HABILITATION À DIRIGER DES RECHERCHES

CENTRALE–SUPÉLEC

Université Paris-Sud 11
École Doctorale STITS

Méthodes d'estimation robuste dans le régime des grandes matrices aléatoires

(Robust estimation methods in the large random matrix regime)

Présentée par:

Romain COUILLET

le **2 février 2015** *à* **CentraleSupélec**.

Membres du jury:

Prof. Florent BENAYCH-GEORGES,	Université Paris–Descartes	Examinateur
Prof. Pascal BONDON,	CNRS/Université Paris Sud	Examinateur
Prof. Alfred HERO,	University of Michigan	Rapporteur
Prof. Matthew MCKAY,	HongKong Univ. of Science and Technology	Examinateur
Dr. Esa OLLILA,	Aalto University	Examinateur
Prof. Leonid PASTUR,	National Academy of Sciences of Ukraine	Rapporteur
Prof. Jean-Yves TOURNERET,	INP-ENSEEIHT	Rapporteur.

Contents

Acknowledgments

Scientific research is a fascinating venture for its providing successive waves of various emotions, ranging from outstanding delight when a several-month attempt on a proof finally comes to light to utter despair when the day after the same proof turns out to be erroneous. But scientific research is all the more thrilling when seen as a teamwork with each partner dedicating hours trying to solve various angles of a problem for the good of the team as a whole. I am quite indebted to many colleagues with whom I sometimes shared those long hours, be it in front of a blackboard, over Skype, or simply by means of endless emails. Among them, Abla Kammoun is most likely the most restless and work-enthusiastic of my close colleagues who, in addition to being an excellent ally when it comes to heavy calculus, has devoted countless hours line-by-line proofreading our common articles. A special thank naturally goes to her. During the past several years I also came to work more closely with Walid Hachem whose brightness, thoroughness, and efficiency are a constant emulation for me. Aside from Abla and Walid, the works presented in this report – or performed meanwhile but not described here – were an opportunity to continue and extend my scientific collaborations with many other colleagues, among whom in alphabetic order Yacine Chitour, Jakob Hoydis, Matthew McKay, Jamal Najim, Frédéric Pascal, Pablo Piantanida, Jack Silverstein, and Gilles Wainrib. The many fruitful discussions we had, on technical but also philosophical grounds, obviously steered my research orientation to its present state. All this would however have been extremely different if it had not been for Mérouane Debbah who, as a PhD advisor first and as a more distant mentor later, drove me into enjoying research and then taking the reins of my young scientific career.

Beyond the scientific realm, I am grateful for my family who supported me all the way before my PhD and show warming pride in my own achievements. Finally, my pleasure to scribble equations for the sake of science on a daily basis runs tightly in pairs with an overall constant feeling of well-being. For that, I am profoundly thankful to my girlfriend Lorraine.

Part I

Curriculum Vitae

0.1 Contact

Name	COUILLET, Romain
Birth	18/03/1983 in Abbeville (Somme, France)
Status	Assistant Professor at CentraleSupélec Telecommunications Department, Group LANEAS, and Division Signals
Email	romain.couillet@centralesupelec.fr
Web	`http://couillet.romain.perso.sfr.fr/`
Address	3 rue Joliot Curie, 91192 Gif sur Yvette.

0.2 Education

Ph.D. in Physics (Telecommunications) Nov. 2010

Location	CentraleSupélec, Gif sur Yvette, France
Subject	Application of random matrix theory to future wireless flexible networks
Advisor	Mérouane Debbah
Topic	Wireless Communications, Cognitive Radio.

M.S. in System of Communications (SiCom) Mar. 2008

Location	Telecom ParisTech, Paris, France
Grade	Very Good (Très Bien)
Topic	Wireless communications, image processing, blind detection.

Engineering degree in Telecommunication Sep. 2007

Location	Telecom ParisTech/Eurecom Institute, Sophia Antipolis, France
Topic	Mobile communications, embedded systems, computer science.

Preparation to Engineering School 2001–2004

Location	Lycée Louis-le-Grand, Paris, France
Topic	Mathematics and Physics.

0.3 Professional background

Assistant Professor Jan. 2011–Present
CentraleSupélec, Gif sur Yvette, France.

Development Engineer and PhD student Sep. 2007–Dec. 2010
ST-Ericsson, Sophia Antipolis, France.

Research Trainee, advisor R. Knopp Jul. 2006–Aug. 2006
Eurecom Institute, Sophia Antipolis, France.

0.4 Teaching activity

ENS Cachan, Cachan, France.		since 2013
Subject	Random matrix theory and applications	
Details	Master 2, lectures, 15 hrs/year	
CentraleSupélec, Gif sur Yvette, France.		since 2011
Subject	Introduction to random matrix theory	
Details	Master 2, seminar lectures, 12 hrs/year	
Subject	Theoretical foundations of flexible radio networks	
Details	Master 2, seminar lectures, 12 hrs/year	
Subject	Techniques of scientific writing	
Details	Undergraduate and PhD, lectures, 12+18 hrs/year	
Subject	Représentation statistique des signaux	
Details	Undergraduate, practical courses, 12 hrs/year	
Subject	Signaux et systèmes	
Details	Undergraduate, practical courses, 12 hrs/year	
Subject	Filtrage numérique + Filtrage analogique	
Details	Undergraduate, practical courses, 32 hrs/year	
Polytech Nice-Sophia, Sophia-Antipolis, France.		2010
Subject	Digital communications	
Details	Master level, lectures, 24 hours	
Subject	Digital filtering	
Details	Master level, practical courses, 60 hours	

The table above details my main teaching activities which amount to an approximate yearly 150 hours of practical course-equivalent hours (équivalent-TD). Most lectures are part of various master programs (MVA at Cachan, SAR at CentraleSupélec), while my undergraduate teaching activities are limited to practical courses along with an average of five student projects per year (not accounted for in the table).

0.5 Research activity

My research activity is mostly involved with the mathematical study of large dimensional random matrix models in view of their applications to signal processing at large. As a PhD student

(2007–2010), the focus was primarily on the performance analysis of multi-user communication systems (multiple access MIMO channels, broadcast channels, cognitive radio settings) as well as on detection and estimation methods for wireless communication purposes. After my graduation in Nov. 2010 I joined CentraleSupélec (by then Supélec) as an assistant professor (Jan. 2011) where I still work within the Telecommunication Department. From 2011 on, I steered my research orientation on random matrix theory towards array processing applications and opened up my scope of research to a larger palette of random matrix models for various uses in signal processing. In 2013, I initiated an important project on the random matrix study of so-called robust estimators of scatter, a subject that had not received any attention at the time. This first led to a fundamental technical result on the limiting behavior of robust estimators of the Maronna type, which was then followed by several applied publications in array processing (with a novel improved MUSIC algorithm for impulsive data and a novel improved GLR detector) as well as in other fields (in finance with an improved method for Markowitz risk minimization for portfolio optimization). These latest results contribute to a major extent to an ERC Grant program (ERC MORE) for the period 2013–2017 that Mérouane Debbah, professor at CentraleSupélec and principal investigator, and I built and received. To a lesser extent, the array processing contributions are part of the ANR project DIONISOS (2012–2016) in collaboration with Telecom ParisTech and the University of Marne la Vallée both located in the Parisian area.

The main findings of the latest two years of my research activity will form the main part of the technical chapter of the present report.

As a result of these breakthroughs in robust statistics, I decided to launch a wider project of my own on the random matrix analysis of system models at the crossroads between signal processing and machine learning, thus extending the scope of my research to the big data paradigm. The ambition is precisely here to understand and then improve on standard methods used in machine learning (such as classification and clustering methods) and that have yet received little attention in the large dimensional regime typical of big data. As a subset of the many problems that this project covers, system models involved with graphs are quite common and in need of a theoretical understanding. The random matrix analysis of such graphical problems constitutes the core of the ANR JCJC RMT4GRAPH project (2014–2017) that I received in 2014. This project shall serve as a ramp for a wider application to a second ERC (Starting) Grant on random matrix tools for big data, with myself as a principal investigator.

0.5.1 Publication Record, Awards, and Projects

0.5.1.1 Publication Record (as of January 2015)

Publications	Book: 1, Book chapters: 3, Journals: 25+, Conferences: 45+, Patents: 4.
Citations	879 (five best: 187, 131, 72, 43, 24)
h-index	15
i10-index	23
Tutorials	6.

(see Section 0.5.6 for a complete list of publications)

The major means of diffusion of my results are through publications to mathematical journals for the most theoretical part of my research (Elsevier Journal of Multivariate Analysis, Markov Processes and Related Fields, etc.) and to information theory and signal processing IEEE journals for the more applied research (IEEE Transactions on Information Theory, on Signal Processing, etc.). I regularly attend international applied conferences and local workshops (particularly the IEEE Asilomar, ICASSP, and SSP conferences) where I present my recent results. For the sake of broader diffusion of my research breakthroughs, I also regularly intervene as a tutorial speaker in international conferences and write articles in larger scope magazines (IEEE Signal Processing Magazine), along with book chapters, and books.

0.5.1.2 Awards

Throughout my activity as a researcher, some of our articles were awarded best article or conference paper prizes. My first major achievement was my being awarded the EEA/GdR ISIS/GRETSI best PhD thesis prize for my work as a PhD student. More recently, I was elected recipient of the 2013 CNRS Bronze Medal in the section "science of information and its interactions" and of the 2013 IEEE ComSoc Outstanding Young Researcher Award (EMEA Region).

CNRS Bronze Medal (section INS2I) 2013
Awarding my work in Signal Processing and Wireless Communications as a young researcher since 2008.

IEEE ComSoc Outstanding Young Researcher Award (EMEA Region) 2013
Awarding my work in Communications-related topics as a young researcher since 2008.

EEA/GdR ISIS/GRETSI 2011 Award of the Best 2010 Thesis 2011
Thesis "Application of random matrix theory to future wireless flexible networks"

Second prize of the IEEE Australia Council Student Paper Contest 2013
G. Geraci, R. Couillet, J. Yuan, M. Debbah, I. B. Collings, "Large System Analysis of Linear Precoding in MISO Broadcast Channels with Confidential Messages"

Best Student Paper Award Final of the 2011 Asilomar Conference 2011
J. Hoydis, R. Couillet, M. Debbah, "Asymptotic Analysis of Double-Scattering Channels"

Best Student Paper Award of the 2008 ValueTools Conference 2008
R. Couillet, S. Wagner, M. Debbah, A. Silva, "The Space Frontier: Physical Limits of Multiple Antenna Information Transfer"

0.5.1.3 Projects

As a PhD student, I contributed to the following projects.

Project Name	Contribution	
ANR SESAME	20%	2008-2012
FP7 NEWCOM++	10%	2009-2011

As of January 2011, as an assistant professor at CentraleSupélec, I took a more active part to several projects, some of which I essentially wrote on my own (ERC MORE, ANR RMT4GRAPH, and HUAWEI RMTin5G projects).

Project Name	Contribution	
HUAWEI RMTin5G	100% (PI)	2015-2016
ANR RMT4GRAPH	100% (PI)	2014-2017
ERC MORE	50%	2012-2017
ANR DIONISOS	25%	2012-2016
FP7 NEWCOM#	10%	2012-2015

0.5.1.4 Visiting Appointments

Mathematics Department, North Carolina State University, NC, USA Nov. 2010

Host	Jack W. Silverstein, professor at NCSU
Duration	1 month
Details	joint work on statistical inference using random matrix theory

0.5.2 PhD thesis

Location	CentraleSupélec, Gif-sur-Yvette, France and ST-Ericsson, Biot, France
Contract	CIFRE (80% at ST-Ericsson, 20% at CentraleSupélec)
Advisor	Mérouane Debbah, Professor at CentraleSupélec, France
Title	Application of random matrix theory to future wireless flexible networks
Defense	November 12th, 2010
Publications	Book: 1, Book chapters: 2, Journals: 9, Conferences: 19, Patents: 4.

The subject of my PhD thesis involved the use of random matrix theory for the performance analysis and the design of wireless communication systems as well as for the estimation and detection of signal sources for cognitive radio protocols.

The fundamental interest behind random matrix theory for wireless communications lies its ability to evaluate various functionals of (often large dimensional) matrices modelling wireless channels, among which the ergodic capacity or sum rate of multi-antenna point-to-point or multipoint fading channels. These evaluations provide an understanding of the performance of sometimes complex (since large dimensional and mobile) communication systems from which improved system designs can be considered. In terms of detection and estimation for wireless communications, random matrix theory rather aims at retrieving hidden parameters in a likely large number of large dimensional vector observations, mostly by means of sample covariance matrix analysis. For all these problems, most of the necessary mathematical background was only loosely in place by 2008, in the sense that the system models of interest to wireless communication and signal processing engineers had not been studied by mathematicians. This led a few groups of researchers of our community, notably in the Parisian area, to tackle these mathematical problems in view of these underlying applications.

My PhD project fell into this setting. My first concern was with the extension of the capacity analysis of point-to-point multi-antenna (MIMO) ergodic channels to multipoint-to-point systems, and particularly with the ergodic rate region of multiple access fading channels with MIMO transmitters and receiver. Assuming correlation structures at both communication ends of the wireless channel (modeled then as a Kronecker fading channel), my main contribution was first to characterize this rate region thanks to new mathematical results for a random matrix model of the type "sum of Gram matrices" and to establish the asymptotically rate optimal precoders at the transmitters (that is the precoders used to closely reach the rate region boundaries). This work was performed in collaboration with Jack W. Silverstein, professor at North Carolina State University and world expert in random matrix theory. Aside conference articles, this work is contained in:

> R. Couillet, M. Debbah, J. W. Silverstein, "A Deterministic Equivalent for the Analysis of Correlated MIMO Multiple Access Channels", IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3493-3514, 2011. [**Citations: 72**]

In parallel, along with my colleague Sebastian Wagner, PhD student at the Eurecom Institute with D. Slock at the time, we investigated linearly-precoded multi-user MIMO broadcast channels which are of fundamental importance to today's 4G and 5G communication systems. Unlike my previous work of a rather theoretical nature, the system models we considered were pushed to an advanced degree of practical relevance. Specifically, we considered a setting in which a base station equipped with multiple antennas serves a possibly large number of users in the downlink through correlated fading channels, while only possessing estimates of the actual channels (i.e., with imperfect channel state information). This study led to a fine characterization of the achievable sum rates of such systems as a function of the channel state information at the base station and as a function of the channel statistics. This in turn allowed for an improvement of the precoding structures employed by such broadcast systems. These results and numerous associated ones are collected in:

> S. Wagner, R. Couillet, M. Debbah, D. T. M. Slock, "Large System Analysis of Linear Precoding in MISO Broadcast Channels with Limited Feedback", IEEE Transactions on Information Theory, vol. 58, no. 7, pp. 4509-4537, 2012. [**Citations: 131**]

In a similar setting, with my colleague Jakob Hoydis, by then PhD student at CentraleSupélec, we studied the performance of a class of systems implementing isometric precoders which, although of lesser practical use, constituted a first theoretical characterization of advanced models of such precoder types. The main relevance of this article lay in the observation that, although different in nature, the equations ruling the performance of precoders built upon matrices of independent entries or of isometric matrices take an outstandingly similar form, allowing one to relate both a priori different system types. The results were reported in:

> R. Couillet, J. Hoydis, M. Debbah, "Random Beamforming over Quasi-Static and Fading Channels: A Deterministic Equivalent Approach", IEEE Transactions on Information Theory, vol. 58 , no. 10, pp. 6392-6425, 2012. [**Citations: 24**]

These works and a few others of least importance constituted the main core of the performance analysis side of my PhD thesis.

In the course of the PhD it then occurred to us, as first random matrix-based results on detection and estimation in signal processing started being brought forward, that such random matrix tools could be developed and used in the context of spectrum sensing for cognitive radios, and especially so for large dimensional systems (e.g., secondary networks made of several sensing nodes capturing data from a primary multi-user MIMO network). As part of a one-month visit to Prof. Jack W. Silverstein in North Carolina State University, NC, USA, we developed a multi-user power estimation method based on recent array processing results for the estimation of angles of arrival. The result we obtained takes a surprisingly simple form but allows for an important performance gain against more classical non-random matrix based approaches. The result, which may be considered as the most theoretically advanced while practically important outcome of the PhD, is contained in:

> R. Couillet, J. W. Silverstein, Z. Bai, M. Debbah, "Eigen-Inference for Energy Estimation of Multiple Sources", IEEE Transactions on Information Theory, vol. 57, no. 4, pp. 2420-2439, 2011. [**Citations: 43**]

In another article, using finite-dimensional random matrix considerations (a topic of rather marginal interest of the PhD thesis), we also studied the performance of Bayesian-optimal Neyman–Pearson tests for signal detection (in a cognitive radio setting although not restricted to it) which we compared against large dimensional random matrix approaches provided a little before by independent teams. The main outcome was to observe that, while the optimality of our novel test obviously led to better performances than achieved by large dimensional approaches, the latter proved to loose so little in performance that their advantageous simplicity easily overcomes this limitation. The results of this study are reported in:

> R. Couillet, M. Debbah, "A Bayesian Framework for Collaborative Multi-Source Signal Sensing", IEEE Transactions on Signal Processing, vol. 58, no. 10, pp. 5186-5195, 2010. [**Citations: 22**]

In parallel to these scientific contributions, an effort was made during the thesis to broadcast and teach random matrix theory to the wireless communication community mainly. This was performed by means of several written tutorials, book chapters, book, and oral tutorials delivered in international conferences during or soon after my PhD thesis. The contribution of utmost importance is the redaction of a 600-page book on random matrix theory for wireless communications covering both theoretical and applied aspects:

R. Couillet, M. Debbah, “Random matrix theory methods for wireless communications”, Cambridge University Press, 2011. **[Citations: 187]**

The tutorials in international wireless communication conferences, listed below, allowed us in particular to reach an interested audience of up to fifty attendees to whom we taught basic theoretical tools as well as advanced practical results.

R. Couillet, M. Debbah, “Random Matrix Theory for Signal Processing Applications”, IEEE International Conference on Acoustics, Speech and Signal Processing, Prague, Czech Republic, 2011.
R. Couillet, M. Debbah, “Random Matrices in Wireless Flexible Networks”, International Conference on Cognitive Radio Oriented Wireless Networks and Communications (Crowncom), Cannes, France, 2010.
R. Couillet, M. Debbah, “Eigen-Inference Statistical methods for Cognitive Radio”, European Wireless, Lucca, Italy, 2010.

I defended my PhD thesis on November 12th, 2011 which was graded “Très horonable” by the following jury:

Mérouane Debbah	CentraleSupélec	PhD advisor
Pierre Duhamel	CNRS/CentraleSupélec	President of the Jury
Walid Hachem	CNRS/Telecom ParisTech	Member of the Jury
Philippe Loubaton	Université de Marne la Vallée	Examiner
Xavier Mestre	CTTC Catalunya	Examiner
Aris Moustakas	University of Athens	Member of the Jury
Jack W. Silverstein	North Carolina State University	Member of the Jury.

0.5.3 Activities as an Assistant Professor

Location	Telecom. Department, CentraleSupélec, Gif-sur-Yvette, France
Position	Assistant Professor (CDI)
Group leader	Hikmet Sari, Professor at CentraleSupélec, France
Publication total	Book: 1, Book chapters: 3, Journals: 28, Conferences: 47, Patents: 4.

Since January 1st, 2011, I hold a position as an assistant professor at CentraleSupélec, Gif-sur-Yvette, within the Telecommunication Department. By then, while maintaining a side occupation for new problems in wireless communication-related random matrix theory, I steered my research orientation towards signal processing applications of random matrix theory and increased my involvement in purely theoretical questions in the field. Generally speaking, I started to realize at that time that the extent of applications that random matrix tools can embrace extends well beyond the sole scope of wireless communications and that much more results are awaited in the wider area of signal processing. This general state of mind, which paralleled the ambition of my former advisor Mérouane Debbah, led the two of us to apply for and be granted an ERC Consolidator Grant (with M. Debbah as the principal investigator) on the development of mathematical tools for large dimensional systems (ERC MORE EC–120133).

In collaboration with colleagues of the Parisian area (notably Walid Hachem, Philippe Loubaton, Jamal Najim, Djalil Chafai, etc.) we also obtained an ANR grant on random matrix theory for array processing (ANR DIONISOS).

Aside from marginal continuity work on my previous activities on the performance analysis of large dimensional communication systems, as in our recent contribution to the design of interference-aware linear precoders for multi-cell broadcast systems:

> A. Müller, R. Couillet, E. Bjornson, S. Wagner, M. Debbah, "Interference-Aware RZF Precoding for Multi-Cell Downlink Systems", (submitted to) IEEE Transactions on Signal Processing, 2014.

my main remaining activity in wireless communications is oriented towards the second-order error probability performance of finite block-size MIMO communications. For such considerations, first order convergence results of large dimensional random matrix functionals, as required in my previous works, are no longer sufficient and one needs to move to second order results (central limit theorems) for technically more complex random matrix models. Our main contribution in this area was the determination of good lower and upper bounds on the error probability performance of point-to-point fading systems for which the number of antennas at both transmit and receive sides and the number of channel uses are of the same order of magnitude. The respective impacts of the fading channel on the one hand and of the finiteness of the block length on the other are easily isolated. Comparisons against practical (suboptimal) schemes were also performed from which intuitive considerations were extracted. These results are reported in:

> J. Hoydis, R. Couillet, P. Piantanida, M. Debbah, "A Random Matrix Approach to the Finite Blocklength Regime of MIMO Fading Channels", IEEE International Symposium on Information Theory, Boston, Massachusetts, USA, 2012. **[Citations: 13]**

> J. Hoydis, R. Couillet, P. Piantanida, "The Second-Order Coding Rate of the MIMO Rayleigh Block-Fading Channel", (to appear in) IEEE Transactions on Information Theory. **[Citations: 5]**

As part of my research activity for the aforementioned projects (ERC-MORE and ANR-DIONISOS), I initiated works on random matrix models of the spike-type; these can be seen as small rank perturbations of well-understood random matrix models which have the same eigenvalue behavior as these well-known matrices but for finitely many eigenvalues that enjoy an exceptional (but of practical importance) behavior. Among the various applications of such spike models, one involves the detection and localization of failures in a network whose interconnected nodes report sequences of correlated data. Upon local failure in the network, the data correlation matrix is perturbed in a rank-one modification fashion which the empirical sample covariance matrix can exhibit and identify. These resulted are reported in the following article:

> R. Couillet, W. Hachem, "Fluctuations of spiked random matrix models and failure diagnosis in sensor networks", IEEE Transactions on Information Theory, vol. 59, no. 1, pp. 509-525, 2013. **[Citations: 15]**

Subsequently, I took part of a more theoretical study of spiked models for more involved matrix models. That is, instead of considering small rank perturbations of identity matrices, we studied a more elaborate small rank perturbation model for generic matrices. This theoretical work served as a support to a novel type of applications in array processing for signal buried in colored noise (or spatially white noise with time correlation, such as a vector time series process). These articles were respectively published in a mathematical and an applied journal:

> F. Chapon, R. Couillet, W. Hachem, X. Mestre, "The outliers among the singular values of large rectangular random matrices with additive fixed rank deformation", Markov Processes and Related Fields, vol. 20, pp. 183-228, 2014. [**Citations: 6**]

> J. Vinogradova, R. Couillet, W. Hachem, "Statistical Inference in Large Antenna Arrays under Unknown Noise Pattern", IEEE Transactions on Signal Processing, vol. 61, no. 22, pp. 5633-5645, 2013. [**Citations: 5**]

While the aforementioned lines of work followed more or less a natural gradual continuation of my earlier works as a PhD student, based on rather classical random matrix tools, I was recently proposed to study the large dimensional behavior of a random matrix model of profound importance in the field of robust statistics but whose structure is quite unlike any work in the random matrix theory literature. Those matrices, referred to as robust M-estimators of scatter, are designed as an improvement of the classical sample covariance matrix against outliers or impulsiveness among the collected data. As an M-estimator, the robust estimator of scatter is the solution of an optimization problem which remains in general implicit, unlike the explicit sample covariance matrix. From a random matrix technical outlook, this induces an involved dependence structure between the entries of the matrix which classical random matrix tools cannot tackle. During the last two to three years, I intensively studied these random matrix models and made a series of substantial contributions which shall be discussed in Chapter II of this report. To start with, initially not managing to handle more interesting models, we began to analyze a very simple model of robust estimators of scatter for vector observations made of independent entries, the interest of which is in fact quite limited in robust estimation theory:

> R. Couillet, F. Pascal, J. W. Silverstein, "Robust Estimates of Covariance Matrices in the Large Dimensional Regime", (to appear in) IEEE Transactions on Information Theory, 2014, arXiv Preprint 1204.5320. [**Citations: 19**]

This work nonetheless allowed us to set the stage for our subsequent major contributions dealing with more fundamental models of high practical relevance. The most fundamental result we obtained deals with the robust estimate of scatter of the Maronna-type for elliptical vector observations (as opposed to vectors made of independent entries). It was precisely discovered that, in the large dimensional regime, such estimators show a close behavior to standard families of random matrices of the "separable covariance" type, while significantly differing from the standard sample covariance matrix obtained on the same samples. In particular, while the latter usually has an asymptotically unbounded eigenvalue spectrum, the robust estimator of scatter has a provably bounded spectrum which can be fully characterized. Moreover, any outlying observation entails a possibly large isolated eigenvalue in the sample covariance matrix spectrum, whereas robust estimates tame down the outlier contribution to the spectrum. The details of this work are found in:

R. Couillet, F. Pascal, J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples", (to appear in) Elsevier Journal of Multivariate Analysis, 2014. [**Citations: 12**]

Theoretical relevance set aside, the ability to approximate the structurally involved robust estimators by technically accessible and well-understood random matrix objects allows for the design of new estimation techniques based on robust estimates of scatter (rather than on sample covariance matrices) for large dimensional systems. The main blessing of these techniques lies in their controlling the natural spectrum spreading induced by heavy-tailed observations and outliers, so for instance in models involving information and impulsive noise, where sample covariance matrix-based approaches are either provably erroneous or at best inefficient. In the simple case of a spiked-model extension of the aforementioned work, we devised such a statistical inference technique along with a practical application to array processing under the form of a novel improved MUSIC algorithm:

R. Couillet, "Robust spiked random matrices and a robust G-MUSIC estimator", (submitted to) Elsevier Journal of Multivariate Analysis, 2014, arXiv Preprint 1404.7685.

As is quite common to signal processing, to evaluate the performance of a given estimator (and subsequently tune it appropriately), central limit theorems and second order variance figures are often required. The asymptotic approximation results obtained in the previous contributions are unfortunately too weak to ensure that central limit theorems for generic functionals of the robust estimators of scatter extend to the same central limit theorems for the same functionals applied to their approximations. In a recent work, we showed that this holds indeed true for functionals of the quadratic form-type. That is, whenever the performance figure of interest is a quadratic form induced by the (necessarily nonnegative definite) robust estimator of scatter, the fluctuations of the quadratic form are in the limit the same as those of the quadratic form induced by the approximating matrix. This result was applied to a fundamental GLR detection problem in array processing and allowed for the introduction of a novel false-alarm minimizing robust detector:

R. Couillet, A. Kammoun, F. Pascal, "Second order statistics of robust estimators of scatter. Application to GLRT detection for elliptical signals", (submitted to) Journal of Multivariate Analysis, 2014.

The works above mainly considered the setting of impulsive noise following a given distribution for all samples but only hand wavingly considered the impact of outliers among the observations. In a recent work, we treated instead the scenario of non-impulsive regular data corrupted by a certain quantity of outliers, and observed precisely the action of the robust estimators of scatter on the individual observations. Our main findings are collected in:

D. Morales, R. Couillet, M. McKay, "The impact of outliers on large dimensional robust estimators of scatter", IEEE Conference on Acoustics, Speech and Signal Processing (ICASSP'15), Brisbane, Australia, 2015.

It was mentioned above that the robust estimators of scatter exhibit a close behavior to random matrices of so-called separable-covariance model type. These models, somewhat studied in the mathematical literature, are usually of lesser interest to practitioners to the point that some fundamental results on the structural properties of the limiting spectrum for these models was left unexplored. As part of a much theoretical project, we devised such an analysis in the following article:

> W. Hachem, R. Couillet, "Analysis of the limiting spectral measure of large random matrices of the separable covariance type", (submitted to) Random Matrix Theory and Applications, 2013. [**Citations: 6**]

The last five listed works, along with the inspirational ideas and a few key technical lemmas from the sixth to last reference, constitute the main core of our findings for this important project.

More theoretically marginal but of practical importance, we extended the analysis of estimators of the Maronna-type to another important class of robust estimators derived from a hybridization of Tyler's robust estimator of scatter and Ledoit–Wolf's famous shrinkage estimate and which I shall refer to as robust-shrinkage estimators. Similar to the Maronna-type estimators, the robust-shrinkage estimates asymptotically behave similar to a well-understood random matrix from which improvement over classical applications may be brought. The interest of the hybridization is to further improve the approximation of the population covariance matrix by harnessing both the impact of outliers or impulsive noise and the impact of the paucity of observations. This then finds applications in several problems related to the optimization of a functional of population covariance matrices for which data are far from Gaussian and scarce. A first main contribution in this area was to reconcile two schools which considered different normalization procedures for the robust shrinkage estimator which we proved to be asymptotically equivalent and in fact equivalent to a mere Ledoit–Wolf estimator for the impulsion-free observations. As a second contribution, we derived an online procedure for minimizing the Frobenius norm between the robust shrinkage estimator and the population covariance matrix. In another article, we provided an application to financial portfolio risk minimization, which proved on actual financial datasets to systematically improve upon existing approaches. These results may be found in:

> R. Couillet, M. McKay, "Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators", Elsevier Journal of Multivariate Analysis, vol. 131, pp. 99-120, 2014. [**Citations: 13**]

> L. Yang, R. Couillet, M. McKay, "Minimum Variance Portfolio Optimization with Robust Shrinkage Covariance Matrix Estimator", Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 2014.

As a side comment, it is interesting to point out that the first of the two works above found an unexpected application to distributed power minimization problems in wireless communications (Sanguinetti et al., 2014).

0.5.4 Present and Future Activities

My recent work on robust estimation brought recently to light the possibility for random matrix theory to become a major enabling tool for the future big data challenge. Robust statistics may indeed be considered as one device among others to handle large sets of heterogeneous data, therefore bridging to some extent the gap between pure signal processing (where data are accurately described in probabilistic terms) and machine learning (where data are merely collections of inputs of possibly various nature, in particular prone to outliers). In the machine learning realm, most questions relate to data clustering, classification in independent components, sudden change detection, etc., where the data, of intrinsic heterogeneous nature, are barely modelled probabilistically. The big data challenge consists in establishing efficient means of performing these tasks for vast quantities of possibly large dimensional such data. Assigning probabilistic models in a first approximation, whose inaccuracies may be corrected by tools such as robust estimates, makes it arguably useful to consider the performance of machine learning algorithms on large dimensional datasets.

This spirit guided the redaction of a proposal to the ANR Jeunes Chercheuses Jeunes Chercheurs program, which I obtained in August 2014 (ANR RMT4GRAPH). In this program, I propose to consider a subclass of the many big data problems that relate to large dimensional graphs. Classical methods of machine learning, among which kernel methods (e.g., spectral clustering) or echo-state neural network approaches, are indeed often associated to the spectral properties of (possibly large and random) graphs. The random matrices involved are of various structures often loosely studied by the random matrix community. I propose in particular to study kernel random matrices for which only prior works of little practical relevance are available and which are therefore waiting yet for useful results. Similar to the robust estimation of scatter framework developed during the last two to three years, I expect to successively analyze the performance of existing machine learning methods from a probabilistic viewpoint in order then to improve these techniques. The objective of RMT4GRAPH and of the expected follow-up programs is thus to develop an original framework of random matrix techniques to address a variety of machine learning problems falling within the big data challenge.

0.5.5 PhD and postdoctoral students advising

0.5.5.1 PhD students having defended by 2015

Julia VINOGRADOVA 2011–2014

Subject	Random matrices and applications to detection and estimation in array processing
Details	50%, with W. Hachem (Telecom ParisTech)
Publications	2 articles in IEEE-TSP, 2 IEEE conferences
Defense	November 27th, 2014
Funding	DIGITEO grant

Axel MÜLLER 2011–2014

Subject	Random matrix models for multi-cell communications
Details	50%, with M. Debbah (CentraleSupélec)
Publications	3 articles in IEEE-JSTSP (published), -TIT, -TSP, 5 IEEE conferences
Awards	1 best student paper award
Defense	November 13th, 2014
Funding	Private funding by Intel
Position	Engineer at HUAWEI, Paris, France.

