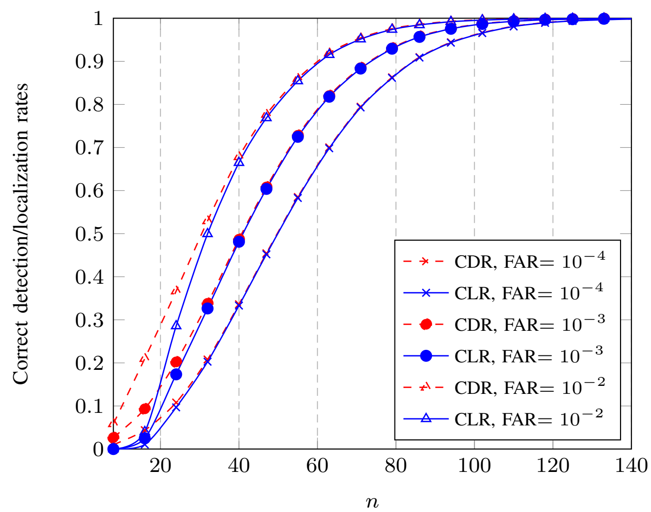

Fig. 2. Correct detection (CDR) and localization (CLR) rates for different levels of false alarm rates (FAR) and different values of n, for failure of parameter θ_1 in the sensor network of Figure 1. The minimal theoretical n for asymptotic observability is $n = 8$.

Carlo simulation curves of sudden change detection and localization for different detection false alarm rates (FAR), with increasing observation window size. The simulation shows that more than $n = 8$ observations are needed to carry out perfect localization, which is explained by the loose approximation of the asymptotic Gaussian fluctuations for perturbation matrices P_k with $\omega_{k,i} \simeq \sqrt{c}$.

III. Conclusion

In this article, a computationally cheap local failure localization framework is designed based on the study of the asymptotic fluctuations of eigenvalues and eigenvector projections of small rank perturbation of Gaussian random matrices. Simulations applied to sudden parameter change localization corroborate the efficiency of the approach but also raise its inherent limitations.

References

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