Axel Müller. The PhD work of Axel Müller involves the use of random matrix theoretical tools to improve the performance of wireless communications in more practical future-oriented scenarios. In particular, while most works prior to the thesis focused on single-cell communications omitting the existence of interfering adjacent cells and often assuming perfect knowledge of the channel states of each point-to-point link, the purpose of the thesis was instead to turn theoretical considerations into practice by incorporating as much complexity in the systems as theoretically tractable. The major contributions (only a subset of all the work carried out) of the thesis went into two specific directions.

The first main result deals with the performance analysis of polynomial precoders in a downlink multi-user MIMO setting, which have the practical advantage to bear less complexity than optimal linear precoders when a large number of antennas are used at the transmitter (such as in a massive MIMO setting), yet perform better than matched-filters. An accurate estimation of the weight parameters intervening in the precoder structure was performed to improve the resulting system performance.

The second contribution is the development of a new precoder structure for multi-cell communications with interference reduction. The latter is named ia-RZF for interference-aware regularized zero-forcing precoder and is parametrized so to balance the power transmitted to the legitimate users in the base station's own cell against the power leakage compromising the other cells' users. By partial or full knowledge of the adjacent cell users' channels, the aforementioned parameters can be selected to optimize the overall cell throughput, which takes on intuitive forms in simple system settings.

These works and others were the objects of two articles submitted to IEEE Transactions on Information Theory and IEEE Transactions on Signal Processing, of one article published in IEEE Journal of Selected Topics in Signal Processing, and of five conference articles presented at the IEEE Global Communications Conference and the Workshop on Spatial Stochastic Models for Wireless Networks in 2012, the IEEE Asilomar Conference on Signals, Systems, and Computers in 2013, the EUSIPCO conference and the IEEE Sensor Array and Multichannel Signal Processing Workshop in 2014. For the latter work, Axel Müller received a best student paper award.

Julia Vinogradova. The PhD thesis of Julia Vinogradova targeted the generalization of recent random matrix methods of detection and estimation for array processing applications, when the additive noise in the system has memory. This appropriately models situations of high data sampling rate, hardware imperfections, large band interference, etc. In this setting, the classical white noise assumption falls apart and so do many detection and estimation schemes that generally assume a temporally uncorrelated (often spatially white) noise setting. Two specific considerations were made which constituted the major two results of the thesis: (i) the noise temporal correlation has an a priori unknown structure and (ii) the noise is a stationary time series with finite-time correlation structure thus modelled as a Toeplitz matrix. In both settings, we assumed the noise i.i.d. across the antenna elements (so that noise correlation is modelled as a single time-covariance operator) and we also supposed that the information carried by the observation signal is a low rank perturbation perturbation of the noise-only signal.

In scenario (i), improved spiked-model based random matrix schemes were designed which allow for detection and estimation of the information parameters, under two (unfortunately rather stringent) assumptions: (i-a) the noise correlation structure is such that the resulting noise-only sample covariance matrix does not exhibit isolated eigenvalues (that could be wrongly detected as signal bearers) and more importantly (i-b) the information time correlation must be white (to ensure that the product of time correlations which naturally intervenes in the calculus prohibits the estimation). The crux of the method lies in gathering a sufficient statistics from the main bulk of eigenvalues of the (time) sample covariance matrix to extract the required knowledge on the noise correlation; this is then used along with the information versus noise "freeness in time" to proceed to estimation upon isolated (signal bearing) eigenvalues.

In scenario (ii), a consistent Toeplitz correlation matrix estimate was designed, based on a simple version of existing methods relying on Toeplitzifying the sample covariance matrix. The major novelty is that this estimation is performed in spite of the presence of a low rank information in the signal and does not require any arbitrary truncation procedure (which might nonetheless improve our method if appropriately exploited). The Toeplitz estimate is then used to approximately whiten the signal data from which, due to consistency in the random matrix regime, the model becomes essentially equivalent to an information-plus-white noise model. Classical detection and estimation schemes can then operate straightforwardly. Unlike (i), no strong assumption is required but for the Toeplitz structure of the noise correlation.

These two works led to two articles published in IEEE Transactions on Signal Processing and to two conference papers in ICASSP'13 and EUSIPCO'14, respectively.

0.5.5.2 PhD students under my current supervision

Azary ABBOUD 2012–2015

Subject	Distributed optimization for smart grids
Details	33%, with M. Debbah and H. Siguerdidjane (CentraleSupélec)
Publications	1 article in progress, 1 IEEE conference
Funding	CentraleSupélec grant
Status	Defense expected in Sept. 2015.

Gil KATZ 2013–2016

Subject	Interactive communication for distributed computing
Details	33%, with M. Debbah, P. Piantanida (CentraleSupélec)
Publications	1 IEEE conference
Funding	ERC MORE
Status	Defense expected in Sept. 2016.

Evgeny KUSMENKO 2015–2018

Subject	Random matrix and machine learning
Details	80%, with M. Debbah (CentraleSupélec)
Funding	ERC MORE
Status	Defense expected in 2018.

Adrien PELLETIER 2012–2014

Subject	New random matrix tools for wireless communications
Details	50%, with J. Najim (University of Marne-la-Vallée)
Publications	1 IEEE conference
Funding	DIGICOSME grant
Status	PhD aborted, looking for PhD in mathematics.

Azary Abboud. In order to bring together various research entities within CentraleSupélec and partly in relation to the ERC MORE on applications to smart grids, a PhD subject was designed by M. Debbah, H. Siguerdidjane, and myself on communication tools for smart grids. The PhD consists precisely in devising advanced distributed communication methods to optimize the regulation of power grids. Azary Abboud, a student with expertise in communication engineering, was selected for this position. Her work focuses on distributed optimization methods, such as the alternating direction method of multipliers (ADMM), which she analyzes from a theoretical viewpoint.

Relying on recent publications on (random) asynchronous distributed adaptations of the standard ADMM, she produced a practical application to asynchronous smart grids communications for power-flow optimization that guarantees convergence to the optimal solution of the problem, while requiring little synchronization between the network nodes. The results of this study were presented at the IEEE ICASSP conference in 2014. An extended version of this work was then submitted to IEEE Transactions on Signal Processing and is currently under revision.

Gil Katz. One of the main directions of the ERC MORE project (aside from the random matrix analysis of large systems) goes towards the theoretical analysis of large dimensional networks from an information theoretic perspective. Being a subject of his own interest, Pablo Piantanida, assistant professor at CentraleSupélec, thus joined Mérouane Debbah and myself to propose a PhD thesis on distributed computing for interactive communications. The idea behind the subject is to establish tight performance lower and upper bounds for network distributed computing problems accounting for the cost of data exchanges within the network. This approach tends to unite information theory, most of which restricted to communication questions, and distributed signal processing, for which communications between nodes is only loosely considered. Gil Katz, former master student at Technion, Israel, specialized in information theory, was selected to carry out this task as PhD student.

A first article was published at the IEEE Allerton 2014 conference in which the exact region of joint detection and estimation by a pair of nodes communicating over a lossy channel was established. This work brought to light the difficulties to extend such considerations to more than two nodes in the network, which the remainder of the PhD shall try to establish.

My personal involvement in the PhD relates to my earlier works on signal processing for large dimensional systems. As of now, since the work concentrates mostly on two-nodes communication networks, my contribution has remained limited, P. Piantanida having handled most of the supervision of the PhD student.

Evgeny Kusmenko. Straddled across the ERC MORE and the RMT4GRAPH project, an objective of present research is to better understand the structure of random matrices describing graphical models. In particular, the focus of the PhD is here on the spectrum of kernel random matrices and their associated Laplacian matrices. Applications are numerous and of particular interest to machine learning topics, such as that of spectrum clustering. Evegeny Kusmenko, who started his PhD in January 2015, will have the dual task of extracting the relevant features and open questions of theoretical machine learning and of analyzing random matrix models about which the current literature is quite scarce. As machine learning is a topic of high practical relevance in various fields (rather than one of theoretical predominance), another aspect of the PhD consists in connecting both practical and theoretical considerations to bring to light fundamental design elements in the so far quite heuristic machine learning methods. Evgeny Kusmenko received a solid background in engineering from TU Dresden, specialized in wireless communications.

Adrien Pelletier. Random matrix theory has provided a large amount of fundamental results for the performance analysis, optimization, and improved design of large dimensional wireless communication systems. The technical tools to tackle the majority of problems of present interest have now been in place for a few years. Nonetheless, these tools are mostly adequate to fast channel fading considerations in local single- or several-cell scenarios with fixed terminals. In parallel the opposite large-scale long-term fading consideration is tackled by tools such as stochastic geometry, however for restrictively simple communication systems. The PhD thesis, under a DIGICOSME grant, intended to devise new tools at the boundary between random matrix theory and stochastic geometry to handle such problems. More generally, the PhD had the ambition to propose a fresh toolbox of random matrix methods for large communication systems. As such, the PhD was of a quite theoretical nature and required the expertise of a student with strong mathematical background. Adrien Pelletier, MSc. at Cachan and MSc. at Paris VI in probability theory, was selected for these skills.

During the first year of his PhD we produced a new result for the joint performance of multiple users in a single-cell communication scenario (uplink and downlink). This (a priori initial) work was published at the 2013 IEEE Asilomar Conference and was in the process of being extended to a journal article (extension for a more general model). However, after a year of work, Adrien Pelletier's involvement in his PhD started to decay dramatically (regular long absences from work, complete absence of production, no answer to email exchanges) so we had to take the decision to interrupt the PhD after two years.

0.5.5.3 PhD student open position

PhD-1 2015–2018

Subject	Random matrix and graphical models
Details	100%
Funding	ANR JCJC RMT4GRAPH
Status	Defense expected in 2018.

PhD-1. The ANR JCJC project RMT4GRAPH is built on two major pillars: the analysis of specific graphical Hermitian random matrix models, such as symmetric kernel matrices, and the study of more fundamental non-Hermitian matrices with i.i.d. entries modelling more elementary random directed graphs. Of interest in particular will be the spectrum analysis (from weak eigenvalue convergence to isolated eigenvalue considerations) of advanced models in which e.g., the matrix entries have a variance profile rather than purely i.i.d. entries or exhibit some correlation pattern. These models find numerous applications in systems surrounding random graphs, such as neural networks. In the latter example, the dynamics of the system is ruled by the amount of isolated eigenvalues of the random matrix which is therefore an object of fundamental relevance. The PhD will tend to produce theoretical results in this direction.

0.5.6 Full publication record (with clickable links)

In this section are listed all publications in book, journals, and conferences. Dark blue titles in the electronic version of the report are clickable URL links allowing for an immediate access to the article. The section starts by a list of the most cited publications overall and restricted to articles produced after my PhD thesis.

0.5.6.1 Five most cited publications overall

Article	*Cites*
R. Couillet, M. Debbah, “Random Matrix Methods for Wireless Communications,” Cambridge University Press, 2011.	**187**
S. Wagner, R. Couillet, M. Debbah, D. T. M. Slock, “Large System Analysis of Linear Precoding in MISO Broadcast Channels with Limited Feedback”, IEEE Transactions on Information Theory, vol. 58, no. 7, pp. 4509-4537, 2012.	**131**
R. Couillet, M. Debbah, J. W. Silverstein, “A Deterministic Equivalent for the Analysis of Correlated MIMO Multiple Access Channels”, IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3493-3514, 2011.	**72**

R. Couillet, J. W. Silverstein, Z. Bai, M. Debbah, "Eigen-Inference for Energy Estimation of Multiple Sources", IEEE Transactions on Information Theory, vol. 57, no. 4, pp. 2420-2439, 2011.	**43**
R. Couillet, J. Hoydis, M. Debbah, "Random beamforming over quasi-static and fading channels: A deterministic equivalent approach," IEEE Transactions on Information Theory, vol. 58, no. 10, pp. 6392-6425, 2012.	**24**

0.5.6.2 Five most cited publications produced after 2011

Article	*Cites*
R. Couillet, F. Pascal, J. W. Silverstein, "Robust Estimates of Covariance Matrices in the Large Dimensional Regime," (in Press) IEEE Transactions on Information Theory, arXiv Preprint 1204.5320, 2014.	**19**
G. Geraci, R. Couillet, J. Yuan, M. Debbah, I. B. Collings, "Large System Analysis of Linear Precoding in MISO Broadcast Channels with Confidential Messages," IEEE Journal on Selected Area in Communications, vol. 31, no. 9, pp. 1660-1671, 2013. **2nd prize of the 2012-2013 IEEE Australia Council Student Paper Contest.**	**18**
R. Couillet, W. Hachem,"Fluctuations of spiked random matrix models and failure diagnosis in sensor networks," IEEE Transactions on Information Theory, vol. 59, no. 1, pp. 509-525, 2013.	**15**
J. Hoydis, A. Müller, R. Couillet, M. Debbah, "Analysis of Multicell Cooperation with Random User Locations Via Deterministic Equivalents," Eighth Workshop on Spatial Stochastic Models for Wireless Networks, Paderborn, Germany, 2012.	**15**
R. Couillet, M. McKay, "Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators" Elsevier Journal of Multivariate Analysis, vol. 131, pp. 99-120, 2014.	**13**

0.5.6.3 Books

- R. Couillet, M. Debbah, "Random Matrix Methods for Wireless Communications," Cambridge University Press, 2011.
  - *Content*: Theoretical random matrix tools (finite dimensional analysis, limiting spectral laws, free probability, deterministic equivalents, statistical inference) and applications to wireless communications (SU-MIMO, MU-MIMO, CDMA, detection, estimation, channel modelling).
  - *Nb. pages*: 600+.

0.5.6.4 Book Chapters

- Chapter "Random matrix theory" in E. Serpedin, T. Chen, D. Rajan, "Mathematical Foundations for Signal Processing, Communications and Networking", CRC Press, Taylor & and Francis Group, 2011.
- Several chapters in J. Palicot, C. Moy, M. Debbah, R. Couillet, H. Tembine, "De la radio logicielle à la radio intelligente", Hermès-Lavoisier Editions, 2010.
- Chapter "Fundamentals of OFDMA Synchronization" in T. Jiang, L. Song, Y. Zhang, "Orthogonal Frequency Division Multiple Access Fundamentals and Applications", Auerbach Publications, CRC Press, Taylor & and Francis Group, 2010.

0.5.6.5 Publications in Journals and Magazines

Mathematics

- R. Couillet, F. Pascal, J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples" (to appear in) Elsevier Journal of Multivariate Analysis, 2013.
- Y. Chitour, R. Couillet, F. Pascal "On the convergence of Maronna's M-estimators of scatter" IEEE Signal Processing Letters, vol. 22, no. 6, pp. 709-712, 2014.
- R. Couillet, W. Hachem, "Analysis of the limiting spectral measure of large random matrices of the separable covariance type", Random Matrix Theory and Applications, vol. 1, pp. 1-23, 2014.
- F. Chapon, R. Couillet, W. Hachem, X. Mestre, "The outliers among the singular values of large rectangular random matrices with additive fixed rank deformation," Markov Processes and Related Fields, vol. 20, pp. 183-228, 2014.
- R. Couillet, F. Pascal, J. W. Silverstein, "Robust Estimates of Covariance Matrices in the Large Dimensional Regime," (to appear in) IEEE Transactions on Information Theory, arXiv Preprint 1204.5320, 2014.

Signal Processing

- R. Couillet, "Robust spiked random matrices and a robust G-MUSIC estimator" (submitted to) Elsevier Journal of Multivariate Analysis, 2014, arXiv Preprint 1404.7685.
- R. Couillet, A. Kammoun, F. Pascal, "Second order statistics of robust estimators of scatter. Application to GLRT detection for elliptical signals" (submitted to) Elsevier Journal of Multivariate Analysis, 2014, arXiv Preprint 1410.0817.
- R. Couillet, M. McKay, "Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators" Elsevier Journal of Multivariate Analysis, vol. 131, pp. 99-120, 2014.

- J. Vinogradova, R. Couillet, W. Hachem, "Estimation of Toeplitz covariance matrices in large dimensional regime with application to source detection large", (to appear in) IEEE Transactions on Signal Processing, 2013.
- J. Vinogradova, R. Couillet, W. Hachem, "Statistical Inference in Large Antenna Arrays under Unknown Noise Pattern," IEEE Transactions on Signal Processing, vol. 61, no. 22, pages 5633-5645, 2013.
- J. Yao, R. Couillet, J. Najim, M. Debbah, "Fluctuations of an Improved Population Eigenvalue Estimator in Sample Covariance Matrix Models," IEEE Transactions on Information Theory, vol. 59, no. 2, pp. 1149-1163, 2013.
- R. Couillet, M. Debbah,"Signal Processing in Large Systems: a New Paradigm," IEEE Signal Processing Magazine, vol. 30, no. 1, pp. 24-39, 2013.
- R. Couillet, W. Hachem,"Fluctuations of spiked random matrix models and failure diagnosis in sensor networks," IEEE Transactions on Information Theory, vol. 59, no. 1, pp. 509-525, 2013.
- R. Couillet, J. W. Silverstein, Z. Bai, M. Debbah, "Eigen-Inference for Energy Estimation of Multiple Sources", IEEE Transactions on Information Theory, vol. 57, no. 4, pp. 2420-2439, 2011.
- R. Couillet, M. Debbah, "A Bayesian Framework for Collaborative Multi-Source Signal Sensing", IEEE Transactions on Signal Processing, vol. 58, no. 10, pp. 5186-5195, 2010.

Information Theory

- J. Hoydis, R. Couillet, P. Piantanida, "The Second-Order Coding Rate of the MIMO Rayleigh Block-Fading Channel," (submitted to) IEEE Transactions on Information Theory, arXiv Preprint 1303.3400, 2013.
- J. Hoydis, R. Couillet, M. Debbah, "Iterative Deterministic Equivalents for the Capacity Analysis of Communication Systems," Technical Report.
- R. Couillet, J. Hoydis, M. Debbah, "Random beamforming over quasi-static and fading channels: A deterministic equivalent approach," IEEE Transactions on Information Theory, vol. 58, no. 10, pp. 6392-6425, 2012.
- R. Couillet, M. Debbah, J. W. Silverstein, "A Deterministic Equivalent for the Analysis of Correlated MIMO Multiple Access Channels", IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3493-3514, 2011.

Wireless Communications

- A. Müller, R. Couillet, E. Bjørnson, S. Wagner, M. Debbah, "Interference-Aware RZF Precoding for Multi-Cell Downlink Systems" (submitted to) IEEE Transactions on Signal Processing, 2014.

- A. Kammoun, R. Couillet, J. Najim, M. Debbah,"Performance of capacity inference methods under colored interference" IEEE Transactions on Information Theory, vol. 59, no. 2, pp. 1129-1148, 2013.
- G. Geraci, R. Couillet, J. Yuan, M. Debbah, I. B. Collings, "Large System Analysis of Linear Precoding in MISO Broadcast Channels with Confidential Messages," IEEE Journal on Selected Area in Communications, vol. 31, no. 9, pp. 1660-1671, 2013. **Second prize of the 2012-2013 IEEE Australia Council Student Paper Contest.**
- S. Wagner, R. Couillet, M. Debbah, D. T. M. Slock, "Large System Analysis of Linear Precoding in MISO Broadcast Channels with Limited Feedback", IEEE Transactions on Information Theory, vol. 58, no. 7, pp. 4509-4537, 2012.
- R. Couillet, A. Ancora, M. Debbah, "Bayesian Foundations of Channel Estimation for Cognitive Radios", Advances in Electronics and Telecommunications, vol. 1, no. 1, pp. 41-49, 2010.
- R. Couillet, M. Debbah,"Le téléphone du futur : plus intelligent pour une exploitation optimale des fréquences" Revue de l'Electricité et de l'Electronique, no. 6, pp. 71-83, 2010.
- R. Couillet, M. Debbah, "Mathematical foundations of cognitive radios", Journal of Telecommunications and Information Technologies, no. 4, 2009.
- R. Couillet, M. Debbah, "Outage performance of flexible OFDM schemes in packet-switched transmissions", Eurasip Journal on Advances on Signal Processing, Volume 2009, Article ID 698417, 2009.

Smart Grids

- R. Couillet, S. Medina Perlaza, H. Tembine, M. Debbah, "Electrical Vehicles in the Smart Grid: A Mean Field Game Analysis," IEEE Journal on Selected Areas in Communications: Smart Grid Communications Series, vol. 30, no. 6, pp. 1086-1096, 2012.

0.5.6.6 International Conference Articles

Signal Processing

- D. Morales-Jimenez, R. Couillet, M. McKay,"Large dimensional analysis of Maronna's M-estimator with outliers", IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'15), Brisbane, Australia, 2015.
- A. Kammoun, R. Couillet, F. Pascal, "Second order statistics of bilinear forms of robust scatter estimators", IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'15), Brisbane, Australia, 2015.

- R. Couillet, M. McKay, “Robust covariance estimation and linear shrinkage in the large dimensional regime”, IEEE International Workshop on Machine Learning for Signal Processing (MLSP’14), Reims, France, 2014.
- L. Yang, R. Couillet, M. McKay, “Minimum variance portfolio optimization with robust shrinkage covariance estimation”, Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 2014.
- P. Vallet, X. Mestre, Ph. Loubaton, R. Couillet, “Asymptotic analysis of Beamspace-MUSIC in the context of large arrays”, IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM’14), A Coruna, Spain, 2014.
- R. Couillet, A. Kammoun, “Robust G-MUSIC”, European Signal Processing Conference (EUSIPCO’14), Lisbon, Portugal, 2014.
- R. Couillet, F. Pascal, “Robust M-estimator of scatter for large elliptical samples”, IEEE Worshop on Statistical Signal Processing (SSP’14), Gold Coast, Australia, 2014.
- R. Couillet, F. Pascal, J. W. Silverstein, “A Joint Robust Estimation and Random Matrix Framework with Application to Array Processing,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP’13), Vancouver, Canada, 2013.
- J. Vinogradova, R. Couillet, W. Hachem, “A new method for source detection, power estimation, and localization in large sensor networks under noise with unknown statistics,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP’13), Vancouver, Canada, 2013.
- R. Couillet, P. Bianchi, J. Jakubowicz, “Decentralized convex stochastic optimization with few constraints in large networks,” IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP’11), San Juan, Puerto Rico, 2011.
- R. Couillet, W. Hachem, “Local Failure Localization in Large Sensor Networks,” IEEE Asilomar Conference (ASILOMAR’11), Pacific Grove, CA, USA, 2011.
- R. Couillet, M. Guillaud, “Performance of Statistical Inference Methods for the Energy Estimation of Multiple Sources,” (Invited Paper) IEEE Statistical Signal Processing Workshop (SSP’11), Nice, France, 2011.
- R. Couillet, J. W. Silverstein, M. Debbah, “Eigen-inference for multi-source power estimation,” IEEE International Symposium on Information Theory, Austin TX, USA, 2010.
- R. Couillet, M. Debbah, “Bayesian inference for multiple antenna cognitive receivers”, IEEE Wireless Communications & Networking Conference, Budapest, Hungary, 2009.

Information Theory

- G. Katz, P. Piantanida, R. Couillet, “Joint Estimation and Detection Against Independence”, Fifty-second Allerton Conference on Communication, Control, and Computing, Allerton, IL, USA, 2014.

- J. Hoydis, R. Couillet, P. Piantanida, “Bounds on the Second-Order Coding Rate of the MIMO Rayleigh Block-Fading Channel,” IEEE International Symposium on Information Theory, Istanbul, Turkey, 2013.

- J. Hoydis, R. Couillet, P. Piantanida, M. Debbah, “A Random Matrix Approach to the Finite Blocklength Regime of MIMO Fading Channels,” IEEE International Symposium on Information Theory, Boston, Massachusetts, USA, 2012.

Wireless Communications

- A. Pelletier, R. Couillet, J. Najim, “Second-Order Analysis of the Joint SINR distribution in Rayleigh Multiple Access and Broadcast Channels,” Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 2013.

- A. Müller, E. Björnson, R. Couillet, M. Debbah, “Analysis and management of heterogeneous user mobility in large-scale downlink systems,” Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 2013.

- G. Geraci, R. Couillet, J. Yuan, M. Debbah, I. Collings, “Secrecy Sum-Rates with Regularized Channel Inversion Precoding under Imperfect CSI at the Transmitter,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP’13), Vancouver, Canada, 2013.

- M. de Mari, R. Couillet, M. Debbah, “Concurrent data transmissions in green wireless networks: when best send one’s packets?,” (Invited paper) IEEE International Symposium on Wireless Communication Systems (ISWCS’12), Paris, France, 2012.

- A. Müller, J. Hoydis, R. Couillet, M. Debbah, “Optimal 3D Cell Planning: A Random Matrix Approach,” IEEE Global Communications Conference (GLOBECOM’12), Anaheim, Caligornia, USA, 2012.

- M. Rezaee, R. Couillet, M. Guillaud, G. Matz, “Sum-Rate Optimization for the MIMO IC under Imperfect CSI: a Deterministic Equivalent Approach,” IEEE International Workshop on Signal Processing Advances for Wireless Communications, Cesme, Turkey, 2012.

- J. Hoydis, A. Müller, R. Couillet, M. Debbah, “Analysis of Multicell Cooperation with Random User Locations Via Deterministic Equivalents,” Eighth Workshop on Spatial Stochastic Models for Wireless Networks, Paderborn, Germany, 2012.

- A. Kammoun, M. Kharouf, R. Couillet, J. Najim, M. Debbah, “On the fluctuations of the SINR at the output of the Wiener filter for non centered channels: the non Gaussian case,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP’12), Kyoto, Japan, 2012.

- J. Hoydis, R. Couillet, M. Debbah, “Asymptotic Analysis of Double-Scattering Channels,” IEEE Asilomar Conference on Signals, Systems, and Computers (ASILOMAR’11), Pacific Grove, CA, USA, 2011. **Best student paper award finalist**

- A. Kammoun, R. Couillet, J. Najim, M. Debbah, “Performance of fast rate adaption techniques in interference-limited networks,” IEEE Global Communications Conference (GLOBECOM’11), Houston, TX, USA, 2011.
- J. Yao, R. Couillet, J. Najim, E. Moulines, M. Debbah, “CLT for eigen-inference methods in cognitive radios,” IEEE International Conference on Acoustics, Speech and Signal Processing, Prague, Czech Republic, 2011.
- J. Hoydis, R. Couillet, M. Debbah, “Deterministic Equivalents for the Performance Analysis of Isometric Random Precoded Systems,” IEEE International Conference on Communications, Kyoto, Japan, 2011.
- J. Hoydis, J. Najim, R. Couillet, M. Debbah, “Fluctuations of the Mutual Information in Large Distributed Antenna Systems with Colored Noise,” Forty-Eighth Annual Allerton Conference on Communication, Control, and Computing, Allerton, IL, USA, 2010.
- R. Couillet, H. V. Poor, M. Debbah, “Self-organized spectrum sharing in large MIMO multiple-access channels,” IEEE International Symposium on Information Theory, Austin TX, USA, 2010.
- S. Wagner, R. Couillet, D. T. M. Slock, M. Debbah, “Optimal Training in Large TDD Multi-user Downlink Systems under Zero-forcing and Regularized Zero-forcing Precoding,” IEEE Global Communication Conference, Miami, 2010.
- S. Wagner, R. Couillet, D. T. M. Slock, M. Debbah, “Large System Analysis of Zero-Forcing Precoding in MISO Broadcast Channels with Limited Feedback” IEEE International Workshop on Signal Processing Advances for Wireless Communications, Marrakech, Morocco, 2010.
- R. Couillet, M. Debbah, “Information theoretic approach to synchronization: the OFDM carrier frequency offset example”, Advanced International Conference on Telecommunications, Barcelona, Spain, 2010.
- R. Couillet, M. Debbah, “Uplink capacity of self-organizing clustered orthogonal CDMA networks in flat fading channels” IEEE Information Theory Workshop Fall’09, Taormina, Sicily, 2009.
- R. Couillet, M. Debbah, J. W. Silverstein, “Asymptotic Capacity of Multi-User MIMO Communications” IEEE Information Theory Workshop Fall’09, Taormina, Sicily, 2009.
- R. Couillet, M. Debbah, J. W. Silverstein, “Rate region of correlated MIMO multiple access channel and broadcast channel” IEEE Workshop on Statistical Signal Processing, Cardiff, Wales, UK, 2009.
- R. Couillet, M. Debbah, “Mathematical foundations of cognitive radios” U.R.S.I.’09, Warsaw, Poland, 2009.
- R. Couillet, M. Debbah, “A maximum entropy approach to OFDM channel estimation”, IEEE International Workshop on Signal Processing Advances for Wireless Communications, Perugia, Italy, 2009.

- R. Couillet, M. Debbah, “Flexible OFDM schemes for bursty transmissions”, IEEE Wireless Communications & Networking Conference, Budapest, Hungary, 2009.
- R. Couillet, S. Wagner, M. Debbah, “Asymptotic Analysis of Correlated Multi-Antenna Broadcast Channels”, IEEE Wireless Communications & Networking Conference, Budapest, Hungary, 2009.
- R. Couillet, S. Wagner, M. Debbah, A. Silva, “The Space Frontier: Physical Limits of Multiple Antenna Information Transfer”, ValueTools, Inter-Perf Workshop, Athens, Greece, 2008. **Best student paper award**
- R. Couillet, M. Debbah, “Free deconvolution for OFDM multicell SNR detection”, IEEE Personal, Indoor and Mobile Radio Communications Symposium, Cognitive Radio Workshop, Cannes, France, 2008.

Smart Grids

- R. Couillet, E. Zio, “A subspace approach to fault diagnostics in large power systems” (Invited Paper) IEEE International Symposium on Communications, Control, and Signal Processing (ISCCSP’12), Rome, Italy, 2012.
- A. Abboud, R. Couillet, M. Debbah, H. Siguerdidjane, “Asynchronous alternating direction method of multipliers applied to the direct-current optimal power flow problem,” IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP’14), Florence, Italy, 2014.
- R. Couillet, S. Medina Perlaza, H. Tembine, M. Debbah, “A mean field game analysis of electric vehicles in the smart grid,” IEEE International Conference on Computer Communications (INFOCOM’12), Orlando, FL, USA, 2012.

0.5.6.7 Patents

All patents listed below belong to the society ST-Ericsson.

- R. Couillet, M. Debbah, **No. 08368028.0** “Process and apparatus for performing initial carrier frequency offset in an OFDM communication system”
- R. Couillet, M. Debbah, **No. 08368023.1** “Method for short-time OFDM transmission and apparatus for performing flexible OFDM modulation”
- R. Couillet, S. Wagner, **No. 09368025.4** “Precoding process for a transmitter of a MU-MIMO communication system”
- R. Couillet, **No. 09368030.4** “Process for estimating the channel in an OFDM communication system, and receiver for doing the same”

0.5.7 Community Life

0.5.7.1 Technical Committees

- Member of the Signal Processing Theory and Methods Technical Committee, since 2014.
- TPC of several wireless communication and signal processing international conferences.

0.5.7.2 Special Issue Editorship

- Special issue "Grandes Matrices Aléatoires" for Revue du Traitement du Signal, 2015.

0.5.7.3 Special Session Organization

- Special session "Random Matrix Advances in Signal Processing", 5 articles, IEEE Symposium on Signal Processing (SSP'14), Gold Coast, Australia, 2014.
- Special session "Random Matrices and Applications", 4 articles, IEEE Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 2013.
- Journée GdR "Estimation et traitement statistique en grande dimension", 6 articles, Telecom ParisTech, Paris, France, 2013.

0.5.7.4 Tutorials

- R. Couillet, "Random Matrices, Robust Estimation, and Applications", IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'15), Brisbane, Australia, 2015.
- R. Couillet, A. Kammoun, "Future Random Matrix Tools for Large Dimensional Signal Processing", European Conference on Signal Processing (EUSIPCO), Lisbon, Portugal, 2014.
- R. Couillet, M. Debbah, "Random Matrix Advances in Signal Processing", IEEE International Workshop on Signal Processing Advances in Wireless Communications, Darmstadt, Germany, 2013.
- R. Couillet, M. Debbah, "Random Matrix Theory for Signal Processing Applications", IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'11), Prague, Czech Republic, 2011.
- R. Couillet, M. Debbah, "Random Matrices in Wireless Flexible Networks", International ICST Conference on Cognitive Radio Oriented Wireless Networks and Communications (Crowncom'10), Cannes, France, 2010.
- R. Couillet, M. Debbah, "Eigen-Inference Statistical methods for Cognitive Radio", European Wireless, Lucca, Italy, 2010.

Part II

Robust Estimation and Random Matrix Theory

Chapter 1

Foreword

1.1 Timetable of the technical contributions

The report introduces a contribution of my recent research activities, which I consider to be the most advanced and elaborate work I have conducted so far and to be an appropriate topic to account for my latest research as an assistant professor at CentraleSupélec. Indeed, quite unlike my other activities which are to some extent in natural continuation to existing works, this work results from an open discussion back in September 2011 with my colleague Frédéric Pascal at CentraleSupélec on an ambitious problem never considered in the literature at that time: that of analyzing the behavior of robust M-estimators of scatter in the random matrix regime. This discussion led to several months of frustrating trials and errors until an important first breakthrough was made in early 2012. This breach in the problem was large enough to produce a first article (Couillet et al., 2014b) which set the stage to future work but for which we had to assume hypotheses of little practical relevance, i.e., that the data vectors upon which the M-estimator is built have independent entries. As such, in spite of the important breakthrough from our outlook, this article suffered a lot of criticism and little understanding from the research community.

It then took yet another year, after again several months of vain explorations, to finally reach the most fundamental result of this work, which anecdotally resulted from a sudden inspirational albeit simple idea. The complete derivation of the main theorem was heavily technical but the crux of the proof still lay in this one simple ingredient. The result consisted in the understanding of the large dimensional behavior of the M-estimator for elliptically distributed data vectors. The main theorem shows that the M-estimator can be asymptotically well approximated by a random matrix model of the separable variance profile type, which we had incidentally recently analyzed in depth in (Couillet and Hachem, 2014). The result got published in (Couillet et al., 2013), a work which we broadly advertised and which received quite positive feedback. If structurally central to the work, the result however remained of a theoretical nature, but applications could easily be derived from it. A first application consisted in appending a small perturbation to the elliptically distributed data model (that in applications stands for impulsive noise observations) to account for the presence of informative signals. Technically, this merely boiled down to developing a spiked version of the model studied in (Couillet et al., 2013). This entailed yet another important article (Couillet, 2014), of practical interest this time, the main contribution of which was the introduction of an improved MUSIC algorithm for array processing which accounts both for large system dimensions and for noise impulsiveness. In recollection, this precise application was considered two years before as the ultimate, most likely infeasible, task I had expected to achieve in this vast project.

All these contributions mainly rely on finding an asymptotically (as the system dimensions get large) close matrix approximation of the M-estimator having the key property of being mathematically tractable, unlike the M-estimator itself. For practical purposes though, when devising consistent estimators (as with the improved MUSIC estimator of (Couillet, 2014)), for fair comparison against other consistent estimators or for appropriate selection of a free parameter in the model, one requires not only consistency results but second-order statistics (i.e., finite dimensional variance or central limit theorems). To delve into such considerations, the aforementioned matrix approximation of the M-estimator needs be very accurate in the sense that the approximation error (in spectral norm difference) must remain small enough not to induce additional fluctuations in the second-order regime of the M-estimator. Unfortunately, simulations suggest that this error is too large to allow for a simple analysis. Nonetheless, for many practical applications, the eventual objective function often deals with quadratic forms involving the M-estimator. For these quadratic forms, it turned out that the approximation error (obtained by replacing the M-estimator by its matrix approximation) is significantly reduced due to a salutary fluctuation self-averaging effect. The latter makes the error smaller than the fluctuations of the quadratic form of interest, which further allows for an easy study of these fluctuations. Being a technically heavy work, instead of studying the fluctuations of the improved MUSIC algorithm (which we however strongly believe to behave the same), we restricted our attention to a simpler application in robust signal detection by generalized likelihood ratio tests. This unfolded into a novel false-alarm rate minimizing estimator which again accounts for both large system sizes and noise impulsiveness, and the second-order performance of which is accurately determined. This work was published in (Couillet et al., 2014a).

The latter contribution constitutes the last technically challenging result of this whole line of work since 2012, which I conducted in collaboration with Frédéric Pascal, assistant professor at CentraleSupélec and expert in robust statistics, Abla Kammoun, postdoctoral student at King Abdullah's University of Science and Technology and expert in advanced technical considerations in random matrix theory, as well as Jack W. Silverstein, professor of mathematics at North Carolina State University and one of the pioneers of the random matrix theory. Frédéric Pascal was at the onset of the whole work and brought to light the intuitive notions behind robust estimation. Abla Kammoun enriched several technical discussions throughout these years and contributed among other things to prove a lemma of fundamental importance in (Couillet et al., 2014a). As for Jack Silverstein, he took part to the early work in relation with the first articles and provided some interesting ideas and references to handle a few nontrivial technical details of these articles.

Once we had obtained a first set of important results in early 2013, we could envision a much wider applicative scope than I had initially anticipated. I started at that point to share these, by then vague, ideas to several colleagues, among which Walid Hachem, Philippe Loubaton, and

Leonid Pastur, the three of whom are known for their contributions to both theoretical and applied random matrix theory. All of them showed a deep interest in the work, although it was not until I later met with Matthew McKay, professor at Hong Kong University of Science and Technology, that the work broadened towards other models and other applications. His own interest in shrinkage estimation methods along with my interest in robust statistics led us then to work on so-called robust shrinkage methods which had been studied at several occasions in the literature but never really took off for lack of appropriate theoretical tools. Together we wrote an article on such robust shrinkage methods based on rather simpler proof techniques almost immediately deriving from (Couillet et al., 2013). Our main finding, published in (Couillet and McKay, 2014), was to show that the two types of robust shrinkage estimators studied in parallel in the literature, each of which claimed by the authors to be better than the other (for rather unclear reasons), turn out to be asymptotically equivalent for elliptical data and to be equivalent to the well-known Ledoit–Wolf shrinkage estimator obtained for a scaled version of the data. An algorithm was devised in the same article to determine the asymptotically best shrinkage approximation of the population covariance matrix in terms of Frobenius norm error. Such techniques have the advantage of accounting for impulsiveness in the vector norms and also of benefiting from the positive effect of linear shrinkage methods à la Ledoit–Wolf when only few observations are available. As a concrete application of this result, along the same lines, a risk minimizing robust estimator for portfolio optimization in finance was then proposed in (Yang et al., 2014) which, from a purely simulation-based standpoint, proved to often (if not always) outperform alternative methods of the literature.

Although these various applications constituted a much lesser theoretical thrill and were restricted to specific models, they underline the important potential that robust estimation in the random matrix regime may bring to many other similar problems relying on sample covariance matrices but expecting instabilities due to impulsiveness in the model.

From an even deeper practical standpoint though, all these contributions kept considering the M-estimation side of the "robust M-estimators", in that data were until that point assumed to be clean elliptically distributed vectors (i.e., impulsiveness in the noise means heavy-tailed randomness in the norm of a conditionally Gaussian noise vector). As we got increasingly acquainted with the subject, we realized that some of our results suggested that (i) under these assumptions the M-estimators are sometimes asymptotically no better than other much simpler estimators (such as per-sample normalized sample covariance matrices) and (ii) when it comes to selecting an optimal M-estimator from a given class of estimators for a specific problem, these assumptions sometimes led to trivial optima (such as finding that the optimal estimator within the open Maronna class is the Tyler's estimator which no longer belongs to the Maronna class). These two negative features however do not take into account the robust aspect of the estimator in the sense that the simpler models of Item (i) and the trivial solutions of Item (ii) are naturally extremely sensitive to the introduction of some types of outliers in the observed data. As such, the results of our works at this point, not integrating the outlier-resilient component, could lead to unfortunate hasty conclusions. In fact, once faced with real data such as financial time series (instead of synthetic data), the robust estimators we devised often proved in stark opposition to these conclusions. It is in particular at first quite surprising to observe that robust estimators acting on supposedly close-to-elliptical real data, and thus presumably no better than some theoretically equivalent methods for elliptical data, often show an outstanding performance gain against these. Since integrating elliptical vectors and outliers in a joint model would lead to too specific models and would ill serve our purpose, in order to bring these observations to light, we recently investigated the action of the robust M-estimators on Gaussian data corrupted by rare (deterministic or random) outliers. We understood through this work the fundamental hinges on which robust estimators adequately tame down the impact of the outliers. A noticeable observation was that, while strong-energy outliers are nicely harnessed by most M-estimator families, outliers of weak amplitude could provoke dramatic performance loss for certain types of M-estimators and particularly for those M-estimators which would systematically perform best in outlier-free scenarios (such as the Tyler estimator which suffers from weak energy outliers). From a technical viewpoint, the latter work does not meet much difficulties but instead follows rather straightforwardly from the results in (Couillet et al., 2013).

1.2 Outline

The report describes in full technical details the aforementioned contributions following the chronological thread described above.

Before getting into the actual contributions, Chapter 2 provides an introduction to the main notations and objects under study in the report along with some mostly hand-waving discussions that should bring some important insights to the reader. This, we expect, will help the reader through the manuscript.

As the main results take the form of an approximation of M-estimation matrices by random matrices of the separable variance type, our first technical section, Chapter 3 shall recollect the latest findings about the spectrum of such important mathematical objects. The (recent) results of this section are extracted from

> W. Hachem, R. Couillet, "Analysis of the limiting spectral measure of large random matrices of the separable covariance type", Random Matrix Theory and Applications (under revision).

Having set a solid theoretical ground for the mathematical objects under study, we shall then be in position to introduce the most fundamental result of the present manuscript and to describe its proof in length. This result, on the asymptotic equivalence between robust estimators of scatter of the Maronna class for elliptical data and a certain random matrix model, along with its associated proof, are presented in Section 4.1 of Chapter 4 and are an excerpt of

> R. Couillet, F. Pascal, J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples", (to appear in) Elsevier Journal of Multivariate Analysis.

The second part of Chapter 4, Section 4.2, describes the extension of this work to a perturbed elliptical noise model of the data and the resulting improved detection and estimation procedures, in particular to array processing. The results from this section closely follow the article

R. Couillet, “Robust spiked random matrices and a robust G-MUSIC estimator”, (submitted to) Elsevier Journal of Multivariate Analysis.

The adaption of the results for the robust M-estimate of the Maronna type to the hybrid robust shrinkage model is subsequently discussed in Chapter 5. The main results and technical proofs are first provided in Section 5.1. It is to be noted that, while the proof for the so-called Chen model first relies on some additional technical ingredients and is moreover rather hard to follow, the proof for the Abramovich–Pascal estimate is quite easy to understand and is in fact much simpler than that of the Maronna M-estimator of the previous section. The innate reason for this simplicity lies in the spectral norm boundedness of the robust estimate inverse ensured by the shrinkage component. The reader finding it hard to follow the proof for the Maronna estimator, the readability of which gets reduced by technical aspects of spectral norm control and other similar subtleties, should be more at ease with the present section. These results, along with some immediate applications to Frobenius norm minimizing robust shrinkage, follow from the article

R. Couillet, M. McKay, “Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators”, Elsevier Journal of Multivariate Analysis, vol. 131, pp. 99-120, 2014.

The adaptation of these results to yet another minimization problem, that of minimizing risk in the Markowitz portfolio optimization problem is treated next in Section 5.2. This result and its complete proof, which follows from classical derivations not fully reproduced here, are found in:

L. Yang, R. Couillet, M. McKay, “Minimum Variance Portfolio Optimization with Robust Shrinkage Covariance Matrix Estimator”, on-going work.

The next section, Chapter 6, introduces our results on second-order statistics for quadratic forms built upon hybrid robust shrinkage estimators and an associated application in signal detection. The choice of hybrid robust shrinkage estimators against Maronna type estimators was mostly made out of mathematical simplicity, for the reasons evoked above, and is mostly instrumental to the understanding of similar second-order results for robust estimators. The proof of the result, however complex and quite involved, is thus at least not overloaded by additional technical aspects. This proof is presented in full length in Section 6.1. The application of this result to a robustness-improved generalized likelihood ratio test appropriate for source detection in impulsive noise environments is then introduced in Section 6.2. These results are originally found in:

R. Couillet, A. Kammoun, F. Pascal, “Second order statistics of robust estimators of scatter. Application to GLRT detection for elliptical signals”, (submitted to) Elsevier Journal on Multivariate Analysis.

Finally, the report will be concluded by an account of the effect of robust estimators of the Maronna type on data collections which, unlike in the previous sections, include outliers. The results in this section, Chapter 7, provide a clear understanding of the “magic in play” that many simulation trials on actual real-world data convey. These are mostly extracted from

D. Morales, R. Couillet, M. McKay, “Large dimensional analysis of Maronna’s M-estimator with outliers”, on-going work.

Chapter 2

Introduction

The present work discusses our first breakthroughs in an, as of 2012, unexplored yet fundamental research field: that of robust statistics in the random matrix regime. Before precisely defining what we mean by the latter and discussing our contributions, let us start by motivating the work by a fifty-year old challenge.

2.1 Motivation

Back in the sixties, John W. Tukey, who became famous for his contributions to modern signal processing, was one of the first to understand the important need for improved mathematical methods to automatically deal with large datasets. Fifty years later, the demand for big data analysis tools has grown fast and has now become a topic of major concern in the statistics and signal processing communities. Of particular importance among big data challenges is the stringent need for breaking the curse of dimensionality that naturally occurs when dealing with exact statistics of large dimensional populations. Recent mathematical tools have made it clear that the curse of dimensionality paradigm can be turned into a "dimensionality blessing", a term coined by the statistician David L. Donoho in (Donoho, 2000). The key idea somehow follows from a similar viewpoint as long taken in statistical mechanics by which the many degrees of freedom (or randomness) in both the system observations and the system dimensions result in a statistical "hardening" effect, such as met in probability theory with the classical law of large numbers.

Random matrix theory is one of today's major driving tools (along for instance with compressive sensing) for bringing dimensionality blessings in signal processing, although its roots date back to the works of Vyacheslav Girko on generalized statistics (Girko, 1987). Let us describe how generalized statistics provide dimensionality blessings. Denoting $X = [x_1, \ldots, x_n] \in \mathbb{C}^{N\times n}$ the matrix of n stacked observed random data $x_1, \ldots, x_n$, with $x_i \in \mathbb{C}^N$, it is assumed that N and n are both large but that the ratio N/n is non trivial (i.e., sufficiently remote from zero or infinity). This regime is appropriate to model engineering systems built upon a large number N of nodes which extract n successive observations of a shortly stationary environment. Statistical methods based on X mostly rely on functionals of the type $\int f(X)\mu(dX)$, with μ the measure of the joint entry distribution of X, which are usually difficult objects to evaluate. Moreover, since the paradigm $n \gg N$ is not valid here, classical asymptotic statistics (law of large numbers and central limit theorems relying on $n \to \infty$ and N fixed) are impractical. This is mostly what characterizes the curse of dimensionality for large data matrix operations. Spurred by the ideas of Girko, the last ten years of random matrix studies for signal processing have successfully turned these curses into blessings, in particular by providing improved methods that supplement and often largely outperform classical asymptotic signal processing techniques; among those methods are notably novel detection and estimation techniques for array processing (Mestre, 2008b; Couillet et al., 2011b). Technically speaking, the dimensionality blessings appear whenever, in addition to independence between the vectors $\{x_i\}_{i=1}^n$, degrees of independence among *the entries* of the vectors are exploited (Bai and Silverstein, 2009) (or more generally when a concentration of measure phenomenon can be exhibited (El Karoui, 2009)). Now, since each x_i may be built upon up to $O(N)$ statistical degrees of freedom and since $N = O(n)$ in the large random matrix assumptions, exploiting these degrees of freedom often results in fast converging methods, with in particular central limit theorems with speed $O(n)$ instead of the standard $O(\sqrt{n})$. This empowers methods originating from random matrix theory against classical asymptotic approaches, even for N not so large as often observed in practice.

Another point made by Tukey was that practical data, upon which statistical treatment is to be performed, often diverge from their expected model. Since practical systems are often designed out of modelling, this lack of control on data negatively affects the system to a point usually not accounted for. Along with Peter Huber, Tukey then became one of pioneers of the then new field of robust statistics. The latter precisely consists in determining the impact of ill-behaved data on systems and in designing improved systems that cope with such shortcomings (Huber, 1964). Today robust statistics form a rich research field, see e.g., (Maronna et al., 2006), the major contribution of which being the elaboration back in the seventies of various classes of robust estimators of mean and variance and of robust regression estimators. These robust estimators owe their appellations to the fact that they are optimal in their being weakly affected by many or arbitrarily ill-behaved outliers among the dataset. Robust statistics have even come up with metrics to gauge the robustness of various estimators, among which the influence function (measuring the degree of negative impact of the introduction of few outliers) and the breaking point (measuring the proportion of outliers that an estimator can tolerate before becoming arbitrarily biased) are the most important representatives. The research in robust statistics, which thrived in the seventies and eighties, however slowly lost momentum in the subsequent decades. A probable explanation relates to the fact that the tools developed in this field inherently take mathematically intractable forms. As a telling example, which we shall study throughout these notes, consider the estimation of population covariance matrices from observed samples. While the known sample covariance matrix is a mathematically simple and thus practical estimate for the population covariance matrix, it is highly not robust against outliers or heavy-tailed distributions; its natural robust counterpart is extremely more powerful in its resilience to outliers, but it suffers from its being only defined as the solution of a fixed-point equation and thus lacks mathematical tractability. Robust statistics have come up with important results on the performance of the estimators themselves, however only in the regime where the number of observed data is extremely large. Also, when used as a plug-in estimator for other purposes than merely estimating population covariance matrices, the resulting estimator statistics quickly become unmanageable.

This report studies the behavior of robust estimators of scatter in the regime where the number of observations n and the population size N are both large. As shall be seen, our main finding is to realize that, for various classes of data distributions of practical interest, these estimators – which we recall have an intrinsic intractable structure – asymptotically behave similar to simple and well-understood classes of random matrices, in particular separable sample covariance matrices. This result alone has major consequences for robust statistics that we believe may allow for a resurgence of its lost interest by the statistics and signal processing communities. Indeed, for one, through the approximation by well-known random matrices, the result turns the intractable robust estimators into asymptotically tractable entities. This result, although of a purely theoretical nature, in turn allows for the development of new detection and estimation schemes derived from now classical random matrix-based algorithms but featuring the robust estimator of scatter instead of the sample covariance matrix as central ingredient. These schemes shall then benefit from the robustness properties carried along by the robust estimator of scatter.

2.2 Sample covariance matrices, robust estimators of scatter, and the random matrix regime

Let us now turn the previous motivational discussion into rigorous mathematical terms.

2.2.1 The downsides of sample covariance matrices.

Letting $x_1, \ldots, x_n \in \mathbb{C}^N$ be n independent realizations of a random variable x with say zero mean and covariance matrix C_N, it is well-established that the sample covariance matrix $\frac{1}{n}\sum_{i=1}^n x_i x_i^*$ is a consistent estimator for C_N in the sense that, as $n \to \infty$, $\frac{1}{n}\sum_{i=1}^n x_i x_i^* \xrightarrow{\text{a.s.}} C_N$. Aside from this fundamental consistency aspect, the major reasons for the almost exclusive use of sample covariance matrices as an estimator for population covariance matrices in the literature are twofold: (i) from a practical outlook, it takes an extremely simple form which allows for easy theoretical analysis and (ii) on a more philosophical viewpoint, it corresponds to the maximum-likelihood estimator for Gaussian x. However, sample covariance matrices come along with at least two main drawbacks. For one, the sample covariance matrix estimate is quite sensible to the introduction of arbitrarily strong outliers. Precisely, the introduction of a single outlying observation of say arbitrarily large norm induces an arbitrarily large bias on the estimation of C_N. Thus, not only may the sample covariance matrix be affected by a single bad observation, but it can become arbitrarily remote from the expected C_N to be estimated. Second, in practical settings where N and n are similar in size, it is a weak estimator for C_N which entails that, when used as a plug-in estimator, functionals of C_N are in general no longer consistently estimated by the same functional of $\frac{1}{n}\sum_{i=1}^n x_i x_i^*$.

These two problems have been tackled with success by independent fields of research: robust statistics which aims in particular at producing alternative estimators for C_N which are resilient to a possibly large number of arbitrarily strong outliers and random matrix theory which considers among other questions that of consistently estimating functionals of C_N when both N and n grow large. As we shall see, the objects of interest produced by robust statistics however do not fall into the realm of classical entities studied in random matrix theory, which thus until recently left open the questions of analyzing the performance of robust estimators as N and n are both large on the one hand and therefore of developing improved algorithms that would be both resilient to outliers and large system sizes on the other. The purpose of this document is to report the recent advances that we performed to reconcile both fields and to derive the aforementioned theoretical analyses and improved algorithms.

2.2.2 The advent of robust statistics.

To start, let us precisely introduce the objects of interest throughout the work, that is robust estimators of scatter.

Huber was historically the first one to provide a definite formulation of a robust estimator. In his landmark article (Huber, 1964), Huber assumes the observation of n independent variables $x_1, \ldots, x_n \in \mathbb{C}$ with identical probability measure $(1-\varepsilon)\mu+\varepsilon\mu'$, where μ is a known and expected distribution fitting the model, μ' some polluting unknown distribution, and $\varepsilon > 0$ small. He then determines, both for the mean (or location parameter) and for the variance (or scale), the estimator among the class of M-estimators[1] which meets the smallest possible asymptotic variance (as $n \to \infty$) in the worst possible choice for μ' (this is thus formulated as a min-max problem). Huber shows that such M-estimators are unique for both location and scale and provides an explicit characterization of these.

Those results easily extend to multivariate data $x_1, \ldots, x_n \in \mathbb{C}^N$ to the estimation of location and scatter (or covariance) matrix. In the case of the M-estimation of scatter for zero mean observations, the estimator, call it $\hat{C}_N \in \mathbb{C}^{N\times N}$, is one solution of the fixed point equation (provided $n > N$ and that all subsets of $N-1$ vectors are linearly independent)

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} u_{\mathrm{H}}\left(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i\right)x_ix_i^* \tag{2.1}$$

where $u_{\mathrm{H}} : \mathbb{R}_+ \to \mathbb{R}_+$ is the continuous function

$$u_{\mathrm{H}}(s) = \min\left\{\alpha, \frac{\beta}{s}\right\}$$

and $\alpha, \beta > 0$ are parameters depending on ε and μ, the exact values of which are of little relevance for our present considerations. The focus of our work shall indeed not be on the technical details of how such a result is obtained but rather on how $\hat{C}_N$ behaves when both N and n are of similar dimensions (and rather large). Note from the expression (2.1) that $\hat{C}_N$ appears as a form of weighted sample covariance matrix with weights being either α for small

[1] Recall that an M-estimator $\hat{z}$ for i.i.d. observations $z_1, \ldots, z_n$ of a random variable z is such that, for some cost function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, $\hat{z}$ is a minimizer of $\sum_{i=1}^{n} \rho(z_i - \hat{z})$. The M-estimation of the location parameter of $x_1, \ldots, x_n$ is that for $z_i = x_i$, while the M-estimator of scale is that for $z_i = \log(|x_i|^2)$.

$x_i^* \hat{C}_N^{-1} x_i$ or smaller (possibly arbitrarily small) positive values for larger $x_i^* \hat{C}_N^{-1} x_i$. Outliers are then considered here as vectors x_i's for which the quadratic form takes large values, which hand-wavingly might be seen as vectors x_i whose alignment to the leading eigenspaces formed by all the x_j's is weak.

As robust statistics and M-estimation are inherently connected, a parallel track of robust estimation theory consists in letting $x_1, \ldots, x_n$ be i.i.d. vectors taken from a quite impulsive but known distribution, the scatter matrix estimate then being the maximum-likelihood estimator for this distribution. In this case, outliers are seen as those few samples extracted from the impulsive vector distribution that show extreme behavior. Maronna realized in (Maronna, 1976) that, for x_1 taken from the class of elliptical distributions, the maximum-likelihood estimate for the scatter matrix takes a specific form which belongs to a generic class of estimators, that we shall from now on refer to as the Maronna class. The estimators from this class are defined as solutions $\hat{C}_N$ of the equation

$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^{n} u \left(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i \right) x_i x_i^* \tag{2.2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-increasing function such that $\phi(x) = xu(x)$ is non-decreasing with $\lim_{x\to\infty} \phi(x) = \phi_\infty > 1$. Under some technical conditions, Maronna shows that, for each N, n with $N < n$, a solution of (2.2) always exists. Besides, if ϕ is increasing (and not only non-decreasing), $\hat{C}_N$ is the unique solution of (2.2), allowing then for a proper definition of $\hat{C}_N$. This result is generalized in (Kent and Tyler, 1991) where the aforementioned technical conditions are relaxed to the mere requirement of linear independence of every family of $N-1$ vectors among the x_i's.

Note that, setting aside the requirement for ϕ to be increasing, the function u_{H} previously defined would belong to the Maronna class. As such, Maronna's class of estimators of scatter encompasses both maximum-likelihood estimators of scatter for elliptical observations and, up to a slight smoothening of Huber's function u_{H}, Huber's robust estimator for outliers. The u functions specified above in particular all behave similar to u_{H} in their attenuating those observations x_i meeting large values for $\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i$. The family of estimators $\hat{C}_N$ defined by (2.2) is as such quite an interesting object, to which we shall dedicate an important part of the report.

Following on Maronna's work, Tyler took a third approach to robust M-estimation in (Tyler, 1987) by assuming that, if impulsiveness is only a matter of vector norm, rather than entire distribution, as in the case where $x_i = \sqrt{\tau_i} y_i$ for i.i.d. y_i of unit norm and random heavy-tailed $\tau_i > 0$, then a so-called scale-free robust estimator of scatter is adequately defined for all $N < n$, as the unique solution with given trace, of the equation

$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i} x_i x_i^*. \tag{2.3}$$

Note the importance here of constraining the trace of $\hat{C}_N$ as it is easily seen that, if $\hat{C}_N$ is a solution to (2.3), then so are every $\alpha \hat{C}_N$ for $\alpha > 0$ (and it is then clear that C_N is only estimated up to a constant). A second remark of importance is that Tyler's estimator behaves again similar to Maronna's estimators but for the fundamental difference that $x \mapsto 1/x$ is not defined at $x = 0$ as opposed to Maronna's u function. This slight change has fundamental consequences which, as shall be seen, makes the analysis of Maronna's estimator much more amenable to our proof approaches than Tyler's.

The aforementioned $\hat{C}_N$ estimators have the downside of not being defined for $N \geq n$ which is a situation of practical interest in the case of large dimensional observations or scarce data. In a similar way that the so-called linear shrinkage estimator $(1-\rho)\frac{1}{n}\sum_{i=1}^n x_i x_i^* + \rho I_N$, $\rho \in [0,1]$, was developed in (Ledoit and Wolf, 2004) to compensate for the poor sample covariance matrix performance in these scenarios (a discussion on this poor performance is provided later in this section), two teams developed in parallel two hybrid robust-shrinkage estimators in (Pascal et al., 2013; Chen et al., 2011), respectively defined as the unique solutions $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ of

$$\hat{C}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{1}{\frac{1}{N}x_i^*\hat{C}_N^{-1}(\rho)x_i} x_i x_i^* + \rho I_N \tag{2.4}$$

for $\rho \in (\max\{0, 1-n/N\}, 1]$ and

$$\begin{aligned} \check{C}_N(\rho) &= \frac{\check{B}_N(\rho)}{\frac{1}{N}\operatorname{tr}\check{B}_N(\rho)} \\ \check{B}_N(\rho) &= (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{1}{\frac{1}{N}x_i^*\check{C}_N^{-1}(\rho)x_i} x_i x_i^* + \rho I_N \end{aligned} \tag{2.5}$$

for $\rho \in (0,1]$. By different means, the authors show that the defining implicit equations have unique solutions for all N, n and all $x_1, \ldots, x_n$. The difference between both estimators lies foremost in the different normalizations, as $\frac{1}{N}\operatorname{tr}\check{C}_N(\rho) = 1$ while $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}(\rho) = 1$. However, it is not clear, even to the authors (to whom we asked), what practical advantage each estimator has over the other. We however concur with the authors on the potential performance gains in applications induced by the free parameter ρ (if properly set) and by the inherent robustness of the estimator.

Note that zero is excluded from the definition space of ρ for both estimators. The case $\rho = 0$ corresponds indeed to the aforementioned Tyler's estimator for which uniqueness no longer holds. Technically speaking, $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ will be much simpler to analyze than Maronna's M-estimator. This is precisely due to $\rho > 0$ which ensures the boundedness in spectral norm of the inverses of the two estimators.

For most of the articles mentioned so far, the authors systematically provided asymptotic properties of the $\hat{C}_N$ (or $\hat{C}_N(\rho)$, $\check{C}_N(\rho)$) estimators as $n \to \infty$, while N is kept constant. The results are that, for well-behaved distributions for x_1, $\hat{C}_N$ (or any other estimator) is a consistent estimator of the covariance or scatter matrix and fluctuation results such as central limit theorems at rate $1/\sqrt{n}$ are established. These results are of key importance to understand in particular the important gains in variance of the estimators against the sample covariance matrix estimator. These gains can in fact be usually made arbitrarily large by selecting a specific distribution for the x_i which induces particularly slow convergence of the sample covariance matrix while being properly harnessed by the robust estimator counterpart. As this is not much informative, the performance in terms of the variance of the robust estimator on purely Gaussian data is often considered instead. The latter allows for an assessment of the loss of performance naturally induced by shifting for a robust – thus not maximum-likelihood for Gaussian inputs – estimate. A classically obtained result is that robust estimators of scatter perform as sample covariance matrices would on a smaller amount of data; thus robustness comes at the cost of a requirement for more data.

2.2.3 The large N, n regime.

The results above are however of little relevance when it comes to studying large dimensional systems for which both N and n are large, or in more practical terms when the system dimension is large but one cannot afford too many observations of it. In this setting, random matrix theory has established for long (Marčenko and Pastur, 1967) that the sample covariance matrix is already no longer a consistent estimator of the population covariance matrix, in the sense that $\|\frac{1}{n}\sum_{i=1}^n x_i x_i^* - C_N\|$ does not converge to zero as $N, n \to \infty$ with $N/n \to c \in (0, \infty)$ for all distributions of x_i of practical relevance. It is thus quite unexpected that $\|\hat{C}_N - C_N\|$ would converge to zero for the various robust estimators described above. It indeed does not.

This being said, in many practical situations, estimating C_N is not the eventual objective. Often in signal processing one is rather concerned with functionals of various statistics of the system under study, and in particular in functionals of C_N. Under the classical large n assumption, the sample covariance matrix thus merely serves the purpose of a plug-in estimator for this functional. That is, calling f this functional, one exploits convergences of the type $f(\frac{1}{n}\sum_{i=1}^n x_i x_i^*) \to f(C_N)$. Due to inconsistency in the random matrix regime, i.e., as both N and n grow large, for most cases of interest such convergence results unfortunately no longer hold in this regime. Recent random matrix works (following mostly from the pioneering works from Girko (Girko, 1987)) have tackled the question of providing such estimates, consistent in the regime $N, n \to \infty$ with $N/n \to c \in (0, \infty)$, as in e.g., (Mestre, 2008b) where consistent estimates of linear functionals of the eigenvalues of C_N are provided. The approach carried out to obtain such estimators consists in a thorough understanding of the limiting spectrum of $\frac{1}{n}\sum_{i=1}^n x_i x_i^*$ as provided early by (Silverstein and Bai, 1995; Silverstein and Choi, 1995), from which a relation expressing the sought for functional $f(C_N)$ in terms of a functional of this limiting spectrum is retrieved. Approximating the latter by the finite (but large) dimensional spectrum of $\frac{1}{n}\sum_{i=1}^n x_i x_i^*$ provides the estimator. It is important to raise a philosophical comment at this point. The method we just presented is based on exploiting the sample covariance matrix, which is no longer a consistent estimator for C_N in the large N, n regime, as a means to retrieve information about C_N. Although this might seem natural, one could have also considered other pre-filtering of the data $x_1, \ldots, x_n$ to obtain the same information. The reason why the sample covariance matrix is used is obviously out of simplicity and surely few researchers ever thought of considering any other starting point. However, note that, in addition to leading to simple technicalities, one may expect that the several innate properties of the sample covariance matrix, i.e., its being maximum-likelihood for Gaussian x_i and its being consistent with C_N for n large alone, should play a role in the performances of the eventual $f(C_N)$ estimator.

In particular, we might expect better performance for Gaussian inputs than non-Gaussian ones, which often turns out to be the case (as confirmed by various central limit theorems that exhibit a variance that is minimal under Gaussian inputs, see e.g., (Hachem et al., 2008b)).

When it comes to extending the above sample covariance matrix-based studies to random vectors $x_1, \ldots, x_n$ following more impulsive distributions (such as elliptical distributions) or containing a certain amount of outliers, many problems are observed. For one, the spectrum of the sample covariance matrix may be of asymptotically wide, if not unbounded, support (in the elliptical case) or may maintain a bounded limiting support but contain isolated eigenvalues purely due to outliers. In both cases, these properties have a negative impact on the resulting estimators of $f(C_N)$, the performance of which are often optimized when the spectra of the sample covariance matrices have a short support and do not exhibit spurious eigenvalues. In extended information-plus-noise models (in which x_i stands for noise observed at time instant i), the largest eigenvalues carry much information and are better found as far away as possible from the remaining eigenvalues. The fact that a pure-noise (only x_i) setting may exhibit arbitrarily large eigenvalues or spurious "signal-like" (and thus deceiving) eigenvalues leads to many problems that are in general not easily dealt with. Following our philosophical line of thought, it is sensible that these concerns might be tamed down if one would use robust estimators of scatter in place of sample covariance matrices. This seems all the more compelling that in the case of elliptical vector observations, Maronna's estimator of scatter is a maximum-likelihood estimator for C_N, which makes its use much more natural than the use of sample covariance matrices in the first place. As we shall see, our intuition will turn out to be correct in that the estimators developed in this report will show unexpectedly powerful properties of outlier control and of adaption to elliptical data.

2.2.4 Harnessing robust M-estimators in the large N, n regime.

As we recalled earlier, deriving estimators of functionals of C_N in the large N, n regime based on the sample covariance matrix as the first building block requires a deep understanding of the limiting spectrum of the latter, from which a connection between some other functional of the sample covariance matrix and the sought for functional of C_N can be drawn to obtain the estimator. Trading the sample covariance matrix for a robust estimator of scale, call it generically $\hat{C}_N$, in this scheme thus demands a prior understanding of the large N, n behavior of $\hat{C}_N$. This task turns out to be quite challenging as $\hat{C}_N$, unlike sample covariance matrices and similar objects of interest in random matrix theory, has a quite involved dependence structure of its entries. In effect, we do not know of any past contributions (be it in mathematics or in applied science) prior to 2012 when we started this study that tackled any such random matrix model. Most results of random matrix theory indeed rely on some clear independence, or simple dependence, structure of the matrix entries; here, even for independent x_i with independent entries, the entries of $\hat{C}_N$ are related in a non-trivial manner. Lifting this difficulty constituted the main challenge of the work. What we shall precisely demonstrate is that, for x_i i.i.d. elliptical and $\hat{C}_N$ the Maronna estimator of scatter, $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ in spectral norm, where $\hat{S}_N$ is a much more classical random matrix with clear entry dependence. This result is proved in full length in Section 4.1.

The strength of the spectral norm convergence induces immediately the possibility to study the spectrum of $\hat{C}_N$ through that of $\hat{S}_N$. Of particular importance is the fact that not only do $\hat{C}_N$ and $\hat{S}_N$ share the same limiting spectrum, but they also share the (approximately) same individual eigenvalues for all finite but large N, n. This point has fundamental consequences in statistical inference procedures discussed below. Technically speaking, the matrix $\hat{S}_N$ precisely belongs to the class of random matrices with a separable covariance. Similar to mere sample covariance matrices, the limiting spectrum of $\hat{S}_N$ is well characterized, although only recently was its study fully completed in (Couillet and Hachem, 2014). Since these elements are key to a detailed study of $\hat{C}_N$ in the random matrix regime, Chapter 3 recalls these elements and shall constitute a good opportunity for introducing the various analytical tools needed in the rest of the report.

Having performed the challenging task of harnessing $\hat{C}_N$ for large N, n, the statistical inference of functionals $f(C_N)$ based on $\hat{C}_N$ then merely consists in applying the classical random matrix tools developed to this aim, albeit for the slightly more complex $\hat{S}_N$ matrix model. The convergence $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ indeed allows for an asymptotically licit substitution of $\hat{C}_N$ by $\hat{S}_N$ in the expression of most estimates of $f(C_N)$. This part of the study does not bring any particular difficulty. As a practical example, having generalized the convergence $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ to an information-plus-noise model of x_i with elliptical noise and small rank information, we obtained consistent estimates, based on $\hat{C}_N$, for two types of functionals of the information part of the signal having important applications in array processing. Precisely, we adapt the classical array processing MUSIC algorithm to evaluate source powers and angles of arrival contained in the deterministic parameters of the model (which is not exactly C_N to be fully correct). Details are provided in Section 4.2.

Following the same approach carried out for $\hat{C}_N$ of the Maronna class, we then similarly study the large dimensional behavior of the robust shrinkage matrices $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$. Although the final results for both matrices shall take a slightly different form than for $\hat{C}_N$ of the Maronna class, the derivations to reach these are essentially the same and, apart from a technical complication for $\check{C}_N(\rho)$, do not present any particular difficulty. The interest of presenting these results in the report mostly lies on practical grounds. Many classical statistical inference applications (in fields as various as statistical biology or finance) where $N > n$ are indeed already known to gain from Ledoit–Wolf shrinkage of the sample covariance matrix. As these applications also often involve outlying or impulsive data, the estimators $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ are expected to further cope with this aspect. A second useful property of $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ lies in their analytical simplicity which shall ease the more elaborate considerations, such as central limit theorems, discussed later on. The theoretical considerations are reported in Section 5.1, while applications, here to portfolio optimization in financial statistics, are the object of Section 5.2.

2.2.5 Large N, n performance of robust consistent estimators.

As is classical not only in random matrix theory but in large dimensional statistics at large, beyond consistency, one is often interested in the second order statistics of the studied estimators. These, usually inaccessible for all finite N, n, are often obtained through central limit theorems.

In random matrix theory, many statistics of practical interest lead to central limit theorems, either with fluctuations of order $O(1/n)$ for linear eigenvalue functionals or of order $O(1/\sqrt{n})$ for quadratic forms. As of today, multiple methods exist to achieve central limit theorems for random matrix estimators based on a sample covariance matrix, as for instance martingale (Bai and Silverstein, 2004) or characteristic function (Pastur and Ŝerbina, 2011) approaches. The fluctuations of functionals of the matrix $\hat{S}_N$, the earlier mentioned random equivalent for $\hat{C}_N$, are in particular amenable to calculus via these methods. But, while $\hat{S}_N$ was seen to be an adequate plug-in estimator for $\hat{C}_N$ allowing for the study of "first order" functionals of $\hat{C}_N$ (among which the aforementioned estimators of $f(C_N)$), it cannot be straightforwardly said that the second order statistics of functionals of $\hat{C}_N$ are asymptotically the same as those of the same functionals with $\hat{S}_N$ in place of $\hat{C}_N$. If one were able to prove the stronger convergence $n^\alpha\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\mathcal{D}} 0$ for some $\alpha > 0$, then one could ensure that most classical functionals of $\hat{C}_N$ fluctuating at rate $O(1/n^\beta)$ for any $\beta < \alpha$ have the same fluctuations as for $\hat{S}_N$. Unfortunately, the latter convergence does not seem to hold for any $\alpha \geq 1/2$ (which is the minimum requested for fluctuations of quadratic forms). This means that finer results are required to ensure that a straightforward substitution of $\hat{C}_N$ by $\hat{S}_N$ in the fluctuations study is authorized.

Since generic fluctuation results are difficult to obtain (even for classical sample covariance matrix based random matrix results), we shall concentrate here on a specific form of functionals of $\hat{C}_N$, that is functionals of the bilinear (or quadratic) form $a^*\hat{C}_N^k b$ for $k \in \mathbb{Z}$ and $\|a\| = \|b\| = 1$. For simplicity and practical interest, our focus will precisely be on bilinear forms for the shrinkage robust estimator $\hat{C}_N(\rho)$ defined in (2.4) rather than for Maronna's estimator. For these bilinear forms, when $x_1, \ldots, x_n$ are i.i.d. elliptical, it is possible to prove that $a^*\hat{C}_N^k(\rho)b - a^*\hat{S}_N^k(\rho)b = o_P(n^{\varepsilon-1})$ for all $\varepsilon > 0$. This and the fact that $a^*\hat{S}_N^k(\rho)b$ fluctuates in $n^{-\frac{1}{2}}$ ensures that so does $a^*\hat{C}_N^k(\rho)b$ with the same limiting distribution. Proving the convergence $a^*\hat{C}_N^k(\rho)b - a^*\hat{S}_N^k(\rho)b = o_P(n^{\varepsilon-1})$ is a difficult task which we handle in much the same way that we prove $\|\hat{C}_N(\rho) - \hat{S}_N(\rho)\| \xrightarrow{\text{a.s.}} 0$, however with some important improvement necessary to gain in convergence speed. The technical details are explored in Section 6.1, while an application to false alarm minimizing source detection under impulsive noise is carried out in Section 6.2.

2.2.6 The impact of outliers.

It was recalled multiple times, although loosely commented, that all large N, n results discussed so far suppose $x_1, \ldots, x_n$ to be i.i.d. elliptically distributed. The reason for such a systematic modelling is twofold: (i) elliptical vectors are easy to study, and (ii) in much the same way that Gaussian variables appropriately model background noise phenomena, elliptical vectors constitute an adequate extension to many impulsive noise scenarios of practical interest. However, they at least do not properly model what Huber initially considered outlying data, that is unknown vectors not necessarily attached to any natural distribution. Outliers may be as various as missing data, data arising from a short-time modification of the system model, or deterministic unknown vectors altogether that arise from a priori unexpected sources. For these scenarios, Huber's estimator, which we haven't considered in itself so far (but only as an example of the Maronna class), is expected to bring advantageous properties. In our last study, proposed in Chapter 7, we undertake this analysis and understand clearly why Huber's estimator is so fundamentally precious and more efficient in particular than Tyler's estimator in taming down the effect of unknown outliers. For this study, instead of considering the x_i's i.i.d., we assume that a large quantity of them are simply Gaussian with covariance C_N (for simplicity) and that the remaining quantity is deterministic but unknown. The limiting behavior of the matrix $\hat{C}_N$, which again can be assimilated to that of a tractable matrix $\hat{S}_N$ (different this time), sheds light on the role played by the interaction between C_N and the deterministic outliers in the weighs affected by u_{H} to each vector. This analysis further allows for an assessment of the robustness performance of the estimators of the Maronna class which we evaluate by means of various spectrum comparison metrics of $\hat{C}_N$ for the outlier-free versus non outlier-free data.

2.3 A word on notations

Robust M-estimates. Throughout the notes, we shall denote $\hat{C}_N$ robust matrices of the Maronna class, defined in (2.2). Despite closeness in notation, these will be distinct from $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$, reserved for robust shrinkage estimators. The Tyler estimator will not be discussed much and shall, in a slight abuse of notation, be referred to as $\hat{C}_N(0)$; indeed, although $\hat{C}_N(\rho)$ as defined in (2.4) only for $\rho > 0$, letting $\rho = 0$ in (2.4) provides the expression (2.3) of Tyler's estimator. Finally, it should be mentioned that in some of the proofs, for notational simplicity, the indexes N and arguments ρ in parentheses might be dropped. From the context, these will never create any possible confusion with other notations.

Vector notations. Although from chapter to chapter our basic system settings will change with in particular variations in the vector dimensions and statistical representations, we will enforce the following vector convention. The notation w_i will mostly stand for vectors with either i.i.d. entries zero mean and unit variance entries or uniformly distributed on the sphere of radius the square root of the dimension of w_i. From a random matrix viewpoint, at least as far as first order convergence is concerned, these two hypotheses are essentially one and the same. We shall then denote z_i a vector of the type Aw_i for A a properly size deterministic matrix, so that z_i has zero mean and covariance AA^*. Then we shall denote x_i the observation vector which often will be of the type $x_i = \sqrt{\tau_i} A w_i$ with $\tau_i > 0$ some real number modelling impulsiveness; alternatively, x_i might follow an extended form of this basic setting, in particular in Section 4.2 where x_i will be of the form $x_i = As_i + \sqrt{\tau_i} w_i$ with As_i standing for the signal part of x_i and $\sqrt{\tau_i} w_i$ for its noise part, or in Section 7 where x_i will be modelled as $x_i = Aw_i$ for some i's and $x_i = a_i$ for deterministic vectors a_i for other i's.

Other notations. As for other notations of importance in this report, we shall take the following conventions. We denote $\lambda_1(X) \leq \ldots \leq \lambda_N(X)$ the ordered eigenvalues of any Hermitian (or symmetric) matrix X. The superscript $(\cdot)^*$ designates transpose conjugate (if complex) for vectors or matrices. The norm $\|\cdot\|$ is the spectral norm for matrices and the Euclidean norm for vectors. The cardinality of a finite discrete set Ω is denoted by $|\Omega|$. Almost sure convergence is written $\xrightarrow{\text{a.s.}}$. We use the set notations $\mathbb{R}_+ = \{x \in \mathbb{R}, x > 0\}$, $A^* = \{x \in A, x \neq 0\}$, $\mathbb{C}_+ = \{z \in \mathbb{C}, \Im[z] > 0\}$, and similarly for $\mathbb{R}_-$, $\mathbb{C}_-$, $\mathbb{R}_+^*$, etc. The Hermitian (or symmetric) matrix order relations are denoted $A \succeq B$ for $A - B$ nonnegative definite and $A \succ B$ for $A - B$

positive definite. The Dirac measure at point $x \in \mathbb{R}$ is denoted by $\boldsymbol{\delta}_x$. Real and imaginary parts of $z \in \mathbb{C}$ are denoted respectively by $\Re[z]$ and $\Im[z]$.

Chapter 3

The limiting spectrum of separable sample covariance matrices

As mentioned in the introduction, the most important results of this report consist in showing that various robust estimators of scatter, call them generically $\hat{C}_N$, defined as the solutions of (2.2), (2.4), (2.5), etc. and constructed from i.i.d. elliptical vectors $x_1, \ldots, x_n$ can be asymptotically well approximated by matrices which we shall denote $\hat{S}_N$. These matrices $\hat{S}_N$ will be shown to belong to a well-known class of random matrices of the separable covariance type. As such, many properties of $\hat{S}_N$ naturally transfer to $\hat{C}_N$, starting with its limiting eigenvalue distribution, when properly defined. While the study of the defining equations for the limiting spectrum of matrices of the type $\hat{S}_N$ dates back to the early nineties, no thorough study of the analytical properties of this spectrum has ever been carried out. This first chapter intends to fill the gap by showing, in a similar fashion as in (Silverstein and Choi, 1995), that the limiting spectrum is continuous away from zero, analytical whenever the density is positive, and its support can be fully characterized.

This chapter will also offer the opportunity to recall the basic analytical tools at the root of many random matrix results and essential to a good understanding of this report.

3.1 Introduction and problem statement

We consider the $N \times n$ random matrix $X = C_N^{\frac{1}{2}} W T^{\frac{1}{2}}$ where W is an $N \times n$ real or complex random matrix having independent and identically distributed elements with mean zero and unit variance, the $N \times N$ matrix C_N is determinisitic, Hermitian and nonnegative, and the $n \times n$ matrix T is also deterministic, Hermitian and nonnegative. We assume that $N, n \to \infty$ such that $N/n = c_N \to c > 0$. We also assume that the spectral measures ν_n of C_N and $\tilde{\nu}_n$ of T converge respectively towards the probability measures ν and $\tilde{\nu}$ as $N \to \infty$ and $n \to \infty$, respectively. We assume that $\nu \neq \boldsymbol{\delta}_0$ and $\tilde{\nu} \neq \boldsymbol{\delta}_0$, where we recall that $\boldsymbol{\delta}_x$ is the Dirac measure at x.

Many contributions showed that the spectral measure of $\hat{S}_N = \frac{1}{n} X X^*$ converges to a deterministic probability measure μ and provided a characterization of this limit measure under various assumptions (Girko, 1990; Shlyakhtenko, 1996; Boutet de Monvel et al., 1996; Hachem et al., 2006), the weakest being found in (Zhang, 2006). We show here that μ has a density away from zero, this density being analytical wherever positive, and it behaves as $\sqrt{|x-a|}$ near an edge a of its support for a large class of measures ν, $\tilde{\nu}$. We also provide a complete characterization of this support along with a thorough analysis of the master equations relating μ to ν and $\tilde{\nu}$.[1] To that end, we follow the general ideas already provided in the classical paper of Marčenko and Pastur (Marčenko and Pastur, 1967) and further developed in (Silverstein and Choi, 1995) and (Dozier and Silverstein, 2007).

In (Silverstein and Choi, 1995), Silverstein and Choi performed this study for the sample covariance matrix model where $T = I_n$. The outline of the present section closely follows that of (Silverstein and Choi, 1995) although at multiple occasions our proofs will depart from those of (Silverstein and Choi, 1995), allowing for a more self-contained analysis. In particular, while Silverstein and Choi benefited from the existence of an explicit expression for the inverse of the Stieltjes transform of μ when $T = I_n$, this property no longer holds in the present more general setting which requires more fundamental analytical tools.

We recall in passing that, for $T = I_n$, if $\max_i \operatorname{dist}(\lambda_i(C_N), \operatorname{supp}(\nu)) \to 0$ and some mild additional conditions are satisfied, it was further shown in (Bai and Silverstein, 1998) that, with probability one, no closed interval outside the support of μ contains an eigenvalue of $\hat{S}_N$ for all large n. In (Bai and Silverstein, 1999), a finer result on the so called exact separation of the eigenvalues of $\hat{S}_N$ between the connected components of the support of μ is shown. Note that the requirement $\max_i \operatorname{dist}(\lambda_i(C_N), \operatorname{supp}(\nu)) \to 0$ is quite fundamental in practice (although not necessary in the mathematical sense) since, if not fulfilled, eigenvalues of $\hat{S}_N$ would be allowed to wander away from the support of μ. This remark will drive the behavior of the eigenvalues of $\hat{C}_N$ in Chapter 4 with major consequences for the applications presented in Section 4.2.

Recently, it has been discovered that the characterization in (Silverstein and Choi, 1995) of the support of μ and the results on the master equations relating μ to ν, beside their own interest, lead in conjunction with the results of (Bai and Silverstein, 1998, 1999) to the design of consistent statistical estimators of some linear functionals of the eigenvalues of C_N or projectors on the eigenspaces of this matrix. Such estimators were developed in particular by Mestre in (Mestre, 2008a,b), the initial idea dating back to the work of Girko (see e.g., (Girko, 2001)).

Turning to the separable covariance matrix ensemble of interest here, the absence of eigenvalues outside the support of μ (under similar fundamental conditions as discussed above) has been established by Paul and Silverstein in (Paul and Silverstein, 2009) without characterizing this support. The results of the present chapter therefore complement those of (Paul and Silverstein, 2009). More importantly, similar to the case $T = I_n$, these results are a necessary first step to devise statistical estimation algorithms of e.g., linear functionals of the eigenvalues of one of the matrices C_N or T. Finally, it has been noticed that there is an intimate connection between the square root behavior of the density of the limit distribution of the (scaled-centered) extreme

[1] Note that the requirement of spectral measures convergence for both C_N and T may seem relatively stringent and one would often prefer letting ν_n and $\tilde{\nu}_n$ wander as N, n grows. In this case, deterministic equivalents of the eigenvalue distribution of $\frac{1}{n}XX^*$ instead of a limit will be considered, which depend on ν_n and $\tilde{\nu}_n$ instead of ν and $\tilde{\nu}$. However, as far as spectrum properties are concerned, these deterministic equivalents will essentially have the same properties as the limiting measures. It is as such not necessary to introduce this complication in the present study.

eigenvalues found at the edges of the support and the Tracy–Widom fluctuations of the eigenvalues near those edges (see in particular (El Karoui, 2007) dealing with the sample covariance matrix case). It can be conjectured that this behavior still holds in the separable covariance case considered here. In this respect, Theorem 3.3.3 introduced below may help guessing the exact form of the Tracy–Widom law at the edges of the support of μ.

We now recall the results describing the asymptotic behavior of the spectral measure of $\hat{S}_N$. We also introduce the basic analytical tools that will be of constant use in this report.

3.1.1 The master equations

We recall that the Stieltjes transform of a probability measure π on $\mathbb{R}$ is the function

$$f(z) = \int \frac{1}{t-z}\pi(dt)$$

defined on $\mathbb{C}_+$. The function $f(z)$ (i) is holomorphic on $\mathbb{C}_+$, (ii) satisfies $f(z) \in \mathbb{C}_+$ for any $z \in \mathbb{C}_+$, and (iii) $\lim_{y\to\infty} |yf(\imath y)| = 1$. In addition, if π is supported by $\mathbb{R}^+$, then (iv) $zf(z) \in \mathbb{C}_+$ for any $z \in \mathbb{C}_+$. Conversely, any function $f(z)$ satisfying (i)–(iv) is the Stieltjes transform of the probability measure π supported by $\mathbb{R}_+$ defined by

$$\pi([a,b]) = \frac{1}{\pi}\lim_{y\to 0}\int_a^b \Im[f(x+\imath y)]dx$$

at all continuity points $a < b$ (Krein and Nudelman, 1997). Finally, observe that the Stieltjes transform of π can be trivially extended from $\mathbb{C}_+$ to $\mathbb{C} \setminus \operatorname{supp}(\pi)$ where $\operatorname{supp}(\pi)$ is the support of π.

In this chapter, a slight generalization of this result will be needed (Krein and Nudelman, 1997, Appendix A). Precisely, we shall use the fact that the following three statements are equivalent:

- The function $f(z)$ satisfies properties (i), (ii), and (iv);
- The function $f(z)$ admits the representation

  $$f(z) = a + \int_0^\infty \frac{1}{t-z}\pi(dt)$$

  where $a \geq 0$ and π is a Radon positive measure on $\mathbb{R}_+$ such that $0 < \int_0^\infty (1+t)^{-1}\pi(dt) < \infty$;
- The function $f(z)$ satisfies the properties (i)–(ii) and is analytical and nonnegative on the negative real axis $(-\infty, 0)$.

We now recall the characterization of the limitation spectrum of $\hat{S}_N$ as it stands so far. Let us denote $\mu_n = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i(\hat{S}_N)}$ the spectral measure of $\hat{S}_N = \frac{1}{n}XX^*$. The spectral measure $\tilde{\mu}_n$ of $\frac{1}{n}X^*X$ is then $\tilde{\mu}_n = (N/n)\mu_n + (1-N/n)\boldsymbol{\delta}_0$. These measures satisfy the following.

Theorem 3.1.1 ((Zhang, 2006), see also (Hachem et al., 2006) for similar notations). *For any $z \in \mathbb{C}_+$, the system of equations*

$$\delta = c \int \frac{t}{-z(1+\tilde{\delta}t)} \nu(dt) \tag{3.1}$$

$$\tilde{\delta} = \int \frac{t}{-z(1+\delta t)} \tilde{\nu}(dt) \tag{3.2}$$

admits a unique solution $(\delta, \tilde{\delta}) \in \mathbb{C}_+^2$. Let $\delta(z)$ and $\tilde{\delta}(z)$ be these solutions. The function

$$m(z) = \int \frac{1}{-z(1+\tilde{\delta}(z)t)} \nu(dt), \ z \in \mathbb{C}_+ \tag{3.3}$$

is the Stieltjes transform of a probability measure μ supported by $\mathbb{R}_+$. The function

$$\tilde{m}(z) = \int \frac{1}{-z(1+\delta(z)u)} \tilde{\nu}(du), \ z \in \mathbb{C}_+$$

is the Stieltjes transform of the probability measure $\tilde{\mu} = c\mu + (1-c)\boldsymbol{\delta}_0$. Moreover, as $N, n \to \infty$ with $N/n \to c \in (0, \infty)$,

$$\int \varphi(\lambda)\mu_n(d\lambda) \xrightarrow{\text{a.s.}} \int \varphi(\lambda)\mu(d\lambda)$$

$$\int \varphi(\lambda)\tilde{\mu}_n(d\lambda) \xrightarrow{\text{a.s.}} \int \varphi(\lambda)\tilde{\mu}(d\lambda)$$

for any continuous and bounded real function φ.

As a side note, observe as a simple corollary of Theorem 3.1.1 that

- for $C_N = I_N$, $m(z) = \delta(z)/c$ is defined as the unique solution in $\mathbb{C}_+$ of

$$m(z) = \left(-z + \int \frac{t}{1+cm(z)t} \tilde{\nu}(dt)\right)^{-1} \tag{3.4}$$

- for $T = I_n$, $\delta(z)$ is the unique solution in $\mathbb{C}_+$ of

$$\delta(z) = c \int \frac{1}{-z + \frac{t}{1+\delta(z)}} \nu(dt). \tag{3.5}$$

These two relations will be extensively used in the proofs of the main results of the following chapters (with possibly some slight changes in notations from chapter to chapter).

Returning to the general setting, we first collect some simple facts and identities that will be often used in this chapter:

- Define

$$F(\tilde{\delta}, z) = \int \frac{t}{-z + ct \int \frac{u}{1+u\tilde{\delta}} \nu(du)} \tilde{\nu}(dt) \; - \tilde{\delta}, \quad (\tilde{\delta}, z) \in \mathbb{C}_+^2. \tag{3.6}$$

  Plugging (3.1) into (3.2), we obtain that $\tilde{\delta}(z)$ can also be defined as the unique solution of $F(\tilde{\delta}, z) = 0$, which will sometimes be more convenient to than than the dual equation (3.1)–(3.2).

- The functions $m(z)$ and $\tilde{m}(z)$ satisfy

$$\begin{aligned} m(z) &= \int \frac{1 + \tilde{\delta}(z)t - \tilde{\delta}(z)t}{-z(1 + \tilde{\delta}(z)t)} \nu(dt) = -z^{-1} - c^{-1}\delta(z)\tilde{\delta}(z) \\ \tilde{m}(z) &= -z^{-1} - \delta(z)\tilde{\delta}(z). \end{aligned} \tag{3.7}$$

- For any $z_1, z_2 \in \mathbb{C}_+$, define

$$\begin{aligned} \gamma(z_1, z_2) &= c \int \frac{t^2}{z_1 z_2 (1 + \tilde{\delta}(z_1)t)(1 + \tilde{\delta}(z_2)t)} \nu(dt) \\ \tilde{\gamma}(z_1, z_2) &= \int \frac{t^2}{z_1 z_2 (1 + \delta(z_1)t)(1 + \delta(z_2)t)} \tilde{\nu}(dt) \end{aligned} \tag{3.8}$$

  (since $|(1+\tilde{\delta}(z_1)t)(1+\tilde{\delta}(z_2)t)| \geq \Im\tilde{\delta}(z_1)\Im\tilde{\delta}(z_2)t^2$ and $|(1+\delta(z_1)t)(1+\delta(z_2)t)| \geq \Im\delta(z_1)\Im\delta(z_2)t^2$, the integrability is guaranteed). By definition of $\tilde{\delta}(z)$, we have

$$\tilde{\delta}(z_1) - \tilde{\delta}(z_2) = \int \frac{(z_1 - z_2)t + (z_1\delta(z_1) - z_2\delta(z_2))t^2}{z_1 z_2 (1 + \delta(z_1)t)(1 + \delta(z_2)t)} \tilde{\nu}(dt)$$

  and by developing the expression of $z_1\delta(z_1) - z_2\delta(z_2)$ using (3.1), we obtain

$$\begin{aligned} &(1 - z_1 z_2 \gamma(z_1, z_2)\tilde{\gamma}(z_1, z_2))\tilde{\delta}(z_1) - \tilde{\delta}(z_2)) \\ &= (z_1 - z_2) \int \frac{t}{z_1 z_2 (1 + \delta(z_1)t)(1 + \delta(z_2)t)} \, \tilde{\nu}(dt). \end{aligned} \tag{3.9}$$

  A similar derivation performed over $z_1 = z$ and $z_2 = z^*$ for $z \in \mathbb{C}_+$ shows that

$$(1 - |z|^2 \gamma(z, z^*)\tilde{\gamma}(z, z^*))\Im\tilde{\delta}(z) = \Im z \int \frac{t}{|z|^2 |1 + \delta(z)t|^2} \, \tilde{\nu}(dt). \tag{3.10}$$

  On $\mathbb{C}_+$, $\Im\tilde{\delta}(z) > 0$. Moreover, the integral at the right hand side is strictly positive. Hence

$$\forall\, z \in \mathbb{C}_+, \quad 1 - |z|^2 \gamma(z, z^*)\tilde{\gamma}(z, z^*) > 0.$$

  This inequality will be of central importance in the sequel.

The two measures introduced by the following proposition share many properties with μ as will be seen below, despite their not being probability measures. They will play an essential role in what follows.

Proposition 3.1.1. *The functions $\delta(z)$ and $\tilde{\delta}(z)$ admit the representations*

$$\delta(z) = \int_0^\infty \frac{1}{t-z}\rho(dt)$$
$$\tilde{\delta}(z) = \int_0^\infty \frac{1}{t-z}\tilde{\rho}(dt), \quad z \in \mathbb{C}_+$$

where ρ and $\tilde{\rho}$ are two Radon positive measures on $\mathbb{R}_+$ such that

$$0 < \int_0^\infty \frac{1}{1+t}\rho(dt) < \infty$$
$$0 < \int_0^\infty \frac{1}{1+t}\tilde{\rho}(dt) < \infty.$$

Proof. One can observe that the function $F(\tilde{\delta}, z)$ defined in (3.6) is holomorphic on $\mathbb{C}_+^2$. Fixing $z_0 \in \mathbb{C}_+$, a short derivation shows that

$$\begin{aligned}\left|\frac{\partial F}{\partial \tilde{\delta}}(\tilde{\delta}, z_0)\right| &= |1 - z_0^2\gamma(z_0, z_0)\tilde{\gamma}(z_0, z_0)| \\ &\geq 1 - |z_0^2\gamma(z_0, z_0)\tilde{\gamma}(z_0, z_0)| \qquad \geq 1 - |z_0|^2\gamma(z_0, z_0^*)\tilde{\gamma}(z_0, z_0^*) > 0\end{aligned}$$

by (3.10). The holomorphic implicit function theorem (Fritzsche and Grauert, 2002, Ch. 1, Th. 7.6) shows then that $\tilde{\delta}(z)$ is holomorphic in a neighborhood of z_0. Since z_0 is chosen arbitrarily in $\mathbb{C}_+$, we get that $\tilde{\delta}(z)$ is holomorphic in $\mathbb{C}_+$. Recall that $\Im\tilde{\delta}(z) > 0$ on $\mathbb{C}_+$. Since we furthermore have

$$\Im(z\tilde{\delta}(z)) = \Im\delta(z) \int \frac{t^2}{|1+\delta(z)t|^2}\tilde{\nu}(dt) > 0$$

on $\mathbb{C}_+$, we get the representation

$$\tilde{\delta}(z) = \tilde{a} + \int \frac{1}{t-z}\tilde{\rho}(dt)$$

where $\tilde{a} \geq 0$ and where $\tilde{\rho}$ satisfies the properties given in the statement. Let us show that $\tilde{a} = 0$. Observe that $\tilde{\delta}(x) \downarrow \tilde{a}$ when x is a real negative number converging to $-\infty$. By a continuation argument, $F(\tilde{\delta}(x), x) = 0$ for any negative value of x. As $x \to -\infty$, we get by the monotone convergence theorem

$$I(\tilde{\delta}) = \int \frac{u}{1+u\tilde{\delta}}\nu(du) \uparrow I(\tilde{a}) = \int \frac{u}{1+u\tilde{a}}\nu(du) \in (0, \infty].$$

When $x < 0$ is far enough from zero, $I(\tilde{\delta}) \geq C$ where $C > 0$ is a constant, and the dominated convergence theorem shows that

$$\tilde{\delta}(x) = \int \frac{t}{-x + ctI(\tilde{\delta}(x))}\nu(dt) \xrightarrow[x\to-\infty]{} 0.$$

A similar argument can be applied to $\delta(z)$. $\square$

3.2 Some elementary properties of μ

In the asymptotic regime where N is fixed and $n \to \infty$, the matrix $\hat{S}_N - (\frac{1}{n} \operatorname{tr} T) C_N$ will converge to zero when the assumptions of the law of large numbers are satisfied. In the large N, n setting, the following result is therefore expected.

Proposition 3.2.1. *Assume that $M_\nu = \int t\nu(dt)$ and $M_{\tilde{\nu}} = \int t\tilde{\nu}(dt)$ are both finite. Then*

$$\mu(dt) \to \nu(M_{\tilde{\nu}}^{-1}\, dt)$$

a.s. as $c \to 0$ where the convergence is understood as the weak convergence of probability measures.

Proof. For any $u \geq 0$ and any $z \in \mathbb{C}_+$, $|z(1 + \tilde{\delta}(z)u)| \geq \Im(z(1 + \tilde{\delta}(z)u)) \geq \Im(z)$, hence $|\delta(z)| \leq cM_\nu / \Im(z)$, which implies that $\delta(z) \to 0$ as $c \to 0$. Similarly, $|z(1 + \delta(z)t)| \geq \Im(z)$ for any $t \geq 0$ and any $z \in \mathbb{C}_+$, hence $\tilde{\delta}(z) \to -M_{\tilde{\nu}}/z$ by dominated convergence. Invoking the dominated convergence theorem again, we get

$$m(z) \xrightarrow[c\to 0]{} \int \frac{1}{M_{\tilde{\nu}} t - z} \nu(dt) = \int \frac{1}{t - z} \nu(M_{\tilde{\nu}}^{-1}\, dt)$$

which shows the result. $\square$

We now characterize $\mu(\{0\})$. Intuitively, $\operatorname{rank}(X) \simeq \min[N(1 - \nu(\{0\})), n(1 - \tilde{\nu}(\{0\}))]$ and $\mu(\{0\}) \simeq 1 - \operatorname{rank}(X)/N$ for large n. The following result is therefore also expected.

Proposition 3.2.2. $\mu(\{0\}) = 1 - \min[1 - \nu(\{0\}), c^{-1}(1 - \tilde{\nu}(\{0\}))]$.

Proof. From the general expression of the Stieltjes transform of a probability measure, it is easily seen using the dominated convergence theorem that $\mu(\{0\}) = \lim_{y\downarrow 0}(-\imath y m(\imath y))$. Moreover, since $|y(t - \imath y)^{-1}| \leq (t^2 + 1)^{-1/2}$ when $|y| \leq 1$, the dominated convergence theorem and Proposition 3.1.1 show that $\tilde{\rho}(\{0\}) = \lim_{y\downarrow 0}(-\imath y \tilde{\delta}(\imath y))$.

Let us write $\nu = \nu(\{0\})\boldsymbol{\delta}_0 + \nu'$ and $\tilde{\nu} = \tilde{\nu}(\{0\})\boldsymbol{\delta}_0 + \tilde{\nu}'$, and let us assume that $1 - \nu(\{0\}) < c^{-1}(1 - \tilde{\nu}(\{0\}))$, or equivalently, that $\nu'(\mathbb{R}_+) < c^{-1}\tilde{\nu}'(\mathbb{R}_+)$. In this case, we will show that $\tilde{\rho}(\{0\}) > 0$. That being true, we get

$$\mu(\{0\}) = \lim_{y\downarrow 0}(-\imath y m(\imath y)) = \nu(\{0\}) + \lim_{y\downarrow 0} \int \frac{1}{1 + \tilde{\delta}(\imath y)t} \nu'(dt) = \nu(\{0\})$$

(since $\Re(\tilde{\delta}(\imath y)) > 0$, see below, the integrand above is bounded in absolute value by 1, and furthermore, it converges to 0 for any $t > 0$ due to the fact that $\tilde{\rho}(\{0\}) > 0$).

We assume that $\tilde{\rho}(\{0\}) = 0$ and raise a contradiction. The equation $F(\tilde{\delta}, \imath y) = 0$ for $y > 0$ can be rewritten as

$$\int \frac{t}{-\imath y \tilde{\delta}(\imath y) + ct \int \frac{u\tilde{\delta}(\imath y)}{1 + u\tilde{\delta}(\imath y)} \nu'(du)} \tilde{\nu}'(dt) = 1.$$

We have

$$\Re(\tilde{\delta}(\imath y)) = \Re \int \frac{1}{t - \imath y} \tilde{\rho}(dt) = \int \frac{t}{t^2 + y^2} \tilde{\rho}(dt) > 0$$

and $\lim_{y\to 0} \Re(\tilde{\delta}(\imath y)) \in (0, \infty]$ by the monotone convergence theorem. Let

$$I(y) = \int \frac{u\tilde{\delta}(\imath y)}{1 + u\tilde{\delta}(\imath y)} \nu'(du).$$

Writing $\tilde{\delta} = \tilde{\delta}(\imath y)$, we have

$$\Re(I(y)) = \int \frac{u(\Re\tilde{\delta})(1 + u\Re\tilde{\delta}) + (u\Im\tilde{\delta})^2}{(1 + u\Re\tilde{\delta})^2 + (u\Im\tilde{\delta})^2} \nu'(du)$$

whose $\liminf$ is positive as $y \downarrow 0$. Furthermore, we have for $y > 0$

$$\Re(-\imath y\tilde{\delta}(\imath y)) = \Re \int \frac{-\imath y}{t - \imath y} \tilde{\rho}(dt) = \int \frac{y^2}{t^2 + y^2} \tilde{\rho}(dt) > 0$$

hence $\liminf_{y\downarrow 0} |-\imath y\tilde{\delta}(\imath y) + ctI(y)| \geq ct \liminf_{y\downarrow 0} \Re I(y)$. Consequently, we have by the assumption $\tilde{\rho}(\{0\}) = 0$ and the dominated convergence theorem again

$$\int \frac{t}{-\imath y\tilde{\delta}(\imath y) + ctI(y)} \tilde{\nu}'(dt) - \frac{\tilde{\nu}'(\mathbb{R}_+)}{cI(y)} \xrightarrow[y\downarrow 0]{} 0.$$

This shows that $\lim_{y\downarrow 0} I(y) = c^{-1}\tilde{\nu}'(\mathbb{R}_+)$. But since $\Re(\tilde{\delta}(\imath y)) > 0$, $|u\tilde{\delta}(\imath y)(1 + u\tilde{\delta}(\imath y))^{-1}| \leq 1$ for $u \geq 0$ hence $|I(y)| \leq \nu'(\mathbb{R}_+)$. Therefore, $c^{-1}\tilde{\nu}'(\mathbb{R}_+) \leq \nu'(\mathbb{R}_+)$ which contradicts the assumption.

If $\nu'(\mathbb{R}_+) > c^{-1}\tilde{\nu}'(\mathbb{R}_+)$, we replace μ, $m(z)$ and $\tilde{\delta}(z)$ with $\tilde{\mu}$, $\tilde{m}(z)$ and $\delta(z)$ respectively in the previous argument.

To deal with the case $\nu'(\mathbb{R}_+) = c^{-1}\tilde{\nu}'(\mathbb{R}_+)$, we use the fact that μ is continuous with respect to $\tilde{\nu}$ in the weak convergence topology (see (Zhang, 2006, Chap. 4)). By approximating $\tilde{\nu}$ by a sequence $\tilde{\nu}_k = \tilde{\nu}_k(\{0\}) + \tilde{\nu}'_k$ such that $\nu'(\mathbb{R}_+) < c^{-1}\tilde{\nu}'_k$, we are led back to the first part of the proof. The result is obtained by continuity. □

3.3 Density and support

3.3.1 Existence of a continuous density

This paragraph is devoted to establishing the following result.

Theorem 3.3.1. *For all $x \in \mathbb{R}_* = \mathbb{R} \setminus \{0\}$, the nontangential limit $\lim_{z\in\mathbb{C}_+\to x} m(z)$ exists. Denoting $m(x)$ this limit, the function $\Im m(x)$ is continuous on $\mathbb{R}_*$, and μ has a continuous derivative $f(x) = \pi^{-1}\Im m(x)$ on $\mathbb{R}_*$. Similarly, the nontangential limits $\lim_{z\in\mathbb{C}_+\to x} \delta(z)$ and $\lim_{z\in\mathbb{C}_+\to x} \tilde{\delta}(z)$ exist. Denoting respectively $\Im\delta(x)$ and $\Im\tilde{\delta}(x)$ these limits, the functions $\Im\delta(x)$ and $\Im\tilde{\delta}(x)$ are both continuous on $\mathbb{R}_*$, and both ρ and $\tilde{\rho}$ have continuous derivatives on $\mathbb{R}_+$. Finally* $\mathrm{supp}(\rho) \cap \mathbb{R}_* = \mathrm{supp}(\tilde{\rho}) \cap \mathbb{R}_* = \mathrm{supp}(\mu) \cap \mathbb{R}_*$.

Since $\tilde{\mu} = c\mu + (1-c)\boldsymbol{\delta}_0$, it is obvious that we can replace m with $\tilde{m}$ in the statement of the theorem.

As soon as the existence of the three limits as $z \in \mathbb{C}_+ \to x$ are established, we know from the Stieltjes inversion formula that the densities exist (see (Silverstein and Choi, 1995)[Th. 2.1]). By a simple passage to the limit argument ((Silverstein and Choi, 1995, Th. 2.2)), we also know that these densities are continuous.

To prove the theorem, we first prove that $\lim_{z\in\mathbb{C}_+\to x} \delta(z)$ and $\lim_{z\in\mathbb{C}_+\to x} \tilde{\delta}(z)$ both exist for all $x \in \mathbb{R}_*$ (Lemmas 3.1 to 3.3). This shows that both ρ and $\tilde{\rho}$ have densities on $\mathbb{R}_*$. Lemma 3.4 shows then that $\lim_{z\in\mathbb{C}_+\to x} m(z)$ exists, and furthermore, that the intersections of the supports of μ, ρ and $\tilde{\rho}$ with $\mathbb{R}_*$ coincide.

Lemma 3.1. *$|\delta(z)|$ and $|\tilde{\delta}(z)|$ are bounded on any bounded region of $\mathbb{C}_+$ lying at a positive distance from the imaginary axis.*

Proof. We first observe that for any $z \in \mathbb{C}_+$,

$$
\begin{aligned}
|\delta(z)| &\leq c\Big(\int \frac{t^2}{|z|^2|1+\tilde{\delta}(z)t|^2}\nu(dt)\Big)^{\frac{1}{2}} = \sqrt{c}\gamma(z,z^*)^{\frac{1}{2}} \\
|\tilde{\delta}(z)| &\leq \tilde{\gamma}(z,z^*)^{\frac{1}{2}}
\end{aligned}
$$

and we recall that $0 < |z|^2\gamma(z,z^*)\tilde{\gamma}(z,z^*) < 1$. Using (3.7), we therefore get that $\sup_{z\in\mathcal{R}} |\tilde{m}(z)| < \infty$ where $\mathcal{R}$ is the region alluded to in the statement of the lemma.

We now assume that $\sup_{z\in\mathcal{R}} |\tilde{\delta}(z)| = \infty$ and raise a contradiction, the case where $\sup_{z\in\mathcal{R}} |\delta(z)|$ being treated similarly. By assumption, there exists a sequence $z_0, z_1, \ldots \in \mathcal{R}$ such that $|\tilde{\delta}(z_k)| \to \infty$. By the inequalities above, we get that $\tilde{\gamma}(z_k, z_k^*) \to \infty$, hence $\gamma(z_k, z_k^*) \to 0$ and therefore $\delta(z_k) \to 0$. In parallel, we have

$$
\begin{aligned}
z_0\tilde{m}(z_0) - z_k\tilde{m}(z_k) &= \int \Big(\frac{-1}{1+\delta(z_0)t} + \frac{1}{1+\delta(z_k)t}\Big)\tilde{\nu}(dt) \\
&= (\delta(z_0) - \delta(z_k)) \int \frac{t}{(1+\delta(z_0)t)(1+\delta(z_k)t)}\tilde{\nu}(dt).
\end{aligned}
$$

Using (3.9), we obtain

$$
\begin{aligned}
&(1 - z_0 z_k \gamma(z_0,z_k)\tilde{\gamma}(z_0,z_k))(\tilde{\delta}(z_0) - \tilde{\delta}(z_k)) \\
&\qquad = (z_k^{-1} - z_0^{-1})\frac{z_0\tilde{m}(z_0) - z_k\tilde{m}(z_k)}{\delta(z_0) - \delta(z_k)}.
\end{aligned}
$$

By what precedes, $\sup_k |(z_k^{-1} - z_0^{-1})(z_0\tilde{m}(z_0) - z_k\tilde{m}(z_k))| < \infty$. Moreover, $\liminf_k |\delta(z_0) - \delta(z_k)| > 0$ since $\Im\delta(z_0) > 0$. By Cauchy–Schwarz, $|\gamma(z_0,z_k)| \leq \gamma(z_0,z_0^*)^{\frac{1}{2}}\gamma(z_k,z_k^*)^{\frac{1}{2}}$ and $|\tilde{\gamma}(z_0,z_k)| \leq \tilde{\gamma}(z_0,z_0^*)^{\frac{1}{2}}\tilde{\gamma}(z_k,z_k^*)^{\frac{1}{2}}$. Therefore,

$$
\begin{aligned}
\inf_k |1 - z_0 z_k\gamma(z_0,z_k)\tilde{\gamma}(z_0,z_k)| &\geq 1 - \sup_k |z_0 z_k \gamma(z_0,z_k)\tilde{\gamma}(z_0,z_k)| \\
&\geq 1 - (|z_0|^2\gamma(z_0,z_0^*)\tilde{\gamma}(z_0,z_0^*))^{\frac{1}{2}} \sup_k (|z_k|^2\gamma(z_k,z_k^*)\tilde{\gamma}(z_k,z_k^*))^{\frac{1}{2}} \\
&> 0
\end{aligned}
$$

which shows that $\sup_k |\tilde{\delta}(z_k)| < \infty$. □

Lemma 3.2. *For $\ell = 1, 2$, the integrals*

$$\int \frac{t^\ell}{|1+\tilde{\delta}(z)t|^2}\nu(dt) \quad \text{and} \quad \int \frac{t^\ell}{|1+\delta(z)t|^2}\tilde{\nu}(dt)$$

are bounded on any bounded region $\mathcal{R}$ of $\mathbb{C}_+$ lying at a positive distance from the imaginary axis.

Proof. We observe that for $\ell = 2$, the integrals given in the statement of the lemma are equal to $c^{-1}|z|^2\gamma(z, z^*)$ and to $|z|^2\tilde{\gamma}(z, z^*)$ respectively. We know that $\sup_{z\in\mathcal{R}} |z|^4\gamma(z, z^*)\tilde{\gamma}(z, z^*) \leq \sup_{z\in\mathcal{R}} |z|^2 < \infty$. Assume that $\tilde{\gamma}(z_n, z_n^*) \to \infty$ along some sequence $z_n \in \mathcal{R}$. Then $\gamma(z_n, z_n^*) \to 0$, which implies that the integrand of $|z_n|^2\gamma(z_n, z_n^*)$ converges to zero ν-almost everywhere. This implies in turn that $|\tilde{\delta}(z_n)| \to \infty$ which contradicts Lemma 3.1. The result is proven for $\ell = 2$.

We now consider the case $\ell = 1$, focusing on the first integral that we write as $\int_0^\infty tI(t)^{-1}\nu(dt)$. Since $\int_0^\infty tI(t)^{-1}\nu(dt) \leq \int_0^1 tI(t)^{-1}\nu(dt) + \int_1^\infty t^2 I(t)^{-1}\nu(dt)$, we only need to bound the first term at the right hand side. Denoting by 1_A the indicator function on the set A, we have

$$\begin{aligned}\int_0^1 \frac{t}{I(t)}\nu(dt) &= \int_0^1 \frac{t}{I(t)} 1_{[0,|2\Re\tilde{\delta}|^{-1}]}(t)\nu(dt) + \int_0^1 \frac{t}{I(t)} 1_{(|2\Re\tilde{\delta}|^{-1},\infty)}(t)\nu(dt) \\ &\leq 4\int_0^1 t\nu(dt) + |2\Re\tilde{\delta}| \int_0^\infty \frac{t^2}{I(t)}\nu(dt)\end{aligned}$$

which is bounded. □

Lemma 3.3. *For any $x \in \mathbb{R}_*$, $\lim_{z\in\mathbb{C}_+\to x} \delta(z)$ and $\lim_{z\in\mathbb{C}_+\to x} \tilde{\delta}(z)$ exist.*

Proof. If both ν and $\tilde{\nu}$ are Dirac probability measures, one can see that $\delta(z)$ and $\tilde{\delta}(z)$ are the Stieltjes transforms of Marčenko–Pastur distributions and the result is straightforward. We shall assume without generality loss that ν is not a Dirac measure.

We showed that δ and $\tilde{\delta}$ are bounded on any bounded region of $\mathbb{C}_+$ lying away from the imaginary axis. Let $z_n \in \mathbb{C}_+$ be a sequence converging to an $x \in \mathbb{R}_*$, and along which $\tilde{\delta}(z_n)$ converges to some $\tilde{\boldsymbol{\delta}}$ and $\delta(z_n)$ converges to some $\boldsymbol{\delta}$. Since $|z_n|^2\gamma(z_n, z_n^*)\tilde{\gamma}(z_n, z_n^*) < 1$, taking the limit we get that $x^2\Gamma(x, \tilde{\boldsymbol{\delta}})\widetilde{\Gamma}(x, \boldsymbol{\delta}) \leq 1$ where

$$\begin{aligned}\Gamma(x, \tilde{\boldsymbol{\delta}}) &= c\int \frac{t^2}{x^2|1+\tilde{\boldsymbol{\delta}}t|^2}\nu(dt) \\ \widetilde{\Gamma}(x, \boldsymbol{\delta}) &= \int \frac{t^2}{x^2|1+\boldsymbol{\delta}t|^2}\tilde{\nu}(dt).\end{aligned}$$

Take two sequences z_n and $\underline{z}_n$ in $\mathbb{C}_+$ which converge to the same $x \in \mathbb{R}_*$, and such that $\tilde{\delta}(z_n)$ and $\tilde{\delta}(\underline{z}_n)$ converge towards $\tilde{\boldsymbol{\delta}}$ and $\underline{\tilde{\boldsymbol{\delta}}}$ respectively, and $\delta(z_n)$ and $\delta(\underline{z}_n)$ converge towards $\boldsymbol{\delta}$ and $\underline{\boldsymbol{\delta}}$ respectively.

We shall show that $\tilde{\boldsymbol{\delta}} = \underline{\tilde{\boldsymbol{\delta}}}$. Writing

$$(1 - z_n\underline{z}_n\gamma(z_n, \underline{z}_n)\tilde{\gamma}(z_n, \underline{z}_n))(\tilde{\delta}(z_n) - \tilde{\delta}(\underline{z}_n)) \quad = (z_n - \underline{z}_n)\int \frac{t}{z_n\underline{z}_n(1+\delta(z_n)t)(1+\delta(\underline{z}_n)t)}\,\tilde{\nu}(dt)$$

the sequence of integrals at the right hand side is bounded by Cauchy–Schwarz and by Lemma 3.2. Moving to the limit, we obtain $(1 - x^2 \mathbf{\Gamma}\widetilde{\mathbf{\Gamma}})(\tilde{\delta} - \underline{\tilde{\delta}}) = 0$ where

$$\mathbf{\Gamma} = c \int \frac{t^2}{x^2(1+\tilde{\delta}t)(1+\underline{\tilde{\delta}}t)} \nu(dt)$$
$$\widetilde{\mathbf{\Gamma}} = \int \frac{t^2}{x^2(1+\boldsymbol{\delta}t)(1+\underline{\boldsymbol{\delta}}t)} \tilde{\nu}(dt).$$

Assume that $\tilde{\delta} \neq \underline{\tilde{\delta}}$. Since ν is different from a Dirac measure, we have $|\mathbf{\Gamma}| < \Gamma(x, \tilde{\delta})^{\frac{1}{2}} \Gamma(x, \underline{\tilde{\delta}})^{\frac{1}{2}}$ by Cauchy–Schwarz. By Cauchy–Schwarz again, we also have $|\widetilde{\mathbf{\Gamma}}| \leq \widetilde{\Gamma}(x, \boldsymbol{\delta})^{\frac{1}{2}} \widetilde{\Gamma}(x, \underline{\boldsymbol{\delta}})^{\frac{1}{2}}$. Consequently,

$$\begin{aligned} |1 - x^2 \mathbf{\Gamma}\widetilde{\mathbf{\Gamma}}| &\geq 1 - x^2 |\mathbf{\Gamma}\widetilde{\mathbf{\Gamma}}| \\ &> 1 - \sqrt{x^2 \Gamma(x, \tilde{\delta})\Gamma(x, \underline{\tilde{\delta}})} \sqrt{x^2 \widetilde{\Gamma}(x, \boldsymbol{\delta}) \widetilde{\Gamma}(x, \underline{\boldsymbol{\delta}})} \\ &\geq 0. \end{aligned}$$

This contradicts $(1 - x^2 \mathbf{\Gamma}\widetilde{\mathbf{\Gamma}})(\tilde{\delta} - \underline{\tilde{\delta}}) = 0$. Hence $\tilde{\delta} = \underline{\tilde{\delta}}$. We prove similarly that $\boldsymbol{\delta} = \underline{\boldsymbol{\delta}}$. □

Lemma 3.4. *For any $x \in \mathbb{R}_*$, $\lim_{z \in \mathbb{C}_+ \to x} m(z)$ exists. Let $m(x) = \lim_{z \in \mathbb{C}_+ \to x} m(z)$, $\delta(x) = \lim_{z \in \mathbb{C}_+ \to x} \delta(z)$ and $\tilde{\delta}(x) = \lim_{z \in \mathbb{C}_+ \to x} \tilde{\delta}(z)$. Then*

$$\Im \delta(x) > 0 \Leftrightarrow \Im \tilde{\delta}(x) > 0 \Leftrightarrow \Im m(x) > 0.$$

Proof. The fact that $\lim_{z \in \mathbb{C}_+ \to x} m(z)$ exists can be immediately deduced from the first identity in (3.7) and the previous lemma. Let us show that $\Im \delta(x) > 0 \Leftrightarrow \Im \tilde{\delta}(x) > 0$. We have

$$\Im \tilde{\delta}(z) = \frac{1}{|z|^2} \int \frac{\Im z t + \Im(z\delta(z)) t^2}{|1 + \delta(z) t|^2} \tilde{\nu}(dt)$$

Assume that $\lim_{z \in \mathbb{C}_+ \to x} \Im \delta(z) = \Im \delta(x) > 0$. By Fatou's lemma, we get

$$\liminf_{z \in \mathbb{C}_+ \to x} \Im \tilde{\delta}(z) \geq \frac{1}{x^2} \int \frac{x \Im \delta(x) t^2}{(1 + \Re \delta(x) t)^2 + t^2 (\Im \delta(x))^2} \tilde{\nu}(dt) > 0.$$

Using this same argument with the roles of δ and $\tilde{\delta}$ interchanged, we get that $\Im \delta(x) > 0 \Leftrightarrow \Im \tilde{\delta}(x) > 0$. Using (3.3) and Fatou's lemma again, we also obtain that $\Im \tilde{\delta}(x) > 0 \Rightarrow \Im m(x) > 0$. Conversely, $\Im m(x) = -c^{-1} \Im(\delta(x) \tilde{\delta}(x)) = -c^{-1}(\Re \delta(x) \Im \tilde{\delta}(x) + \Im \delta(x) \Re \tilde{\delta}(x))$. Therefore, $\Im m(x) > 0 \Rightarrow (\Im \delta(x) > 0 \text{ or } \Im \tilde{\delta}(x) > 0) \Leftrightarrow \Im \tilde{\delta}(x) > 0$. □

3.3.2 Determination of $\operatorname{supp}(\mu)$

In the remainder of the chapter, we characterize $\operatorname{supp}(\mu) \cap \mathbb{R}_* = \operatorname{supp}(\tilde{\rho}) \cap \mathbb{R}_*$, focusing on the measure $\tilde{\rho}$. In the following, we let

$$\mathcal{D} = \begin{cases} \{0\} \cup \{\boldsymbol{\delta} \in \mathbb{R}_* : -\boldsymbol{\delta}^{-1} \notin \operatorname{supp}(\tilde{\nu})\} \text{ if } \operatorname{supp}(\tilde{\nu}) \text{ is compact,} \\ \{\boldsymbol{\delta} \in \mathbb{R}_* : -\boldsymbol{\delta}^{-1} \notin \operatorname{supp}(\tilde{\nu})\} \text{ otherwise,} \end{cases}$$

and

$$\widetilde{\mathcal{D}} = \begin{cases} \{0\} \cup \{\tilde{\delta} \in \mathbb{R}_* : -\tilde{\delta}^{-1} \notin \operatorname{supp}(\nu)\} \text{ if } \operatorname{supp}(\nu) \text{ is compact,} \\ \{\tilde{\delta} \in \mathbb{R}_* : -\tilde{\delta}^{-1} \notin \operatorname{supp}(\nu)\} \text{ otherwise.} \end{cases}$$

Note that $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are both open sets.

Proposition 3.3.1. *If $\boldsymbol{x} \in \mathbb{R}_*$ does not belong to* $\operatorname{supp}(\mu)$*, then $\delta(\boldsymbol{x}) \in \mathcal{D}$, $\tilde{\delta}(\boldsymbol{x}) \in \widetilde{\mathcal{D}}$, and* $1 - \boldsymbol{x}^2\gamma(\boldsymbol{x}, \boldsymbol{x})\tilde{\gamma}(\boldsymbol{x}, \boldsymbol{x}) > 0$.

Proof. Since $\operatorname{supp}(\mu) \cap \mathbb{R}_* = \operatorname{supp}(\rho) \cap \mathbb{R}_* = \operatorname{supp}(\tilde{\rho}) \cap \mathbb{R}_*$ and since the Stieltjes transform of a positive measure is real and increasing on the real axis outside the support of this measure, $\delta(\boldsymbol{x}) \in \mathbb{R}$, $\tilde{\delta}(\boldsymbol{x}) \in \mathbb{R}$, and $\tilde{\delta}'(\boldsymbol{x}) > 0$. Extending (3.9) to a neighborhood of $\boldsymbol{x}$, we get

$$\tilde{\delta}'(\boldsymbol{x}) = \frac{1}{1 - \boldsymbol{x}^2\gamma(\boldsymbol{x}, \boldsymbol{x})\tilde{\gamma}(\boldsymbol{x}, \boldsymbol{x})} \int \frac{t}{\boldsymbol{x}^2(1 + \delta(\boldsymbol{x})t)^2}\, \tilde{\nu}(dt)$$

hence $1 - \boldsymbol{x}^2\gamma(\boldsymbol{x}, \boldsymbol{x})\tilde{\gamma}(\boldsymbol{x}, \boldsymbol{x}) > 0$. We now show that $\delta(\boldsymbol{x}) \in \mathcal{D}$. Assume $\delta(\boldsymbol{x}) \neq 0$. Denoting by $m_{\tilde{\nu}}$ the Stieltjes transform of $\tilde{\nu}$, (3.2) can be rewritten as $m_{\tilde{\nu}}(-\delta(z)^{-1}) = \delta(z) + z\delta^2(z)\tilde{\delta}(z)$. Making z converge from $\mathbb{C}_+$ to a point x lying in a small neighborhood of $\boldsymbol{x}$ in $\mathbb{R}$, the right hand side of this equation converges to a real number, and $-\delta(z)^{-1}$ converges from $\mathbb{C}_+$ to a point in a neighborhood of $-\delta(\boldsymbol{x})^{-1}$ in $\mathbb{R}$. Since $m_{\tilde{\nu}}$ is real on this neighborhood, the load of this neighborhood by $\tilde{\nu}$ is zero, which implies that $\delta(\boldsymbol{x}) \in \mathcal{D}$. Assume now that $\delta(\boldsymbol{x}) = 0$. Then there exists $\boldsymbol{x}_0 \notin \operatorname{supp}(\rho)$ such that $\boldsymbol{x}_0 < \boldsymbol{x}$ and $\delta(x)$ increases from $\delta(\boldsymbol{x}_0)$ to zero on $[\boldsymbol{x}_0, \boldsymbol{x}]$. The argument above shows that $\tilde{\nu}([-\delta^{-1}(\boldsymbol{x}_0), -\delta^{-1}(x)]) = 0$ for any $x \in [\boldsymbol{x}_0, \boldsymbol{x})$. Making $x \uparrow \boldsymbol{x}$, we obtain that $\tilde{\nu}([-\delta^{-1}(\boldsymbol{x}_0), \infty)) = 0$, in other words, $\tilde{\nu}$ is compactly supported. It results that $\delta(\boldsymbol{x}) \in \mathcal{D}$. The same argument shows that $\tilde{\delta}(\boldsymbol{x}) \in \widetilde{\mathcal{D}}$. $\square$

Proposition 3.3.2. *Given $\tilde{\delta} \in \widetilde{\mathcal{D}}$, assume there exists $\boldsymbol{x} \in \mathbb{R}_*$ for which*

$$\begin{aligned} \boldsymbol{\delta} &= c \int \frac{t}{-\boldsymbol{x}(1 + \tilde{\delta}t)} \nu(dt) \ \in \mathcal{D}, \\ \tilde{\delta} &= \int \frac{t}{-\boldsymbol{x}(1 + \boldsymbol{\delta}t)} \tilde{\nu}(dt), \end{aligned} \tag{3.11}$$

and

$$1 - \boldsymbol{x}^2\boldsymbol{\gamma}(\boldsymbol{x}, \tilde{\boldsymbol{\delta}})\tilde{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{\delta}) > 0 \tag{3.12}$$

where

$$\begin{aligned} \boldsymbol{\gamma}(\boldsymbol{x}, \tilde{\delta}) &= c \int \frac{t^2}{\boldsymbol{x}^2(1 + \tilde{\delta}t)^2} \nu(dt) \\ \tilde{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{\delta}) &= \int \frac{t^2}{\boldsymbol{x}^2(1 + \boldsymbol{\delta}t)^2} \tilde{\nu}(dt). \end{aligned}$$

Then $\boldsymbol{x} \notin \operatorname{supp}(\mu)$.

Proof. Let $(\tilde{\delta}, \boldsymbol{x})$ be a solution of Equations (3.11) such that $\tilde{\delta} \in \widetilde{\mathcal{D}}$, $\boldsymbol{\delta} \in \mathcal{D}$, and (3.12) is satisfied. Define on a small enough open neighborhood of $(\tilde{\delta}, \boldsymbol{x})$ in $\mathbb{R}^2$ the function

$$\boldsymbol{F}(\tilde{\delta}, x) = \int \frac{t}{-x + ct \displaystyle\int \frac{u}{1+u\tilde{\delta}} \nu(du)} \tilde{\nu}(dt) - \tilde{\delta}. \tag{3.13}$$

Clearly, $\boldsymbol{F}(\tilde{\delta}, \boldsymbol{x}) = 0$, and a short calculus reveals that

$$\frac{\partial \boldsymbol{F}}{\partial \tilde{\delta}}(\tilde{\delta}, \boldsymbol{x}) = -1 + \boldsymbol{x}^2 \boldsymbol{\gamma}(\boldsymbol{x}, \tilde{\boldsymbol{\delta}}) \tilde{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{\delta}) < 0$$

(in this calculus, integration and differentiation can be exchanged since $\tilde{\delta} \in \widetilde{\mathcal{D}}$ and $\boldsymbol{\delta} \in \mathcal{D}$). By the implicit function theorem, there is a real function $\underline{\tilde{\delta}}(x)$ defined on a real neighborhood V of $\boldsymbol{x}$ such that $\underline{\tilde{\delta}}(\boldsymbol{x}) = \tilde{\delta}$ and every couple $(x, \underline{\tilde{\delta}}(x))$ for $x \in V$ satisfies the assumptions of the statement of the proposition. To establish the proposition, it will be enough to show that for any $x \in V$, $\underline{\tilde{\delta}}(x) = \lim_{z \in \mathbb{C}_+ \to x} \tilde{\delta}(z)$.

Fix $x \in V$. For $z \in \mathbb{C}_+$, let

$$A(z) = \int \frac{t}{xz(1+\delta(z)t)(1+\underline{\delta}(x)t)} \, \tilde{\nu}(dt).$$

By the Cauchy–Schwarz inequality, Lemma 3.2 and the fact that $\boldsymbol{\delta} \in \mathcal{D}$, $|A(z)|$ remains bounded as $z \to x$. Let $(\delta(x), \tilde{\delta}(x))$ be the limit of $(\delta(z), \tilde{\delta}(z))$ as $z \in \mathbb{C}_+ \to x$. Repeating the derivations made in the proof of Lemma 3.3, using the fact that $|A(z)|$ is bounded, and letting $z \to x$, we obtain that $(1 - x^2 \boldsymbol{\Gamma} \widetilde{\boldsymbol{\Gamma}})(\tilde{\delta}(x) - \underline{\tilde{\delta}}(x)) = 0$ where

$$\boldsymbol{\Gamma} = c \int \frac{t^2}{x^2(1+\tilde{\delta}(x)t)(1+\underline{\tilde{\delta}}(x)t)} \nu(dt)$$
$$\widetilde{\boldsymbol{\Gamma}} = \int \frac{t^2}{x^2(1+\delta(x)t)(1+\underline{\delta}(x)t)} \tilde{\nu}(dt)$$

and $\underline{\delta}(x) = -cx^{-1} \int t(1+\underline{\tilde{\delta}}(x)t)^{-1} \nu(dt)$. As in the proof of Lemma 3.3, we show that $1 - x^2 \boldsymbol{\Gamma} \widetilde{\boldsymbol{\Gamma}} > 0$, resulting in $\tilde{\delta}(x) = \underline{\tilde{\delta}}(x)$. □

Proposition 3.3.1 shows that for any $\boldsymbol{x} \in \operatorname{supp}(\mu)^c \cap \mathbb{R}_*$, there exists a couple $(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})$ that satisfies the assumptions of Proposition 3.3.2. The reverse is shown by Proposition 3.3.2.

These observations suggest a practical procedure for determining the support of μ. We let $\tilde{\delta}$ run through $\widetilde{\mathcal{D}}$. For every one of these $\tilde{\delta}$, we compute

$$\psi(\tilde{\delta}) = c \int \frac{t}{1+\tilde{\delta}t} \nu(dt)$$

then we find numerically the solutions of the equation in $\boldsymbol{x}$

$$\tilde{\delta} = \int \frac{t}{-\boldsymbol{x} + \psi(\tilde{\delta})t} \tilde{\nu}(dt)$$

for which $-\boldsymbol{x}^{-1}\psi(\tilde{\delta}) \in \mathcal{D}$. Among these solutions, we retain those points $\boldsymbol{x}$ for which

$$1 - c\int \frac{t^2}{(1+\tilde{\delta}t)^2}\nu(dt)\int \frac{t^2}{(\boldsymbol{x}-\psi(\tilde{\delta})t)^2}\tilde{\nu}(dt) > 0.$$

What is left after making $\tilde{\delta}$ run through $\widetilde{\mathcal{D}}$ is $\mathrm{supp}(\mu)\cap\mathbb{R}_*$. Figure 3.1 gives an idea of the result.

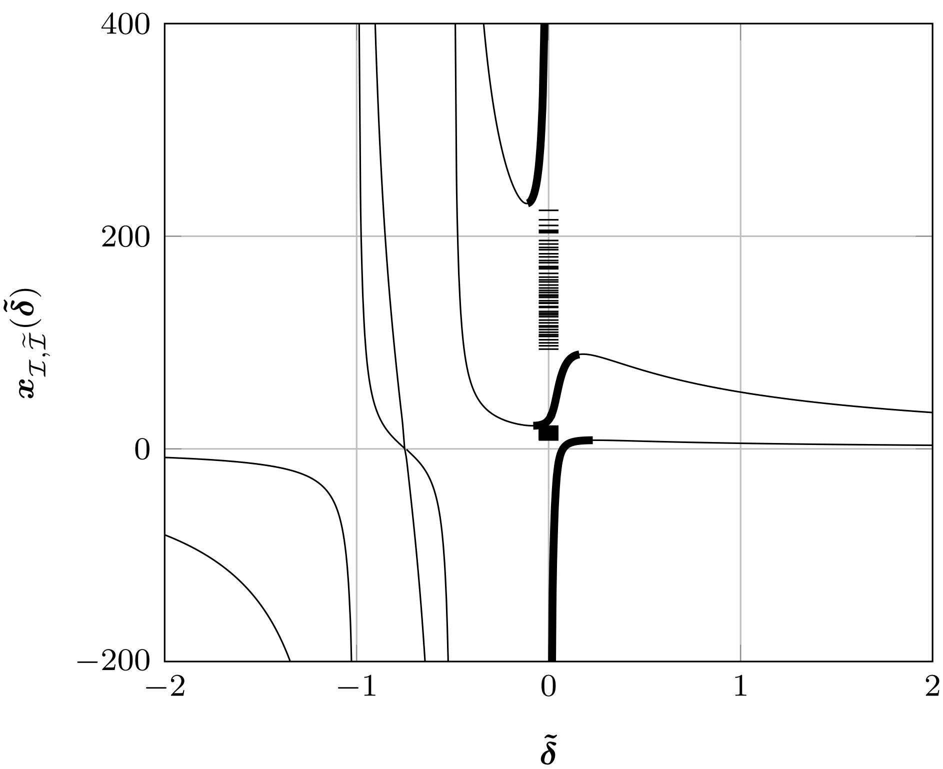


Figure 3.1: $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ for each component pairs $\mathcal{I}$ of $\mathcal{D}$ and $\widetilde{\mathcal{I}}$ of $\widetilde{\mathcal{D}}$. In thick line, positions for which $1 - \boldsymbol{x}^2\boldsymbol{\gamma}(\boldsymbol{x},\tilde{\delta})\tilde{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{\delta}) > 0$. On the vertical axis, in black dashes, empirical eigenvalue positions for $N = 1000$. Setting: $c = 10$, $\nu = 1/2(\delta_1+\delta_2)$, $\tilde{\nu} = 1/2(\delta_1+\delta_{10})$.

The two following propositions will help us bring out some of the properties of the graph of $\boldsymbol{x}$ versus $\tilde{\delta}$. In their statements, we assume that the triples $(\tilde{\delta}_1, \boldsymbol{\delta}_1, \boldsymbol{x}_1)$ and $(\tilde{\delta}_2, \boldsymbol{\delta}_2, \boldsymbol{x}_2)$ satisfy both the statement of Proposition 3.3.2.

Lemma 3.5. *$\tilde{\delta}_1 \neq \tilde{\delta}_2 \Rightarrow \boldsymbol{x}_1 \neq \boldsymbol{x}_2$ and $\boldsymbol{\delta}_1 \neq \boldsymbol{\delta}_2 \Rightarrow \boldsymbol{x}_1 \neq \boldsymbol{x}_2$.*

Proof. We know that $\tilde{\delta}_i = \lim_{z\in\mathbb{C}_+\to\boldsymbol{x}_i}\tilde{\delta}(z)$ for $i = 1, 2$. Assume that $\tilde{\delta}_1 \neq \tilde{\delta}_2$. Then having $\boldsymbol{x}_1 = \boldsymbol{x}_2$ would violate this convergence. ☐

Lemma 3.6. *If $\tilde{\delta}_1 < \tilde{\delta}_2$, if $\boldsymbol{x}_1\boldsymbol{x}_2 > 0$, and if $[\boldsymbol{\delta}_1 \wedge \boldsymbol{\delta}_2, \boldsymbol{\delta}_1 \vee \boldsymbol{\delta}_2] \subset \mathcal{D}$, then $\boldsymbol{x}_1 < \boldsymbol{x}_2$.*

Proof. We use the identity

$$\Big(1 - \boldsymbol{x}_1\boldsymbol{x}_2\gamma(\boldsymbol{x}_1,\boldsymbol{x}_2)\tilde{\gamma}(\boldsymbol{x}_1,\boldsymbol{x}_2)\Big)(\tilde{\delta}_1 - \tilde{\delta}_2)$$
$$= (\boldsymbol{x}_1 - \boldsymbol{x}_2)\int \frac{t}{\boldsymbol{x}_1\boldsymbol{x}_2(1+\boldsymbol{\delta}_1 t)(1+\boldsymbol{\delta}_2 t)}\,\tilde{\nu}(dt)$$

(see (3.9)). By the Cauchy–Schwarz inequality, $1 - \boldsymbol{x}_1\boldsymbol{x}_2\gamma(\boldsymbol{x}_1,\boldsymbol{x}_2)\tilde{\gamma}(\boldsymbol{x}_1,\boldsymbol{x}_2) > 0$.

Let us show that the integral I at the right hand side of the equation above is positive. Assume that for some $t \in \mathrm{supp}(\tilde{\nu})$, the numbers $1 + \boldsymbol{\delta}_1 t$ and $1 + \boldsymbol{\delta}_2 t$ do not have the same sign. Then there exists $\boldsymbol{\delta} \in (\boldsymbol{\delta}_1 \wedge \boldsymbol{\delta}_2, \boldsymbol{\delta}_1 \vee \boldsymbol{\delta}_2)$ such that $1 + \boldsymbol{\delta} t = 0$. But this contradicts $[\boldsymbol{\delta}_1 \wedge \boldsymbol{\delta}_2, \boldsymbol{\delta}_1 \vee \boldsymbol{\delta}_2] \subset \mathcal{D}$. Hence $I > 0$, which shows that $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\tilde{\delta}_1 - \tilde{\delta}_2$ have the same sign. □

In order to better understand the incidence of these propositions, let us describe more formally the procedure for determining the support of Proposiation 3.3.2. Equations (3.11) can be rewritten as $-\boldsymbol{x}\boldsymbol{\delta}\tilde{\delta} = g(\tilde{\delta}) = \tilde{g}(\boldsymbol{\delta})$ where

$$g(\tilde{\delta}) = c\int \frac{\tilde{\delta}t}{1+\tilde{\delta}t}\nu(dt) \quad \text{and} \quad \tilde{g}(\boldsymbol{\delta}) = \int \frac{\boldsymbol{\delta}t}{1+\boldsymbol{\delta}t}\tilde{\nu}(dt)$$

are both increasing on any interval of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ respectively. Let $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ be two connected components of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ respectively[2]. Assume that $\tilde{g}(\mathcal{I}) \cap g(\widetilde{\mathcal{I}}) \neq \emptyset$. Since $\tilde{g}$ is increasing, it has a local inverse $\tilde{g}^{-1}_{\mathcal{I},\widetilde{\mathcal{I}}}$ on $g(\widetilde{\mathcal{I}})$. Let $\boldsymbol{\delta} = \tilde{g}^{-1}_{\mathcal{I},\widetilde{\mathcal{I}}} \circ g(\tilde{\delta})$ and consider the function

$$\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) = -\frac{g(\tilde{\delta})}{\boldsymbol{\delta}\tilde{\delta}} = -\frac{g(\tilde{\delta})}{\tilde{\delta} \times \tilde{g}^{-1}_{\mathcal{I},\widetilde{\mathcal{I}}} \circ g(\tilde{\delta})}, \tag{3.14}$$

with domain the open set $\mathrm{dom}(\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}) = \{\tilde{\delta} \in \widetilde{\mathcal{I}} : \exists \boldsymbol{\delta} \in \mathcal{I} \text{ such that } \tilde{g}(\boldsymbol{\delta}) = g(\tilde{\delta}) \text{ and } \boldsymbol{\delta} \neq 0\}$. Computing $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ on all connected components $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ and dropping the values of $\boldsymbol{x}$ for which $1 - \boldsymbol{x}^2\boldsymbol{\gamma}(\boldsymbol{x},\tilde{\boldsymbol{\delta}})\tilde{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{\delta}) > 0$, we are of course left with $\mathrm{supp}(\mu) \cap \mathbb{R}_*$.

Thanks to Lemmas 3.5–3.6, the functions $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ have the following properties:

1. For any $\boldsymbol{x}_0 \in \mathbb{R}_*$, at most one function $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ satisfies $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) = \boldsymbol{x}_0$ and $\boldsymbol{x}'_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) > 0$ by Lemma 3.5. Note that more than one function $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ can be possibly increasing at a given $\tilde{\delta} \in \widetilde{\mathcal{D}}$, as the figure shows.

2. We show below that there is exactly one couple $(\mathcal{I},\widetilde{\mathcal{I}})$ for which $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ has negative values and is increasing from $-\infty$ to zero where it is negative. Moreover, for any couple $(\mathcal{I},\widetilde{\mathcal{I}})$ and for any $[\tilde{\delta}_1,\tilde{\delta}_2] \in \widetilde{\mathcal{I}}$ such that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_i) > 0$ and $\boldsymbol{x}'_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_i) > 0$, $i = 1, 2$, the function $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ never decreases between $\tilde{\delta}_1$ and $\tilde{\delta}_2$ by Lemma 3.6.

   In summary, if a branch of a $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ is increasing at two points $\tilde{\delta}_1$ and $\tilde{\delta}_2$, then it never decreases between these two points.

[2] To give an example, assume that $\mathrm{supp}(\nu)\cap\mathbb{R}_* = [a_1,b_1]\cup[a_2,b_2]\cup\cdots\cup[a_K,b_K]$ where $0 < a_1 \le b_1 < a_2 \le b_2 < \cdots < a_K \le b_K < \infty$. Then the connected components of $\widetilde{\mathcal{D}}$ are $(-\infty,-a_1^{-1}), (-b_1^{-1},-a_2^{-1}), \ldots, (-b_{K-1}^{-1},-a_{K-1}^{-1})$, and $(-b_K^{-1},\infty)$.

3. Let $b = \sup(\mathrm{supp}(\nu)) \in (0, \infty]$ and $\tilde{b} = \sup(\mathrm{supp}(\tilde{\nu})) \in (0, \infty]$, and let us study the behavior of $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ when $\widetilde{\mathcal{I}} = (-b^{-1}, \infty)$ and $\mathcal{I} = (-\tilde{b}^{-1}, \infty)$. Assume $b = \tilde{b} = \infty$. By the fact that the functions $\delta(x)$ and $\tilde{\delta}(x)$ are both positive and increasing on $(-\infty, 0)$ and by Lemma 3.5, the branch $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ is increasing where it is negative, it is the only branch having this property, and $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) \to -\infty$ as $\tilde{\delta} \downarrow 0$.

   Assume now that $b = \infty$ and $\tilde{b} < \infty$. Here it is easy to notice that $g((-\tilde{b}^{-1}, 0)) \cap \tilde{g}((0, \infty)) = \emptyset$ which implies that we can replace $\mathcal{I}$ with $(0, \infty)$. As in the former case, the graph of $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ consists in one branch that has the same properties as regards the negative values of $\boldsymbol{x}$. The same conclusion holds when $b < \infty$ and $\tilde{b} = \infty$.

   Finally, assume that $b, \tilde{b} < \infty$. Here $g(\tilde{\delta})/\tilde{\delta} \approx C$ and $\boldsymbol{\delta} \approx C'\tilde{\delta}$ near zero, where $C, C' > 0$. Consequently, the graph of $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ consists in two branches, one on $(-b^{-1}, 0)$ and one on $(0, \infty)$. The first branch converges to infinity as $\tilde{\delta} \uparrow 0$, showing that μ is compactly supported, and the second branch behaves below zero as its analogues above. These two branches appear Figure 3.1.

4. Assume that $a = \inf(\mathrm{supp}(\nu) \cap \mathbb{R}_*) > 0$ and let $\widetilde{\mathcal{I}} = (-\infty, -a^{-1})$. Then $g(\tilde{\delta})$ increases from c as $\tilde{\delta}$ increases from $-\infty$. If $\boldsymbol{\delta} < 0$, then $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) < 0$ since $g(\tilde{\delta})/\tilde{\delta} < 0$, and the conclusions of Item (3) show that the branches $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ need not be considered for determining $\mathrm{supp}(\mu)$ when $\mathcal{I} \subset (-\infty, 0)$. It remains to study $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ for $\mathcal{I} = (-\tilde{b}^{-1}, \infty)$. On $(0, \infty)$, the function $\tilde{g}(\boldsymbol{\delta})$ increases from 0 to 1, hence $\tilde{g}((0, \infty)) \cap g(\widetilde{\mathcal{I}}) \neq \emptyset$ if and only if $c < 1$. In that case, it can be checked that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ increases from 0 as $\tilde{\delta}$ increases from $-\infty$. In conclusion, if $a > 0$ and $c < 1$, then $\inf(\mathrm{supp}(\mu) \cap \mathbb{R}_*) > 0$, and the location of this infimum is provided by the branch $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$.

   Similarly, if $\tilde{a} = \inf(\mathrm{supp}(\tilde{\nu}) \cap \mathbb{R}_*) > 0$, $\mathcal{I} = (-\infty, -\tilde{a}^{-1})$ and $\widetilde{\mathcal{I}} \subset (-\infty, 0)$, then the branches $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ need not be considered. If in addition $c > 1$, then $\inf(\mathrm{supp}(\mu) \cap \mathbb{R}_*) > 0$, and the location of this infimum is provided by the branch $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ for $\mathcal{I} = (-\infty, -\tilde{a}^{-1})$ and $\widetilde{\mathcal{I}} = (-b^{-1}, \infty)$.

We terminate this paragraph with the following two results.

Proposition 3.3.3. *Assume that* $\mathrm{supp}(\nu) \cap \mathbb{R}_*$ *and* $\mathrm{supp}(\tilde{\nu}) \cap \mathbb{R}_*$ *consist in* K *and* $\widetilde{K}$ *connected components respectively. Then* $\mathrm{supp}(\mu) \cap \mathbb{R}_*$ *consists in at most* $K\widetilde{K}$ *connected components.*

Proof. When ν is compactly supported, $\mathrm{supp}(\nu - \nu(\{0\})\boldsymbol{\delta}_0) = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_K, b_K]$ where $0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_K \leq b_K < \infty$ or $0 = a_1 < b_1 < a_2 \leq b_2 < \cdots < a_K \leq b_K < \infty$. In the first case, the connected components of $\widetilde{\mathcal{D}}$ are $\widetilde{\mathcal{I}}_0 = (-\infty, -a_1^{-1})$, $\widetilde{\mathcal{I}}_1 = (-b_1^{-1}, -a_2^{-1}), \ldots, \widetilde{\mathcal{I}}_K = (-b_K^{-1}, \infty)$. In the second case, these connected components are $\widetilde{\mathcal{I}}_1, \ldots, \widetilde{\mathcal{I}}_K$. If ν is not compactly supported, $a_K < b_K = \infty$ and the expressions of the connected components of $\widetilde{\mathcal{D}}$ are unchanged. With similar notations, the connected components of $\mathcal{D}$ are $\mathcal{I}_0, \ldots, \mathcal{I}_{\widetilde{K}}$ or $\mathcal{I}_1, \ldots, \mathcal{I}_{\widetilde{K}}$ according to whether $\inf(\mathrm{supp}(\tilde{\nu}) \cap \mathbb{R}_*)$ is positive or not. Let $s = \inf(\mathrm{supp}(\mu) \cap \mathbb{R}_*)$ and $S = \sup(\mathrm{supp}(\mu))$. Following the observations we just made, we notice that the only possible $\boldsymbol{x}_{\mathcal{I}_k,\widetilde{\mathcal{I}}_{\tilde{k}}}(\tilde{\delta}) \in (s, S)$ such that $\boldsymbol{x}'_{\mathcal{I}_k,\widetilde{\mathcal{I}}_{\tilde{k}}}(\tilde{\delta}) > 0$ are those for which $1 \leq k \leq K$,

$1 \leq \tilde{k} \leq \widetilde{K}$, and $(k, \tilde{k}) \neq (K, \widetilde{K})$. Therefore, the number of intervals of $\text{supp}(\mu)^c \cap (s, S)$ is upper bounded by $K\widetilde{K} - 1$, hence the result. ☐

Proposition 3.3.4. $\text{supp}(\mu)$ *is compact if and only if* $\text{supp}(\nu)$ *and* $\text{supp}(\tilde{\nu})$ *are compact.*

Proof. The "if" part has been shown by Item (3) above. Assume $\text{supp}(\mu)$ is compact. The fact that $\text{supp}(\rho) \cap \mathbb{R}_* = \text{supp}(\tilde{\rho}) \cap \mathbb{R}_* = \text{supp}(\mu) \cap \mathbb{R}_*$ and the equation $m_{\tilde{\nu}}(-\delta(z)^{-1}) = \delta(z) + z\delta^2(z)\tilde{\delta}(z)$ show that $m_{\tilde{\nu}}(z)$ can be analytically extended to (A, ∞) for A large enough, hence the compactness of $\text{supp}(\tilde{\nu})$. A similar conclusion holds for $\text{supp}(\nu)$. ☐

3.3.3 Properties of the density of μ on $\mathbb{R}_*$

Theorem 3.3.2. *The density $f(x)$ specified in the statement of Theorem 3.3.1 is analytic for every $x \neq 0$ for which $f(x) > 0$.*

Proof. As in the proof of Lemma 3.3, we assume that ν is not a Dirac measure. Let $x_0 \neq 0$ be such that $f(x_0) > 0$. We start by showing that $\tilde{\delta}(z)$ can be analytically extended from $\mathbb{C}_+$ to a neighborhood of x_0 in $\mathbb{C}$. Write

$$\gamma(x_0, x_0) = \lim_{z \in \mathbb{C}_+ \to x_0} \gamma(z, z), \quad \tilde{\gamma}(x_0, x_0) = \lim_{z \in \mathbb{C}_+ \to x_0} \tilde{\gamma}(z, z),$$
$$\Gamma(x_0, x_0) = \lim_{z \in \mathbb{C}_+ \to x_0} \gamma(z, z^*), \quad \widetilde{\Gamma}(x_0, x_0) = \lim_{z \in \mathbb{C}_+ \to x_0} \tilde{\gamma}(z, z^*).$$

Making $z \in \mathbb{C}_+$ converge to x_0 in (3.10) and recalling that the integral at the right hand side of this equation remains bounded and that $\Im\tilde{\delta}(x_0) > 0$, we get that $x_0^2\Gamma(x_0, x_0)\widetilde{\Gamma}(x_0, x_0) = 1$. Any integrable random variable X satisfies $|\mathrm{E}X| \leq \mathrm{E}|X|$, the equality being achieved if and only if $X = \theta|X|$ almost everywhere, where θ is a modulus one constant. Consequently, $|\gamma(x_0, x_0)| < \Gamma(x_0, x_0)$ since ν is not a Dirac measure, and $|\tilde{\gamma}(x_0, x_0)| \leq \widetilde{\Gamma}(x_0, x_0)$. Therefore, $|x_0^2\gamma(x_0, x_0)\tilde{\gamma}(x_0, x_0)| < 1$. Now, since $\Im\tilde{\delta}(x_0) > 0$, it is easy to see by inspecting Equation (3.6) that the function $F(\tilde{\delta}, z)$ which is holomorphic on $\mathbb{C}_+^2$ can be analytically extended to a neighborhood of $(\tilde{\delta}(x_0), x_0)$ in $\mathbb{C}_+ \times \mathbb{C}_*$ where $\mathbb{C}_* = \mathbb{C} - \{0\}$. Observing that

$$\frac{\partial F}{\partial \tilde{\delta}}(\tilde{\delta}(x_0), x_0) = -1 + x_0^2\gamma(x_0, x_0)\tilde{\gamma}(x_0, x_0) \neq 0$$

and invoking the holomorphic implicit function theorem, we get that there exists a neighborhood $V \subset \mathbb{C}_*$ of x_0, a neighborhood $V' \subset \mathbb{C}_+$ of $\tilde{\delta}(x_0)$ and a holomorphic function $\underline{\tilde{\delta}} : V \to V'$ such that

$$\{(z, \tilde{\delta}) \in V \times V' \,:\, F(\tilde{\delta}, z) = 0\} = \{(z, \underline{\tilde{\delta}}(z)) \,:\, z \in V\}.$$

Since $\tilde{\delta}(z)$ and $\underline{\tilde{\delta}}(z)$ coincide on $V \cap \mathbb{C}_+$, the function $\underline{\tilde{\delta}}(z)$ is an analytic extension of $\tilde{\delta}(z)$ on V.

This result shows in conjunction with (3.3) that $m(z)$ can be extended analytically to V. Therefore, writing $m(z) = \sum_{\ell \geq 0} a_\ell(z - x_0)^\ell$ we get that $f(x) = \pi^{-1}\sum_{\ell \geq 0} \Im a_\ell\,(x - x_0)^\ell$ near x_0. ☐

We now study the behavior of the density $f(x)$ near a boundary point $a > 0$ of $\text{supp}(\mu)$. The observations made above show that when a is a left end point (resp. a right end point) of $\text{supp}(\mu)$, it is a local supremum (resp. a local infimum) of one of the functions $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$. Parallelling the assumptions made in (Marčenko and Pastur, 1967), (Silverstein and Choi, 1995), and (Dozier and Silverstein, 2007), we restrict ourselves to the case where $a = \boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a)$ for some $\tilde{\delta}_a \in \text{dom}(\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}})$. In that case, $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ is of course analytical around $\tilde{\delta}_a$ and $\boldsymbol{x}'_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a) = 0$. Note that this assumption might not be satisfied for some choices of the measures ν and $\tilde{\nu}$. Assuming $a > 0$ is a left end point of $\text{supp}(\mu)$, it is for instance possible that the function $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ increases to a as $\tilde{\delta} \uparrow \tilde{\delta}_a$ with $-\tilde{\delta}_a^{-1} \in \partial\nu$. We however note that our assumption is valid when the measures ν and $\tilde{\nu}$ are both discrete.

Theorem 3.3.3. *Let $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ be two connected components of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ respectively, and assume that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ reaches a maximum at a point $\tilde{\delta}_a \in \text{dom}(\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}})$. Then $\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a) < 0$. Furthermore, for $\varepsilon > 0$ small enough, $f(x) = H(\sqrt{x-a})$ on $(a, a+\varepsilon)$ where $H(x)$ is a real analytical function near zero, $H(0) = 0$, and*

$$H'(0) = \frac{1}{\pi a}\sqrt{\frac{-2}{\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a)}}\int \frac{t}{(1+\tilde{\delta}_a t)^2}\nu(dt).$$

Assume now that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ reaches a minimum at a point $\tilde{\delta}_a \in \text{dom}(\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}})$. Then $\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a) > 0$. Furthermore, for $\varepsilon > 0$ small enough, $f(x) = H(\sqrt{a-x})$ on $(a-\varepsilon, a)$ where $H(x)$ is a real analytical function near zero, $H(0) = 0$, and

$$H'(0) = \frac{1}{\pi a}\sqrt{\frac{2}{\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a)}}\int \frac{t}{(1+\tilde{\delta}_a t)^2}\nu(dt).$$

To prove the theorem, we start with the following lemma which is proven in Section 3.4.

Lemma 3.7. *Assume that either ν or $\tilde{\nu}$ is not a Dirac measure. Let $(\tilde{\delta}_a, a)$ with $a \neq 0$ satisfy*

$$\boldsymbol{F}(\tilde{\delta}_a, a) = 0, \quad \frac{\partial \boldsymbol{F}}{\partial \tilde{\delta}}(\tilde{\delta}_a, a) = 0$$

where the function $\boldsymbol{F}(\tilde{\delta}, \boldsymbol{x})$ is defined by (3.13)*. Then*

$$\frac{\partial^2 \boldsymbol{F}}{\partial \tilde{\delta}^2}(\tilde{\delta}_a, a) = 0 \;\Rightarrow\; \frac{\partial^3 \boldsymbol{F}}{\partial \tilde{\delta}^3}(\tilde{\delta}_a, a) \neq 0.$$

Proof of Theorem 3.3.3. We follow the argument of (Marčenko and Pastur, 1967). We first assume that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}$ reaches a maximum at $\tilde{\delta}_a \in \widetilde{\mathcal{I}}$ and prove that $\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}_a) < 0$. Observe that $\boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})$ satisfies $\boldsymbol{F}(\tilde{\delta}, \boldsymbol{x}_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta})) = 0$, and that $\partial\boldsymbol{F}/\partial\boldsymbol{x} = \int t(\boldsymbol{x}(1+\boldsymbol{\delta}t))^{-2}\tilde{\nu}(dt) > 0$. By the chain rule for differentiation,

$$\begin{aligned}
0 &= \frac{\partial \boldsymbol{F}}{\partial \tilde{\delta}} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\boldsymbol{x}'_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) \\
0 &= \frac{\partial^2 \boldsymbol{F}}{\partial \tilde{\delta}^2} + \left(\frac{\partial^2 \boldsymbol{F}}{\partial \boldsymbol{x}^2} + 2\frac{\partial^2 \boldsymbol{F}}{\partial \tilde{\delta}\partial \boldsymbol{x}}\right)\boldsymbol{x}'_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}) + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\boldsymbol{x}''_{\mathcal{I},\widetilde{\mathcal{I}}}(\tilde{\delta}).
\end{aligned}$$

If we assume that $\boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a) = 0$, then $(\partial^2 \boldsymbol{F}/\partial\tilde{\delta}^2)(\tilde{\delta}_a, a) = 0$ and it is furthermore easy to check that

$$\boldsymbol{x}^{(3)}(\tilde{\delta}_a) = -\frac{\partial^3 \boldsymbol{F}/\partial\tilde{\delta}^3}{\partial \boldsymbol{F}/\partial \boldsymbol{x}}(\tilde{\delta}_a, a).$$

By Lemma 3.7, $\boldsymbol{x}^{(3)}(\tilde{\delta}_a) \neq 0$, but this contradicts the fact that the first non zero derivative of a function at a local extremum is of even order. Hence $\boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a) < 0$.

Equation (3.14) shows that $\boldsymbol{x}_{\mathfrak{I},\tilde{\mathfrak{I}}}$ can be analytically extended to a function $z_{\mathfrak{I},\tilde{\mathfrak{I}}}$ in a neighborhood of $\tilde{\delta}_a$ in the complex plane. Since $\boldsymbol{x}'_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a) = 0$ and $\boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a) < 0$, we can write $z_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}) - a = \varphi(\tilde{\delta})^2$ in this neighborhood where φ is an analytical function satisfying $\varphi(\tilde{\delta}_a) = 0$ and $(\varphi'(\tilde{\delta}_a))^2 = \boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a)/2$. We choose φ such that $\varphi'(\tilde{\delta}_a) = -\imath(-\boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a)/2)^{\frac{1}{2}}$. If we choose $x > a$ such that $x - a$ is small enough, then $z_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}(x)) - a = \varphi(\tilde{\delta}(x))^2$, and moreover $z_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}(x)) = x$. Considering the local inverse Φ of φ in a neighborhood of $\tilde{\delta}_a$, we get that $\tilde{\delta}(x) = \Phi(\sqrt{x-a})$ where the analytic function Φ satisfies $\Phi(0) = \tilde{\delta}_a$ and $\Phi'(0) = 1/\varphi'(\tilde{\delta}_a) = \imath(-2/\boldsymbol{x}''_{\mathfrak{I},\tilde{\mathfrak{I}}}(\tilde{\delta}_a))^{\frac{1}{2}}$ (thus the choice of $\varphi'(\tilde{\delta}_a)$ ensures that $\Im\tilde{\delta}(x) > 0$). Using the equation $\Im m(x) = -x^{-1}\int \Im((1+\tilde{\delta}(x)t)^{-1})\nu(dt)$, we get the result. The case where $\boldsymbol{x}_{\mathfrak{I},\tilde{\mathfrak{I}}}$ reaches a minimum at $\tilde{\delta}_a$ is treated similarly. □

3.4 Proof of Lemma 3.7

First recall that

$$\frac{\partial \boldsymbol{F}}{\partial\tilde{\delta}}(\tilde{\delta}, \boldsymbol{x}) = \boldsymbol{x}^2 \boldsymbol{\gamma}(\boldsymbol{x}, \tilde{\delta})\tilde{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{\delta}) - 1 \tag{3.15}$$

so that $a^2\boldsymbol{\gamma}_a\tilde{\boldsymbol{\gamma}}_a = 1$, with $\boldsymbol{\gamma}_a = \boldsymbol{\gamma}(a, \tilde{\delta}_a)$, $\tilde{\boldsymbol{\gamma}}_a = \tilde{\boldsymbol{\gamma}}(a, \boldsymbol{\delta}_a)$, and

$$\boldsymbol{\delta}_a = c\int \frac{t}{-a(1+\tilde{\delta}_a t)}\nu(dt).$$

Differentiating (3.15), the equation $(\partial^2 \boldsymbol{F}/\partial\tilde{\delta}^2)(\tilde{\delta}_a, a) = 0$ reads

$$\tilde{\boldsymbol{\gamma}}_a c\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt) + a\boldsymbol{\gamma}_a^2 \int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt) = 0 \tag{3.16}$$

where we used

$$\frac{\partial}{\partial\tilde{\delta}}\left(c\int \frac{t}{-\boldsymbol{x}(1+\tilde{\delta}t)}\nu(dt)\right)(\tilde{\delta}_a, a) = a\boldsymbol{\gamma}_a.$$

Assume now that $(\partial^3 \boldsymbol{F}/\partial\tilde{\delta}^3)(\tilde{\delta}_a, a) = 0$. A second differentiation of (3.15) leads then to

$$\begin{aligned} 0 = {} & 2\frac{\boldsymbol{\gamma}_a}{a}c\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt) \\ & + \tilde{\boldsymbol{\gamma}}_a c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) + a^2\boldsymbol{\gamma}_a^3\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt). \end{aligned}$$

Using $a^2\boldsymbol{\gamma}_a\tilde{\boldsymbol{\gamma}}_a = 1$, replace now $\boldsymbol{\gamma}_a/a$ by $1/(a^3\tilde{\boldsymbol{\gamma}}_a)$ in the leftmost term and $a^2\boldsymbol{\gamma}_a^3$ by $\boldsymbol{\gamma}_a^2/\tilde{\boldsymbol{\gamma}}_a$ in the rightmost term. Multiplying the result by $\tilde{\boldsymbol{\gamma}}_a$ leads to

$$\begin{aligned} 0 &= 2\frac{c}{a^3}\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt) \\ &\quad + \tilde{\boldsymbol{\gamma}}_a^2 c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) + \boldsymbol{\gamma}_a^2\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt). \end{aligned} \tag{3.17}$$

We now use (3.16) and $a^2\boldsymbol{\gamma}_a\tilde{\boldsymbol{\gamma}}_a = 1$ to write the two equations:

$$\begin{aligned} 2\frac{c}{a^3}\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt) &= -\frac{2}{a^2}\frac{\boldsymbol{\gamma}_a^2}{\tilde{\boldsymbol{\gamma}}_a}\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt) \\ 2\frac{c}{a^3}\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt) &= -\frac{2c^2}{a^2}\frac{\tilde{\boldsymbol{\gamma}}_a^2}{\boldsymbol{\gamma}_a}\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt). \end{aligned}$$

Replacing the corresponding terms in the leftmost term of (3.17) leads to the two equations

$$\begin{aligned} \frac{2}{a^2}\frac{\boldsymbol{\gamma}_a^2}{\tilde{\boldsymbol{\gamma}}_a}\left(\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\right)^2 - \tilde{\boldsymbol{\gamma}}_a^2 c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) - \boldsymbol{\gamma}_a^2\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt) &= 0 \\ \frac{2}{a^2}\frac{\tilde{\boldsymbol{\gamma}}_a^2}{\boldsymbol{\gamma}_a}\left(c\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt)\right)^2 - \tilde{\boldsymbol{\gamma}}_a^2 c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) - \boldsymbol{\gamma}_a^2\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt) &= 0. \end{aligned}$$

Multiplying each equation by $\boldsymbol{\gamma}_a\tilde{\boldsymbol{\gamma}}_a$ and averaging then gives:

$$\begin{aligned} 0 &= \frac{1}{a^2}\boldsymbol{\gamma}_a^3\left(\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\right)^2 + \frac{1}{a^2}\tilde{\boldsymbol{\gamma}}_a^3\left(c\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt)\right)^2 \\ &\quad - \tilde{\boldsymbol{\gamma}}_a^3\boldsymbol{\gamma}_a c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) - \boldsymbol{\gamma}_a^3\tilde{\boldsymbol{\gamma}}_a\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt). \end{aligned} \tag{3.18}$$

Remark now, by expanding the definition of $\tilde{\gamma}_a$ that

$$\begin{aligned} &\frac{1}{a^2}\boldsymbol{\gamma}_a^3\left(\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\right)^2 - \boldsymbol{\gamma}_a^3\tilde{\boldsymbol{\gamma}}_a\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt) \\ &= \frac{\boldsymbol{\gamma}_a^3}{a^2}\left[\left(\int \frac{t^3}{(1+\boldsymbol{\delta}_a t)^3}\tilde{\nu}(dt)\right)^2 - \int \frac{t^2}{(1+\boldsymbol{\delta}_a t)^2}\tilde{\nu}(dt)\int \frac{t^4}{(1+\boldsymbol{\delta}_a t)^4}\tilde{\nu}(dt)\right] \\ &\le 0 \end{aligned}$$

with the inequality arising from Cauchy–Schwarz. The case of equality holds only if $\tilde{\nu}$ is a Dirac measure. Similarly,

$$\begin{aligned} &\frac{1}{a^2}\tilde{\boldsymbol{\gamma}}_a^3\left(c\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt)\right)^2 - \tilde{\boldsymbol{\gamma}}_a^3\boldsymbol{\gamma}_a c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt) \\ &= \frac{\tilde{\boldsymbol{\gamma}}_a^3}{a^2}\left[\left(c\int \frac{t^3}{(1+\tilde{\delta}_a t)^3}\nu(dt)\right)^2 - \left(c\int \frac{t^2}{(1+\tilde{\delta}_a t)^2}\nu(dt)\right)\left(c\int \frac{t^4}{(1+\tilde{\delta}_a t)^4}\nu(dt)\right)\right] \\ &\le 0 \end{aligned}$$

with equality only if ν is a Dirac measure. Therefore, to ensure (3.18), both ν and $\tilde{\nu}$ must be Dirac measures, which goes against the hypothesis.

Chapter 4

Robust estimates of scatter for elliptical data

In the first section of this chapter, we explore the random matrix asymptotics of Maronna's robust estimator of scatter introduced succinctly in Chapter 2 as the unique solution of (2.2), which we will apply to improve array processing subspace methods in the second section. The assumed model for the observations $x_1, \ldots, x_n \in \mathbb{C}^N$ from now on until Chapter 6 is that of i.i.d. data following a slightly generalized version of the elliptical distribution.

It is to be noted that the technical approach to achieve our main result, Theorem 4.1.2, for Maronna's estimator can and will be extended to robust shrinkage estimators in Chapter 5 but cannot be adapted to Tyler's estimator $\hat{C}_N(0)$ defined as the (unique up to a scale) solution to (2.3). Since the publication of our first articles, the problem has been solved using a different technique in (Zhang et al., 2014). Their approach is based on defining $\hat{C}_N(0)$ as the argument of the minimum of a potential function whose derivative at the minimum is exactly (2.3). Using concentration inequalities for random matrices, they show that this minimum has a decreasingly low probability of falling away from some random equivalent $\hat{S}_N(0)$. Our own approach in the present section rather relies on exploiting the properties of Maronna's functions u and ϕ to contain $\hat{C}_N$ between two asymptotically tight bounds (in the Hermitian ordering sense) related to $\hat{S}_N$ that are amenable to random matrix analysis.

4.1 Theory

In this section, we provide the most fundamental theoretical results of this report, with technical details and some basic lemmas moved to the Appendix A.

Before discussing our main results, we first introduce the notations and assumptions taken in this chapter, some of which being valid for most of the report. We let $x_1, \ldots, x_n \in \mathbb{C}^N$ be n random vectors defined by $x_i = \sqrt{\tau_i} A_N w_i$, where $\tau_1, \ldots, \tau_n \in \mathbb{R}_+$ and $w_1, \ldots, w_n \in \mathbb{C}^{\bar{N}}$ are random and $A_N \in \mathbb{C}^{N \times \bar{N}}$ is deterministic. We denote $c_N \triangleq N/n$ and $\bar{c}_N \triangleq \bar{N}/N$ and shall consider the following growth regime.

Assumption 4.1. *For each N, $c_N < 1$, $\bar{c}_N \geq 1$ and*

$$c_- < \liminf_n c_N \leq \limsup_n c_N < c_+$$

where $0 < c_- < c_+ < 1$.

The robust estimator under consideration in this section is Maronna's M-estimator $\hat{C}_N$ defined, when it exists, as a (possibly unique) solution to the equation in $Z \in \mathbb{C}^{N\times N}$

$$Z = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{1}{N}x_i^* Z^{-1} x_i\right) x_i x_i^* \tag{4.1}$$

where u satisfies the following properties:

(i) $u : [0,\infty) \to (0,\infty)$ is nonnegative continuous and non-increasing

(ii) $\phi : x \mapsto xu(x)$ is increasing and bounded with $\lim_{x\to\infty} \phi(x) \triangleq \phi_\infty > 1$

(iii) $\phi_\infty < c_+^{-1}$.

Note that (ii) is stronger than Maronna's original assumption (Maronna, 1976, Condition (C) p. 53) as ϕ cannot be constant on any open interval. The assumption (iii) is also not classical in robust estimation but obviously compliant with the large n assumption made in classical works (for which $c_+ = 0$). The importance of both assumptions will appear clearly in the proof of the main results.

The statistical hypotheses on $x_1, \ldots, x_n$ are detailed below.

Assumption 4.2. *The vectors $x_i = \sqrt{\tau_i} A_N w_i$, $i \in \{1, \ldots, n\}$, satisfy the following hypotheses:*

1. *the (random) empirical measure $\tilde{\nu}_n = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\delta}_{\tau_i}$ satisfies $\int \tau \tilde{\nu}_n(d\tau) \xrightarrow{\text{a.s.}} 1$*
2. *there exist $\varepsilon < 1 - \phi_\infty^{-1} < 1 - c_+$ and $m > 0$ such that, for all large n a.s. $\tilde{\nu}_n([0,m)) < \varepsilon$*
3. *defining $C_N \triangleq A_N A_N^*$, $C_N \succ 0$ and $\limsup_N \|C_N\| < \infty$*
4. *$w_1, \ldots, w_n \in \mathbb{C}^{\bar{N}}$ are independent unitarily invariant complex (or orthogonally invariant real) zero-mean vectors with, for each i, $\|w_i\|^2 = \bar{N}$, and are independent of $\tau_1, \ldots, \tau_n$.*

Item 1 is merely a normalization condition which, along with Item 3, ensures the proper scaling and asymptotic boundedness of the model parameters. Note in particular that Item 1 ensures a.s. tightness of $\{\tilde{\nu}_n\}_{n=1}^{\infty}$, i.e., for each $\varepsilon > 0$, there exists $M > 0$ such that, with probability one, $\tilde{\nu}_n([M,\infty)) < \varepsilon$ for all n. Item 2 mainly ensures that no heavy mass of τ_i concentrates close to zero; this will ensure that $\hat{C}_N$ does not have too many eigenvalues close to zero and thus has a stable asymptotic behavior.

Note that Item 4 could be equivalently stated as $w_i = \sqrt{\bar{N}}\frac{\tilde{w}_i}{\|\tilde{w}_i\|}$ with $\tilde{w}_i \in \mathbb{C}^{\bar{N}}$ standard complex Gaussian (or standard real Gaussian). This remark will be used throughout the proofs of the main results which rely in part on random matrix identities for matrices with independent entries.

All these conditions are met in particular if the τ_i are independent and identically distributed (i.i.d.) with common unit mean distribution $\tilde{\nu}$ (in which case $\int x\tilde{\nu}_n(dx) \xrightarrow{\text{a.s.}} 1$ by the strong law of large numbers) such that $\tilde{\nu}(\{0\}) = 0$. If in addition $N = \bar{N}$, then $x_1, \ldots, x_n$ are i.i.d. zero-mean complex (or real) elliptically distributed with full rank (Ollila et al., 2012, Theorem 3). In particular, if $2N\tau_1$ is chi-squared distributed with $2N$ degrees of freedom, x_1 is complex zero mean Gaussian. If $1/\tau_1$ is chi-squared distributed with arbitrary degrees of freedom, x_1 is instead zero mean complex Student distributed (see (Ollila et al., 2012) for further discussions and recent results on elliptical distributions).

For simplicity of exposition, most of the following, and in particular the proofs of the main results, will assume the case of complex x_i; the results remain however valid in the case of real random variables.

We shall now take a last technical assumption, which we believe is necessary to our main result.

Assumption 4.3. *For each $a > b > 0$, a.s.*

$$\limsup_{t\to\infty} \frac{\limsup_n \tilde{\nu}_n((t,\infty))}{\phi(at) - \phi(bt)} = 0.$$

Assumption 4.3 controls the relative speed of the tail of $\tilde{\nu}_n$ versus the flattening speed of $\phi(x)$ as $x \to \infty$. Practical examples satisfying Assumption 4.3 are:

- There exists $M > 0$ such that, for all n, $\max_{1\leq i\leq n} \tau_i < M$ a.s. In this case, $\tilde{\nu}_n((t,\infty)) = 0$ a.s. for $t > M$ while $\phi(at) - \phi(bt) \neq 0$ since ϕ is increasing.

- For $u(t) = (1+\alpha)/(\alpha+t)$ for some $\alpha > 0$, it is easily seen that it is sufficient that $\limsup_n \tilde{\nu}_n((t,\infty)) = o(1/t)$ a.s. for Assumption 4.3 to hold. In particular, if the τ_i are i.i.d. with distribution $\tilde{\nu}$, $\limsup_n \tilde{\nu}_n((t,\infty)) = \tilde{\nu}((t,\infty))$ a.s. (for all t continuity points of $\tilde{\nu}$) and, by Markov inequality, it suffices that $\int x^{1+\varepsilon}\tilde{\nu}(dx) < \infty$ for some $\varepsilon > 0$.

The main contribution of this section is twofold: we first present a result on existence and uniqueness of $\hat{C}_N$ as a solution to (4.1) (Theorem 4.1.1) and then study the limiting spectral behavior of $\hat{C}_N$ as $N, n \to \infty$ (Theorem 4.1.2). With respect to existence and uniqueness, we recall that for $\bar{N} = N$ (Maronna, 1976, Theorem 1) ensures the existence and uniqueness of a solution to (4.1) under the statistical hypothesis that each N-subset of $x_1, \ldots, x_n$ spans $\mathbb{C}^N$ and that $\phi_\infty > n/(n-N)$. While the first condition is met with probability one for continuous distributions of x_i, the second condition is restrictive under Assumption 4.1 as it imposes $\phi_\infty > 1/(1-c_-)$ which brings a loss in robustness for c_- close to one.[1] Our first result is a probabilistic alternative to (Maronna, 1976, Theorem 1) which states that for all large n,

[1] As commented in (Maronna, 1976), small values of ϕ_∞ induce increased robustness to the expense of accuracy.

a.s.,[2] (4.1) has a unique solution. This result uses the probability conditions on $x_1, \ldots, x_n$ and also uses $\phi_\infty < c_+^{-1}$ which, as opposed to (Maronna, 1976, Theorem 1), enforces more robust estimators. The uniqueness part of the result also imposes that ϕ be strictly increasing, while (Maronna, 1976, Theorem 1) allows $\phi(x) = \phi_\infty$ for all large x.[3]

As for the large dimensional behavior of $\hat{C}_N$, in the fixed N large n regime and for i.i.d. τ_i, it is of the form $\hat{C}_N \xrightarrow{\text{a.s.}} V_N$ where V_N is the unique solution to $V_N = \mathrm{E}[u(\frac{1}{N}x_1^* V_N^{-1} x_1) x_1 x_1^*]$ (Maronna, 1976, Theorem 5). When the x_i are i.i.d. elliptically distributed and u is such that $\hat{C}_N$ is the maximum-likelihood estimator for C_N, then $V_N = C_N$, leading to a consistent estimator for C_N. In the random matrix regime of interest here, we show that $\hat{C}_N$ does not converge in any classical sense to a deterministic matrix but satisfies $\|\hat{C}_n - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ in spectral norm, where $\hat{S}_N$ follows a random matrix model of the separable covariance type studied in the previous chapter. As such, the spectral behavior of $\hat{C}_N$ is easily analyzed from that of $\hat{S}_N$ for N, n large.

In the next subsection, we introduce some new notations that simplify the analysis of $\hat{C}_N$ and provide an insight on the derivation of our main result, Theorem 4.1.2.

4.1.1 Preliminaries

First note from the expression of $\hat{C}_N$ as a (hypothetical) solution to (4.1) that we can assume $C_N = I_N$ by studying $C_N^{-\frac{1}{2}} \hat{C}_N C_N^{-\frac{1}{2}}$ in place of $\hat{C}_N$. Therefore, here and in all major proofs in the section, without generality restriction, we place ourselves under the assumption $C_N = A_N A_N^* = I_N$.

Our objective is to prove that $\hat{C}_N$ is a well behaved solution of (4.1) (for all large n, a.s.) and to study the spectral properties of $\hat{C}_N$ as N, n grow large. However, the structure of dependence between the rank-one matrices $u(\frac{1}{N}x_i^* \hat{C}_N^{-1} x_i) x_i x_i^*$, $i = 1, \ldots, n$, makes the large dimensional analysis of $\hat{C}_N$ via standard random matrix methods impossible (see e.g., (Pastur and Ŝerbina, 2011; Bai and Silverstein, 2009; Anderson et al., 2010)) as these methods fundamentally rely on the independence (or simple dependence) of the structuring rank-one matrices. We propose here to show that, in the large N, n regime, $\hat{C}_N$ behaves similar to a matrix $\hat{S}_N$ whose structure is more standard and easily analyzed through classical random matrix results. For this we first need to rewrite the fundamental equation (4.1) in order to exhibit a sufficiently "weak" dependence structure in the expression of $\hat{C}_N$. This rewriting is performed in Section 4.1.1.1 below. This being done, we then prove that some weakly dependent terms can be well approximated by independent ones in the large N, n regime. Since the final result does not take an insightful form, we provide below in Section 4.1.1.2 a hint on how to obtain it intuitively.

[2] As is common in random matrix theory, the probability space under consideration is that engendered by the growing sequences $\{x_1, \ldots, x_n\}_{n=1}^\infty$, with N, n satisfying Assumption 4.1, so that an event E_n holds true "for all large n, a.s." whenever, with probability one, there exists n_0 for which E_n is true for all $n \geq n_0$, this n_0 possibly depending on the sequence.

[3] We should point out here that a more general proof than Maronna's and to which we were unaware at the time of publication of this work was provided in (Kent and Tyler, 1991) which somewhat encompasses our present result; nonetheless our proof is quite instrumental to the understanding of the structure of $\hat{C}_N$ in the large dimensional regime and therefore has interest in its own.

4.1.1.1 **Rewriting** (4.1)

We need to introduce some new notations that will simplify the coming considerations. Write $x_i = \sqrt{\tau_i} A_N w_i \triangleq \sqrt{\tau_i} z_i$ and recall that $C_N = I_N$ for the moment (in particular, $\|z_i\|$ is of order $\sqrt{N}$ for most z_i). If $\hat{C}_N$ is well-defined, we denote $\hat{C}_{(i)} \triangleq \hat{C}_N - \frac{1}{n} u(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i) x_i x_i^*$.

Remark that $\hat{C}_{(i)}$ depends on x_i only through the terms $u(\frac{1}{N} x_j^* \hat{C}_N^{-1} x_j)$, $j \neq i$, in which the term $\hat{C}_N$ is built on x_i. But since x_i is only one among a growing number n of x_j vectors, this dependence structure looks intuitively "weak". This informal weak dependence between x_i and $\hat{C}_{(i)}$, along with classical random matrix theory considerations, suggests that the quadratic forms $\frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$, $i = 1, \ldots, n$, are all well approximated by $\frac{1}{N} \operatorname{tr} \hat{C}_N^{-1}$ (more precisely, this would roughly be a consequence of Lemma A.3 and Lemma A.2 in the Appendix if z_i and $\hat{C}_{(i)}$ were truly independent).

With this in mind, let us rewrite $\hat{C}_N$ as a function of $\frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$ instead of $\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i$, $i = 1, \ldots, n$. For this, let $Z \in \mathbb{C}^{N \times N}$ be positive definite such that for each i, $Z_{(i)} \triangleq Z - \frac{1}{n} u(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i) \tau_i z_i z_i^*$ is positive definite. Using the identity $(A + \tau z z^*)^{-1} z = A^{-1} z / (1 + \tau z^* A^{-1} z)$ for invertible A, vector z, and positive scalar τ, observe that

$$\frac{1}{N} z_i^* Z^{-1} z_i = \frac{\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i}{1 + \tau_i u\left(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i\right) \frac{1}{n} z_i^* Z_{(i)}^{-1} z_i}.$$

Hence,

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \left(1 - c_N \tau_i u\left(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i\right) \frac{1}{N} z_i^* Z^{-1} z_i\right) = \frac{1}{N} z_i^* Z^{-1} z_i$$

which, by the definition of ϕ, is

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \left(1 - c_N \phi\left(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i\right)\right) = \frac{1}{N} z_i^* Z^{-1} z_i.$$

Using Assumption 4.1 and $\phi_\infty < c_+^{-1}$, taking n large enough to have $\phi(x) \leq \phi_\infty < 1/c_N$, this can be rewritten

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i = \frac{\frac{1}{N} z_i^* Z^{-1} z_i}{1 - c_N \phi\left(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i\right)}. \tag{4.2}$$

Now, since ϕ is increasing, $g : [0, \infty) \to [0, \infty)$, $x \mapsto x / (1 - c_N \phi(x))$ is increasing, nonnegative, and maps $[0, \infty)$ onto $[0, \infty)$. Thus, g is invertible with inverse denoted g^{-1}. In particular, from (4.2),

$$\tau_i \frac{1}{N} z_i^* Z^{-1} z_i = g^{-1}\left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i\right).$$

Call now $v : [0, \infty) \to [0, \infty)$, $x \mapsto u \circ g^{-1}$. Since g is increasing and nonnegative and u is non-increasing, v is non-increasing and positive. Moreover, $\psi : x \mapsto x v(x)$ satisfies:

$$\psi(x) = x u(g^{-1}(x)) = g(g^{-1}(x)) u(g^{-1}(x)) = \frac{\phi(g^{-1}(x))}{1 - c_N \phi(g^{-1}(x))}$$

which is increasing, nonnegative, and has limit $\psi_\infty^N \triangleq \phi_\infty/(1-c_N\phi_\infty)$ as $x\to\infty$. Hence, v and ψ keep the same properties as u and ϕ, respectively.

With these notations, to prove the existence and uniqueness of a solution to (4.1), it is equivalent to prove that the equation in Z

$$Z=\frac{1}{n}\sum_{i=1}^n \tau_i v\left(\tau_i\frac{1}{N}z_i^*Z_{(i)}^{-1}z_i\right)z_iz_i^*$$

has a unique positive definite solution. But for this, it is sufficient to prove the uniqueness of $d_1,\ldots,d_n\geq 0$ satisfying the n equations:

$$d_j=\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}\tau_i v\left(\tau_i d_i\right)z_iz_i^*\right)^{-1}z_j,\ 1\leq j\leq n. \tag{4.3}$$

Indeed, if these d_i are uniquely defined, then so is the matrix

$$\hat{C}_N=\frac{1}{n}\sum_{i=1}^n \tau_i v\left(\tau_i d_i\right)z_iz_i^* \tag{4.4}$$

with $d_i=\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i$, $\hat{C}_{(i)}=\hat{C}_N-\frac{1}{n}u(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i)x_ix_i^*$ (the existence follows from taking the d_i solution to (4.3) and write $\hat{C}_N$ as in (4.4), while uniqueness follows from the fact that (4.4) cannot be written with a different set of d_i from the uniqueness of the solution to (4.3)).

This is the approach that is pursued to prove Theorem 4.1.1, based on the results from (Yates, 1995). Equation (4.4), which is equivalent to (4.1) (with $\hat{C}_N$ in place of Z), will be preferably used in the remainder of the section.

4.1.1.2 Hint on the main result

Assume here that the d_i above are indeed unique for all large n so that $\hat{C}_N$ is well defined. We provide some intuition on the main result.

From the discussion in Section 4.1.1.1, we may expect the terms d_i to be all close to $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}$ for N,n large enough. We may also expect $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}$ to have a deterministic equivalent γ_N, i.e., there should exist a deterministic sequence $\{\gamma_N\}_{N=1}^\infty$ such that $|\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}-\gamma_N|\xrightarrow{\text{a.s.}}0$. Let us say that all this is true. Since $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}$ is the Stieltjes transform $\frac{1}{N}\operatorname{tr}(\hat{C}_N-zI_N)^{-1}$ of the empirical spectral distribution of $\hat{C}_N$ at point $z=0$, and since $\hat{C}_N$ is expected to be close to $\frac{1}{n}\sum_i\tau_iv(\tau_i\gamma_N)z_iz_i^*$ with now $v(\tau_i\gamma_N)$ independent of $z_1,\ldots,z_n$, from (3.4) in the previous chapter (extended to $z=0$), we would expect that one such γ_N be given by (recall that $C_N=I_N$)

$$\gamma_N=\left(\frac{1}{n}\sum_{i=1}^n\frac{\tau_iv(\tau_i\gamma_N)}{1+c_N\tau_iv(\tau_i\gamma_N)\gamma_N}\right)^{-1}$$

if this fixed-point equation makes sense at all. This can be equivalently written as

$$1 = \frac{1}{n}\sum_{i=1}^{n} \frac{\psi(\tau_i\gamma_N)}{1 + c_N\psi(\tau_i\gamma_N)}. \tag{4.5}$$

We in fact prove in Section 4.1.2 that such a positive γ_N is well defined, unique, and satisfies $\max_{1\leq i\leq n} |d_i - \gamma_N| \xrightarrow{\text{a.s.}} 0$ (under correct assumptions). Proving this result is the main technical difficulty of the proof.

This convergence, along with (4.4), will then ensure that for all large n, a.s.

$$\left\|\hat{C}_N - \hat{S}_N\right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N = \frac{1}{n}\sum_{i=1}^{n} v\left(\tau_i\gamma_N\right)\tau_i z_i z_i^*$$

with γ_N the unique positive solution to (4.5). It will then be immediate under Assumption 4.2–3 to see that the result holds true also for $C_N \neq I_N$.

The major interest of this convergence in spectral norm is that $\hat{S}_N$ is a known and easily manipulable object, as opposed to $\hat{C}_N$. The result therefore conveys a lot of information about $\hat{C}_N$ among which the fact that its largest and smallest eigenvalues are almost surely bounded and bounded away from zero for all large n (which is not in general the case of $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$ for τ_i with unbounded support).

4.1.2 Main results

We now make the statements of Section 4.1.1.2 rigorous. The first result ensures the existence and uniqueness of a solution $\hat{C}_N$ to (4.1) for n large enough.

Theorem 4.1.1 (Existence and Uniqueness)**.** *Let Assumptions 4.1 and 4.2 hold, with* $\limsup_N \|C_N\|$ *non necessarily bounded. Then, for all large n a.s.,* (4.1) *has a unique solution $\hat{C}_N$ given by*

$$\hat{C}_N = \lim_{\substack{t\to\infty \\ t\in\mathbb{N}}} Z^{(t)}$$

where $Z^{(0)} \succ 0$ is arbitrary and, for $t \in \mathbb{N}$,

$$Z^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{1}{N}x_i^*\left(Z^{(t)}\right)^{-1}x_i\right)x_i x_i^*.$$

Having defined $\hat{C}_N$, our main result provides a random matrix equivalent to $\hat{C}_N$, much easier to study than $\hat{C}_N$ itself.

Theorem 4.1.2 (Asymptotic Behavior)**.** *Let Assumptions 4.1–4.3 hold, and let $\hat{C}_N$ be given by Theorem 4.1.1 when uniquely defined as the solution of* (4.1) *or chosen arbitrarily if not. Then*

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^{n} v(\tau_i \gamma_N) x_i x_i^*$$

and γ_N is the unique positive solution of the equation in γ

$$1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)}$$

with the functions $v : x \mapsto (u \circ g^{-1})(x)$, $\psi : x \mapsto x v(x)$, and $g : \mathbb{R}_+ \to \mathbb{R}_+,\ x \mapsto x/(1 - c_N \phi(x))$.

The fact that $\hat{C}_N$ is well approximated by $\hat{S}_N$, which follows the separable covariance random matrix model studied in the previous chapter, has important consequences. From a purely mathematical standpoint, this provides a full characterization of the spectral behavior of $\hat{C}_N$ for large N, n (see in particular Corollary 4.1 below). For application purposes, this first enables the performance analysis in the large N, n horizon of standard signal processing methods already relying on $\hat{C}_N$ (these methods were so far analyzed solely in the fixed N large n regime). A second, more important, consequence for signal processing application is the possibility to fully exploit the structure of $\hat{C}_N$ for large N, n to improve existing robust schemes (see next Section 4.2 for an example in array processing). Note importantly here that $\hat{S}_N$ is a matrix of the separable covariance class studied in the previous chapter, the spectrum of which we therefore can completely characterize. However, $\hat{S}_N$ is not a directly observable matrix since γ_N and the τ_i's are not directly readable from the x_i's so that $\hat{S}_N$ has a purely analytical purpose and cannot be used as a substitute for $\hat{C}_N$ in practice.

As a consequence of Theorem 4.1.2 and of the results of Chapter 3, we now have the following corollary.

Corollary 4.1 (Spectrum)**.** *Let Assumptions 4.1–4.3 hold. Then*

$$\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\delta}_{\lambda_i(\hat{C}_N)} - \mu_N \xrightarrow{\text{a.s.}} 0 \tag{4.6}$$

where the convergence is in the weak probability measure sense, with μ_N a probability measure with continuous density and Stieltjes transform $m_N(z)$ given, for $z \in \mathbb{C}_+$, by

$$m_N(z) = -\frac{1}{z} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{1 + \tilde{\delta}_N(z) \lambda_i(C_N)}$$

where $\tilde{\delta}_N(z)$ is the unique solution in $\mathbb{C}_+$ of the equations in $\tilde{\delta}$

$$\tilde{\delta} = -\frac{1}{z}\frac{1}{n}\sum_{i=1}^{n} \frac{\psi(\tau_i\gamma_N)}{\gamma_N + \psi(\tau_i\gamma_N)\delta}$$

$$\delta = -\frac{1}{z}\frac{1}{n}\sum_{i=1}^{N} \frac{\lambda_i(C_N)}{1 + \lambda_i(C_N)\tilde{\delta}}$$

and where γ_N is defined in Theorem 4.1.2. Besides, the support $\mathcal{S}_N$ of μ_N is uniformly bounded. If $C_N = I_N$, $m_N(z)$ is the unique solution in $\mathbb{C}_+$ of the equation in m

$$m = \left(-z + \gamma_N^{-1}\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma_N)}{1 + c\gamma_N^{-1}\psi(\tau_i\gamma_N)m}\right)^{-1}.$$

Also, for each $N_0 \in \mathbb{N}$ and each closed set $\mathcal{A} \subset \mathbb{R}$ with $\mathcal{A} \cap \left(\bigcup_{N \geq N_0} \mathcal{S}_N\right) = \emptyset$,

$$\left|\left\{\lambda_i(\hat{C}_N)\right\}_{i=1}^{N} \cap \mathcal{A}\right| \xrightarrow{\text{a.s.}} 0 \tag{4.7}$$

so that, in particular,

$$\limsup_N \|\hat{C}_N\| < \infty. \tag{4.8}$$

Proof. Equation (4.6) is obtained from the results of (Zhang, 2006) and Chapter 3, with $\tilde{\delta}_N(z)$ and $m_N(z)$ the deterministic equivalents extension of $\tilde{\delta}(z)$ and $m(z)$ defined in (3.2) and (3.3), i.e., the fundamental equations attached to $(\nu_n, \tilde{\nu}_n)$ instead of $(\nu, \tilde{\nu})$, where $\nu_n = \frac{1}{N}\sum_i \boldsymbol{\delta}_{\lambda_i(C_N)}$. The characterization of μ_N follows immediately from the results of Chapter 3. The uniform boundedness of the support is a consequence of the boundedness of ψ and γ_N, Lemma 4.1 in Section 4.1.3. Finally, the results (4.7) and (4.8) are an application of (Paul and Silverstein, 2009) along with $\limsup_N \|\hat{S}_N\| \leq v(0)\limsup_N \|C_N\| \limsup_N \|\frac{1}{n}\sum_{i=1}^n x_i x_i^*\| < \infty$ by Assumption 4.2–3 and (Bai and Silverstein, 1998). □

A consequence of Theorem 4.1.2 and Corollary 4.1 in the i.i.d. elliptical case is as follows.

Corollary 4.2 (Elliptical case). *Let Assumptions 4.1–4.3 hold and in addition, let τ_i be i.i.d. with law $\tilde{\nu}$ and let $c_N \to c$. Then*

$$\left\|\hat{C}_N - \frac{1}{n}\sum_{i=1}^{n} v(\tau_i\gamma^\infty)x_i x_i^*\right\| \xrightarrow{\text{a.s.}} 0$$

where γ^∞ is the unique positive solution to the equation in γ

$$1 = \int \frac{\psi_c(t\gamma)}{1 + c\psi_c(t\gamma)}\tilde{\nu}(dt)$$

with $\psi_c = \lim_{c_N \to c} \psi$. *Moreover, if* $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\lambda_i(C_N)} \to \nu$ *weakly, then*

$$\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\lambda_i(\hat{C}_N)} \xrightarrow{\text{a.s.}} \mu$$

weakly with μ *a probability measure with continuous density of bounded support* $\mathcal{S}$, *the Stieltjes transform* $m(z)$ *of which is given for* $z \in \mathbb{C}_+$ *by*

$$m(z) = -\frac{1}{z}\int \frac{1}{1+\tilde{\delta}(z)t}\nu(dt)$$

where $\tilde{\delta}(z)$ *is the unique solution in* $\mathbb{C}_+$ *of the equations in* $\tilde{\delta}$

$$\begin{aligned}\tilde{\delta} &= -\frac{1}{z}\int \frac{\psi_c(t\gamma^\infty)}{\gamma^\infty + \psi_c(t\gamma^\infty)\delta}\tilde{\nu}(dt)\\ \delta &= -\frac{c}{z}\int \frac{t}{1+t\tilde{\delta}}\nu(dt).\end{aligned}$$

Finally, for every closed set $\mathcal{A} \subset \mathbb{R}$ *with* $\mathcal{A} \cap \mathcal{S} = \emptyset$,

$$\left|\left\{\lambda_i(\hat{C}_N)\right\}_{i=1}^N \cap \mathcal{A}\right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We use the fact that $\gamma_N \xrightarrow{\text{a.s.}} \gamma^\infty$ (γ_N defined in Theorem 4.1.2) which is a consequence of $\psi/(1+c_N\psi)$ being monotonous and γ_N uniformly bounded, Lemma 4.1. The rest unfolds from classical random matrix results presented in part in Chapter 3. □

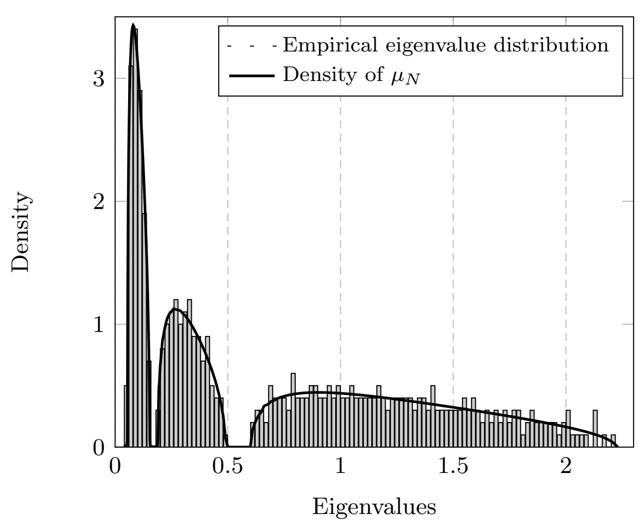


Figure 4.1: Histogram of the eigenvalues of $\hat{C}_N$ for $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

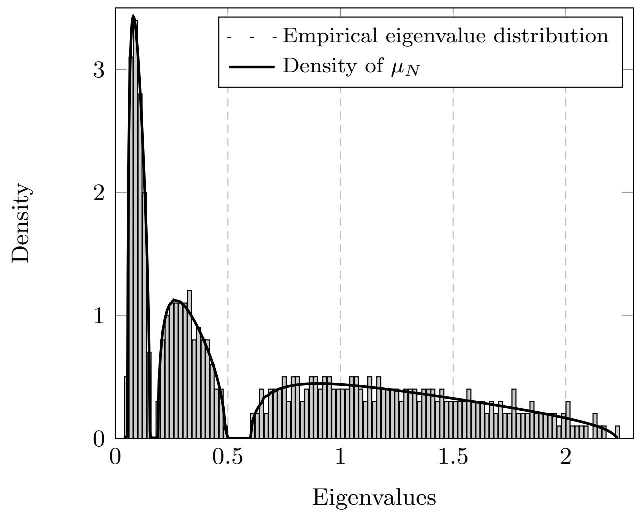


Figure 4.2: Histogram of the eigenvalues of $\hat{S}_N$ for $n = 2500$, $N = 500$, $C_N = \operatorname{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

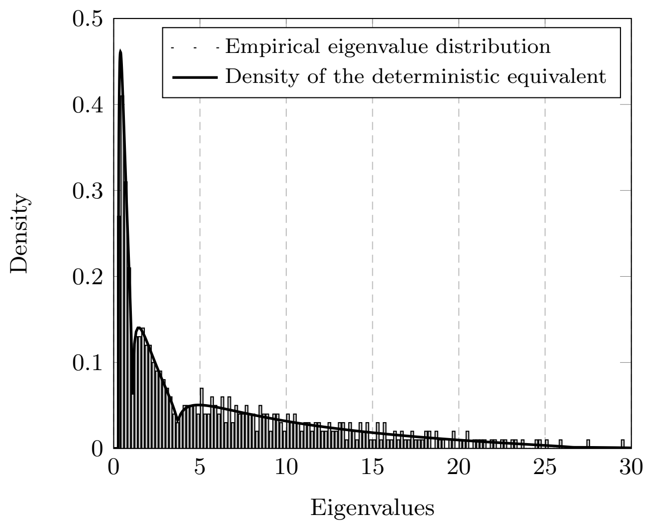


Figure 4.3: Histogram of the eigenvalues of $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$ for $n = 2500$, $N = 500$, $C_N = \operatorname{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(.5, 2)$-distribution.

Figures 4.1 and 4.2 depict the empirical histogram of the eigenvalues of $\hat{C}_N$ and $\hat{S}_N$, for $N = 500$ and $n = 2500$ with $u(t) = (1 + \alpha)/(t + \alpha)$, $\alpha = 0.1$, $C_N = \operatorname{diag}(I_{125}, 3I_{125}, 10I_{250})$, and $\tau_1, \ldots, \tau_n$ i.i.d. with $\Gamma(.5, 2)$ distribution. In thick line is also depicted the density of μ_N in Corollary 4.1 which shows an accurate match to the empirical spectrum as predicted by (4.6). As a comparison, Figure 4.3 shows the empirical histogram of the eigenvalues of the sample covariance matrix $\frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$ under the same parametrization against the deterministic equivalent density for this model in thick line (Zhang, 2006). This graph presents a large eigenvalue spectrum support, seemingly unboundedly growing with N, which is indeed expected according to Proposition 3.3.4 in the previous chapter as τ_1 has unbounded support; this is to be compared against the provably uniformly bounded spectrum of $\hat{C}_N$ (owing again to Proposition 3.3.4 and the uniform boundedness of $v(x)$ and $\|C_N\|$). Also note the gain of separability in the spectrum of $\hat{C}_N$ which exhibits clearly three compacts subsets of eigenvalues, reminiscent of the three masses in the eigenvalue distribution of C_N, while $\frac{1}{n}\sum_{i=1}^n x_i x_i^*$ exhibits a single compact set of eigenvalues.

Both remarks have major consequences for detection and estimation purposes in signal processing applications of robust estimation, where relevant system information is often carried in the largest eigenvalues, ideally found sufficiently far from the "noise" eigenvalues. As such, from a practical standpoint, it is expected that robust estimators would allow for an improved separation between information and noise in impulsive data settings. This behavior will be confirmed by the application carried out in Section 4.2.

In the next section, we present the proofs of Theorem 4.1.1 and Theorem 4.1.2.

4.1.3 Proof of the main results

For the sake of definition, we take all variables to be complex here although the arguments are also valid for real random variables.

4.1.3.1 Proof of Theorem 4.1.1

As mentioned in Section 4.1.1, we can assume without generality restriction that $C_N = I_N$. Indeed, if $\hat{C}_N$ is the unique solution to (4.1) assuming $C_N = I_N$, then, for any other choice of $C_N \succ 0$, $C_N^{\frac{1}{2}}\hat{C}_N C_N^{\frac{1}{2}}$ is the unique solution to the corresponding model in (4.1). Hence, we only need to prove the result for $C_N = I_N$.

Consider a growing sequence $\{x_1,\ldots,x_n\}_{n=1}^\infty$ according to Assumptions 4.1 and 4.2. Since $|\{\tau_i = 0\}| = n\tilde{\nu}_n(\{0\}) < n(1-c_+)$ for all large n a.s. (Assumption 4.2–2), $n - |\{\tau_i = 0\}| > c_+ n > N+1$ which, along with $z_1,\ldots,z_n$ being normalized Gaussian vectors, ensures that $\{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n\}$ spans $\mathbb{C}^N$ for all j for all large n a.s. As long as n is large enough, we can therefore almost surely define $h = (h_1,\ldots,h_n)$ with $h_j : \mathbb{R}_+^n \to \mathbb{R}_+$ given by

$$h_j(q_1,\ldots,q_n) = \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v\left(\tau_i q_i\right) z_i z_i^* \right)^{-1} z_j.$$

As shown in Section 4.1.1.1, in order to show that $\hat{C}_N$ is uniquely defined, it suffices to show that there exists a unique $q_1,\ldots,q_n$ such that for each j, $q_j = h_j(q_1,\ldots,q_n)$. For this, we show first that h satisfies the following properties with probability one:

(a) Nonnegativity: For each $q_1,\ldots,q_n \geq 0$ and each i, $h_i(q_1,\ldots,q_n) > 0$

(b) Monotonicity: For each $q_1 \geq q_1',\ldots,q_n \geq q_n'$ and each i, $h_i(q_1,\ldots,q_n) \geq h_i(q_1',\ldots,q_n')$

(c) Scalability: For each $\alpha > 1$ and each i, $\alpha h_i(q_1, \ldots, q_n) > h_i(\alpha q_1, \ldots, \alpha q_n)$.

Item (a) is obvious since the matrix inverse is well defined for all n large and $z_i \neq 0$ almost surely. Item (b) follows from the fact that, for two Hermitian matrices $A \succeq B \succ 0$, $B^{-1} \succeq A^{-1} \succ 0$ ((Horn and Johnson, 1985, Corollary 7.7.4)), and from v being non-increasing, entailing h_i to be a non-decreasing function of each q_j. As for Item (c), it follows also from the previous matrix inverse relation and from ψ being increasing, entailing in particular that, for $\alpha > 1$, $\psi(\alpha q_i) > \psi(q_i)$ if $q_i \neq 0$ so that $v(\alpha q_i) > v(q_i)/\alpha$ for $q_i \geq 0$.

According to Yates (Yates, 1995, Theorem 2), h is then a *standard interference function* and, if there exists $q_1, \ldots, q_n$ such that for each i, $q_i > h_i(q_1, \ldots, q_n)$ (feasibility condition), then there is a unique $\{q_1, \ldots, q_n\}$ satisfying $q_i = h_i(q_1, \ldots, q_n)$ for each i, which is given by $q_i = \lim_{t\to\infty} q_i^{(t)}$ with $q_i^{(0)} \geq 0$ arbitrary and, for $t \geq 0$, $q_i^{(t+1)} = h_i(q_1^{(t)}, \ldots, q_n^{(t)})$ (which would then conclude the proof). To obtain the feasibility condition, note that the function $q \mapsto \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{i\neq j} \psi(\tau_i q) z_i z_i^*\right)^{-1} z_j$ decreases with limit $\frac{1-c_N\phi_\infty}{\phi_\infty}\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{i\neq j,\tau_i\neq 0} z_i z_i^*\right)^{-1} z_j$ as $q \to \infty$. As $\{\tau_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ are independent and $\limsup_n N/|\{\tau_i \neq 0\}| = \limsup c_N/(1 - \tilde{\nu}_n(\{0\})) < 1$ a.s. (Assumption 4.2 and Assumption 4.1), for all large n a.s., we fall within the hypotheses of Lemma A.4 in the Appendix and we can then write,[4]

$$\max_{1\leq j\leq n}\left|(1-\tilde{\nu}_n(\{0\}))\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0} z_i z_i^*\right)^{-1} z_j - 1\right| \xrightarrow{\text{a.s.}} 0.$$

Assume first that $\tau_j \neq 0$. Then, using the relation

$$\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0, i\neq j} z_i z_i^*\right)^{-1} z_j = \frac{\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0} z_i z_i^*\right)^{-1} z_j}{1 - c_N \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0} z_i z_i^*\right)^{-1} z_j}$$

and the fact that for all large n a.s. $1 - \tilde{\nu}_n(\{0\}) > c_+$, we have

$$\max_{j,\tau_j\neq 0}\left|\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0, i\neq j} z_i z_i^*\right)^{-1} z_j - \frac{1}{1-\tilde{\nu}_n(\{0\}) - c_N}\right| \xrightarrow{\text{a.s.}} 0.$$

Therefore, using the fact that $\tilde{\nu}_N(\{0\}) < 1 - \phi_\infty^{-1}$ for all n large a.s. (Assumption 4.2–2), we have that for all j with $\tau_j \neq 0$

$$\frac{1 - c_N\phi_\infty}{\phi_\infty}\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\tau_i\neq 0, i\neq j} z_i z_i^*\right)^{-1} z_j < 1. \tag{4.9}$$

[4]To be more exact, since $|\{\tau_i \neq 0\}|$ is random with probability space $\mathcal{T}$ producing the τ_i's, Lemma A.4 applies only on a subset of probability one of $\mathcal{T}$. It then suffices to apply Tonelli's theorem (Billingsley, 1995) to ensure that Lemma A.4 can be extended and still holds with probability one on the product space producing the (τ_i, z_i).

If instead $\tau_j = 0$, then

$$\max_{j,\tau_j=0} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0} z_i z_i^* \right)^{-1} z_j - \frac{1}{1-\tilde{\nu}_n(\{0\})} \right| \xrightarrow{\text{a.s.}} 0.$$

and we find also the inequality (4.9) for all large n a.s. and for all j with $\tau_j = 0$, using once more $\tilde{\nu}_N(\{0\}) < 1 - \phi_\infty^{-1}$. As such, (4.9) is valid for all $j \in \{1, \ldots, n\}$.

We can then choose n large enough so that (4.9) holds for all j, after which, taking q sufficiently large,

$$\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \psi(\tau_i q) z_i z_i^* \right)^{-1} z_j < 1$$

which is equivalent to

$$\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} v(\tau_i q) \tau_i z_i z_i^* \right)^{-1} z_j < q$$

for all j, i.e., $h_j(q, \ldots, q) < q$. This ensures feasibility for all large n a.s. and concludes the proof.

4.1.3.2 Proof of Theorem 4.1.2

Similar to the proof of Theorem 4.1.1, we can restrict ourselves to the assumption that $C_N = I_N$. The generalization to C_N satisfying Assumption 4.2-3) will follow straightforwardly. We therefore take $C_N = I_N$ in what follows.

We start the proof by introducing the following fundamental lemmas (note that these lemmas in fact hold true irrespective of $C_N \succ 0$).

Lemma 4.1. *Let Assumption 4.1 hold and let $h : [0, \infty) \to [0, \infty)$ be given by*

$$h(\gamma) = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\tau_i v(\tau_i \gamma)}{1 + c_N \tau_i v(\tau_i \gamma) \gamma} \right)^{-1} = \begin{cases} \gamma \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} \right)^{-1} & , \ \gamma > 0 \\ \frac{1}{v(0)} \left(\frac{1}{n} \sum_{i=1}^{n} \tau_i \right)^{-1} & , \ \gamma = 0. \end{cases}$$

Then, for all large n a.s., there exists a unique $\gamma_N > 0$ satisfying $\gamma_N = h(\gamma_N)$, given by

$$\gamma_N = \lim_{t \to \infty} \gamma_N^{(t)}$$

with $\gamma_N^{(0)} \geq 0$ arbitrary and, for $t \geq 0$, $\gamma_N^{(t+1)} = h(\gamma_N^{(t)})$. Moreover, with probability one,

$$\gamma_- < \liminf_N \gamma_N \leq \limsup_N \gamma_N < \gamma_+$$

for some $\gamma_-, \gamma_+ > 0$ finite.

Proof. As in the proof of Theorem 4.1.1, we show that h (scalar-valued this time) is a standard interference function. We show easily positivity, monotonicity and scalability of h. Indeed, for $\gamma \geq 0$, $h(\gamma) > 0$. For $\gamma \geq \gamma' \geq 0$,

$$\frac{h(\gamma) - h(\gamma')}{h(\gamma)h(\gamma')} = \frac{1}{n}\sum_{i=1}^{n} \frac{\tau_i\left(v(\tau_i\gamma') - v(\tau_i\gamma)\right) + (\gamma - \gamma')c_N\tau_i^2 v(\tau_i\gamma)v(\tau_i\gamma')}{(1 + c_N\tau_i v(\tau_i\gamma)\gamma)(1 + c_N\tau_i v(\tau_i\gamma')\gamma')} \geq 0$$

which follows from v being nonnegative decreasing. Finally, for $\alpha > 1$, $\alpha h(0) > h(0)$ and for $\gamma \neq 0$,

$$\begin{aligned} h(\alpha\gamma) &= \alpha\gamma\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\alpha\gamma)}{1 + c_N\psi(\tau_i\alpha\gamma)}\right)^{-1} < \alpha\gamma\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma)}{1 + c_N\psi(\tau_i\gamma)}\right)^{-1} \\ &= \alpha h(\gamma) \end{aligned}$$

which follows from $\gamma \mapsto \psi(\tau_i\gamma)(1 + c_N\psi(\tau_i\gamma))^{-1}$ being increasing as long as $\tau_i \neq 0$. It remains to prove the existence of a γ such that $\gamma > h(\gamma)$, inducing by (Yates, 1995, Theorem 2) the uniqueness of the fixed-point γ_N given by $\gamma_N = \lim_{t\to\infty}\gamma_N^{(t)}$ as stated in the theorem. For this, we use again the fact that $\gamma \mapsto \psi(\tau_i\gamma)(1 + c_N\psi(\tau_i\gamma))^{-1}$ is increasing and that (Assumption 4.2–2), for all large n a.s.

$$\lim_{\gamma\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma)}{1 + c_N\psi(\tau_i\gamma)} = \frac{(1 - \tilde{\nu}_n(\{0\}))\psi_\infty^N}{1 + c_N\psi_\infty^N} = (1 - \tilde{\nu}_n(\{0\}))\phi_\infty > 1.$$

Therefore, there exists γ_0 (a priori dependent on the set $\{\tau_1, \ldots, \tau_n\}$) such that, for all $\gamma > \gamma_0$, $h(\gamma) < \gamma$.

To prove uniform boundedness of γ_N, let $\varepsilon > 0$ and $m > 0$ be such that $(1 - \varepsilon)\phi_\infty > 1$ and $\tilde{\nu}_n((m, \infty)) > 1 - \varepsilon$ for all n large a.s. (always possible from Assumption 4.2–2). Then, for all n large a.s.

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma)}{1 + c_N\psi(\tau_i\gamma)} > (1 - \varepsilon)\frac{\psi(m\gamma)}{1 + c_N\psi(m\gamma)} \to (1 - \varepsilon)\phi_\infty > 1$$

as $\gamma \to \infty$. Similar to γ_0 above, we can therefore choose γ_+ large enough, now independent of n large, such that, a.s. $\gamma \geq \gamma_+ \Rightarrow \gamma > h(\gamma)$, implying $\gamma_N < \gamma_+$ for these n large since $\gamma_N = h(\gamma_N)$. Also, $h(0) > 1/(2v(0))$ for all large n a.s. since $\frac{1}{n}\sum_{i=1}^{n}\tau_i \xrightarrow{\text{a.s.}} 1$ by Assumption 4.2. Hence, by the continuous growth of h, we can take $\gamma_- = 1/(2v(0)) > 0$ which is such that $\gamma \leq \gamma_- \Rightarrow h(\gamma) \geq h(0) > \gamma$ for all large n a.s. This implies $\gamma_N > \gamma_-$ for all large n a.s., which concludes the proof. $\square$

Remark 4.1. *For further use, note that Lemma 4.1 can be refined as follows. Let (η, M_η) be couples indexed by η with $0 < \eta < 1$ and $M_\eta > 0$ such that $\tilde{\nu}_n((M_\eta, \infty)) < \eta$ for all large n a.s. (possible by tightness of $\tilde{\nu}_n$). Then, for sufficiently small η, the equation in γ*

$$\gamma = \left(\frac{1}{n}\sum_{\tau_i \leq M_\eta}\frac{\tau_i v(\tau_i\gamma)}{1 + c_N\tau_i v(\tau_i\gamma)\gamma}\right)^{-1}$$

has a unique solution γ_N^η for all large n a.s. and there exists $\gamma_-, \gamma_+ > 0$ such that, for all $\eta < \eta_0$ small, $\gamma_- < \gamma_N^\eta < \gamma_+$ for all large n a.s.

Proof. The uniqueness is clear as long as $(1-\eta_0)(1-\limsup_n \tilde{\nu}_n(\{0\}))\phi_\infty > 1$ since then, exploiting the fact that $\lim_n \frac{|\{\tau_i \leq M_\eta\}|}{n} > 1-\eta_0$ a.s.,

$$\lim_{\gamma\to\infty} \frac{1}{n} \sum_{\tau_i \leq M_\eta} \frac{\psi(\tau_i\gamma)}{1+c_N\psi(\tau_i\gamma)} = \frac{|\{\tau_i \leq M_\eta\}|}{n}(1-\tilde{\nu}_n(\{0\}))\phi_\infty > 1$$

for all n large a.s. and the proof follows from the proof of Lemma 4.1. For uniform boundedness, taking $M_{\eta_0} < M_\eta$ large enough (or equivalently $\eta_0 > \eta$ small enough) such that $\liminf_n \frac{|\{m<\tau_i\leq M_\eta\}|}{n} > \liminf_n \frac{|\{m<\tau_i\leq M_{\eta_0}\}|}{n} > 1-\varepsilon$ a.s. in the proof of Lemma 4.1 leads to the same upper bound result for all small $\eta < \eta_0$. As for the lower bound, we still have $h(0) > 1/(2v(0))$ for all large n a.s. independently of η so the result is maintained. □

Lemma 4.2. *Let Assumption 4.1 hold and define γ_N as in Lemma 4.1. Then, as $n \to \infty$,*

$$\max_{1\leq j\leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i\gamma_N) z_i z_i^* \right)^{-1} z_j - \gamma_N \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We first introduce some notations to simplify readability. First, we will write $z_j = \sqrt{N} A_N \tilde{w}_j / \|\tilde{w}_j\| \triangleq \sqrt{N}\tilde{z}_j/\|\tilde{w}_j\|$ with $\tilde{w}_j$ zero-mean $I_{\bar{N}}$-covariance Gaussian, hence $\tilde{w}_j$ is zero-mean I_N-covariance Gaussian. With this notation, in what follows, we denote $A = \frac{1}{n}\sum_{i=1}^n \tau_i v(\tau_i\gamma_N) z_i z_i^*$, $A_{(j)} = A - \frac{1}{n}\tau_j v(\tau_j\gamma_N) z_j z_j^*$, $\tilde{A} = \frac{1}{n}\sum_{i=1}^n \tau_i v(\tau_i\gamma_N)\tilde{z}_i\tilde{z}_i^*$ and $\tilde{A}_{(j)} = \tilde{A} - \frac{1}{n}\tau_j v(\tau_j\gamma_N)\tilde{z}_j\tilde{z}_j^*$.

We first show that there exists $\eta > 0$ such that, for all large n a.s.

$$\min_{1\leq j\leq n} \lambda_1\left(A_{(j)}\right) > \eta \tag{4.10}$$

(recall that λ_1 stands for the smallest eigenvalue). For this, take $0 < \varepsilon < 1 - c_+$ and $m > 0$ be such that $\tilde{\nu}_n((m,\infty)) > 1-\varepsilon$ for all n large a.s. (Assumption 4.2–2). Using the fact that $xv(x) = \psi(x)$ is non-decreasing and that any subtraction of a nonnegative definite matrix cannot increase the smallest eigenvalue, we have

$$\begin{aligned}
\min_{1\leq j\leq n} \lambda_1\left(A_{(j)}\right) &\geq \min_{1\leq j\leq n} \lambda_1\left(\frac{1}{n} \sum_{i\neq j, \tau_i\geq m} \frac{\psi(\tau_i\gamma_N)}{\gamma_N} z_i z_i^* \right) \\
&\geq \frac{\psi(m\gamma_N)}{\gamma_N} \min_{1\leq j\leq n} \lambda_1\left(\frac{1}{n} \sum_{i\neq j, \tau_i\geq m} z_i z_i^* \right).
\end{aligned} \tag{4.11}$$

Since $\tilde{\nu}_n((m,\infty)) > 1-\varepsilon$ for all n large a.s.,

$$0 < c_- < \liminf_n \frac{N}{|\{\tau_i \geq m\}|} \leq \limsup_n \frac{N}{|\{\tau_i \geq m\}|} < \frac{c_+}{1-\varepsilon} < 1.$$

From Lemma A.4 in the Appendix (see footnote in the proof of Theorem 4.1.1 for details), we can then write

$$\begin{aligned}\min_{1\le j\le n} \lambda_1\left(A_{(j)}\right) &\ge \frac{\psi(m\gamma_N)}{\gamma_N}\tilde{\nu}_n((m,\infty)) \min_{1\le j\le n} \lambda_1\left(\frac{1}{|\{\tau_i\ge m\}|}\sum_{i\neq j,\tau_i\ge m} z_i z_i^*\right)\\ &> \frac{\psi(m\gamma_N)}{\gamma_N}(1-\varepsilon)\eta'\end{aligned}$$

for some $\eta'>0$ which, along with the almost sure boundedness of γ_N (Lemma 4.1) proves (4.10).

Now that (4.10) is acquired, let $\mathrm{E}_{\tilde{w}_j}$ denote the expectation with respect to $\tilde{w}_j$ (i.e., conditionally on the sigma-field engendered by the $\tilde{w}_i$, $i\neq j$, and the τ_i) and $\kappa_j \triangleq 1_{\{\lambda_1(A_{(j)})>\eta\}}$ with η as defined in (4.10). From (Bai and Silverstein, 2009, Lemma B.26) (which applies here since $\tilde{z}_j$ and $\kappa_j^{1/p}A_{(j)}^{-1}$ are independent), for $p>2$,

$$\mathrm{E}_{\tilde{w}_j}\left[\kappa_j\left|\frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j - \frac{1}{N}\operatorname{tr} A_{(j)}^{-1}\right|^p\right] \le \frac{\kappa_j K_p}{N^{\frac{p}{2}}}\left[\left(\frac{\zeta_4}{N}\operatorname{tr} A_{(j)}^{-2}\right)^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}}}\operatorname{tr} A_{(j)}^{-p}\right]$$

for ζ_ℓ any upper bound on $\mathrm{E}[|\tilde{z}_{ij}|^\ell]$ and K_p a constant dependent only on p. From the definition of κ_j, we have $\kappa_j\|A_{(j)}^{-1}\|<\eta^{-1}$, so that, using $\frac{1}{N}\operatorname{tr} B\le\|B\|$ for nonnegative definite $B\in\mathbb{C}^{N\times N}$,

$$\mathrm{E}_{\tilde{w}_j}\left[\kappa_j\left|\frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j - \frac{1}{N}\operatorname{tr} A_{(j)}^{-1}\right|^p\right] \le \frac{K_p}{\eta^p N^{\frac{p}{2}}}\left(\zeta_4^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}-1}}\right).$$

This bound being irrespective of all z_i and τ_i, $i\neq j$, we can take the expectation with respect to all w_i, $i\neq j$, and all τ_i to obtain

$$\mathrm{E}\left[\kappa_j\left|\frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j - \frac{1}{N}\operatorname{tr} A_{(j)}^{-1}\right|^p\right] = \mathcal{O}\left(\frac{1}{N^{\frac{p}{2}}}\right).$$

Taking $p>4$ and applying the union bound, Markov inequality, and Borel Cantelli lemma finally shows that

$$\max_{1\le j\le n}\kappa_j\left|\frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j - \frac{1}{N}\operatorname{tr} A_{(j)}^{-1}\right| \xrightarrow{\text{a.s.}} 0. \tag{4.12}$$

With the same arguments on κ_j and with the same p as above, now remark that

$$\begin{aligned}\mathrm{E}_{\tilde{w}_j}\left[\kappa_j\left|\frac{1}{N}z_j^* A_{(j)}^{-1}z_j - \frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j\right|^p\right] &= \mathrm{E}_{\tilde{w}_j}\left[\kappa_j\left|\frac{1}{N}z_j^* A_{(j)}^{-1}z_j\left(1-\frac{\|\tilde{w}_j\|^2}{\bar{N}}\right)\right|^p\right]\\ &\le \frac{1}{\eta^p}\mathrm{E}_{\tilde{w}_j}\left[\left|1-\frac{\|\tilde{w}_j\|^2}{\bar{N}}\right|^p\right] = \mathcal{O}\left(\frac{1}{N^{p/2}}\right)\end{aligned}$$

since $\bar{N}\ge N$, again by (Bai and Silverstein, 2009, Lemma B.26). Therefore, by the union bound, Markov inequality, and Borel Cantelli lemma,

$$\max_{1\le j\le n}\kappa_j\left|\frac{1}{N}z_j^* A_{(j)}^{-1}z_j - \frac{1}{N}\tilde{z}_j^* A_{(j)}^{-1}\tilde{z}_j\right| \xrightarrow{\text{a.s.}} 0. \tag{4.13}$$

Combining (4.12) and (4.13) along with the fact that $\min_{1\leq j\leq n}\kappa_j \xrightarrow{\text{a.s.}} 1$ (from (4.10)) finally gives

$$\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^* A_{(j)}^{-1}z_j - \frac{1}{N}\operatorname{tr} A_{(j)}^{-1}\right| \xrightarrow{\text{a.s.}} 0.$$

By (4.10), $A_{(j)} = (A_{(j)} - \frac{\eta}{2}I_N) + \frac{\eta}{2}I_N$ with $\liminf_n \lambda_1(A_{(j)} - \frac{\eta}{2}I_N) > 0$ a.s., so we are in the conditions of Lemma A.2 and we have

$$\max_{1\leq j\leq n}\left|\frac{1}{N}\operatorname{tr} A_{(j)}^{-1} - \frac{1}{N}\operatorname{tr} A^{-1}\right| \xrightarrow{\text{a.s.}} 0.$$

It remains to find a deterministic equivalent for $\frac{1}{N}\operatorname{tr} A^{-1}$. Similar to above, note first that, for all large n a.s.

$$\left|\frac{1}{N}\operatorname{tr} A^{-1} - \frac{1}{N}\operatorname{tr}\tilde{A}^{-1}\right| \leq \frac{1}{\eta^2}\frac{\psi_\infty}{\gamma_N}\max_{1\leq j\leq n}\left|\frac{1-\bar{N}^{-1}\|\tilde{w}_j\|^2}{\bar{N}^{-1}\|\tilde{w}_j\|^2}\right|\left\|\frac{1}{n}\sum_{i=1}^n \tilde{z}_j\tilde{z}_j^*\right\|$$

where we used the definition and boundedness of ψ and standard matrix inversion formulas. From (Bai and Silverstein, 1998), the right hand side converges almost surely to zero, so that it is equivalent to consider z_i or $\tilde{z}_i$. Now, the trace $\frac{1}{N}\operatorname{tr}\tilde{A}^{-1}$ is exactly the Stieltjes transform $\hat{m}_N(z)$ of the matrix $\tilde{A}$ evaluated at point $z=0$. Since $\lambda_1(\tilde{A}) \geq \lambda_1(\tilde{A}_{(1)}) > \eta$ for all large n a.s. and since $\tau_i v(\tau_i\gamma_N) = \psi(\tau_i\gamma_N)\gamma_N^{-1}$ is uniformly bounded across i and n (from the boundedness of ψ and Lemma 4.1), from standard random matrix results (e.g., (Couillet et al., 2011a) which extends (3.4) to deterministic equivalents instead of limits)[5], we have

$$\hat{m}_N(0) - m_N(0) \xrightarrow{\text{a.s.}} 0$$

where $m_N(0)$ is the unique nonnegative solution to the equation in m (as long as at least one τ_i is non-zero)

$$m = \left(\frac{1}{n}\sum_{i=1}^n \frac{\tau_i v(\tau_i\gamma_N)}{1+c_N\tau_i v(\tau_i\gamma_N)m}\right)^{-1}.$$

Now, by definition, γ_N coincides with such a solution. By uniqueness of $m_N(0)$, one must then have $m_N(0)=\gamma_N$ so that, gathering all results together,

$$\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^* A_{(j)}^{-1}z_j - \gamma_N\right| \xrightarrow{\text{a.s.}} 0$$

which completes the proof. □

[5] More precisely, (Couillet et al., 2011a) shows that $\hat{m}_N(z) - m_N(z) \xrightarrow{\text{a.s.}} 0$ for all points z with $\Im[z] > 0$. Using $\lambda_1(\tilde{A}) > \eta$ for all large n a.s., the proof can be generalized to all $z \in \mathbb{C}$ with positive distance to $[\eta,\infty)$ by turning the bounds in $1/|\Im[z]|$ into $1/d(z,[\eta,\infty))$ with d denoting the Hausdorff distance, so for $z=0$. The existence of $m_N(0)$ is in particular already obtained in the generalization of the existence result of (Couillet et al., 2011a, Appendix A-C), this time for $z=0$. The proof of uniqueness of $m_N(0)$ can then be checked by standard interference function arguments, where feasibility follows in particular from the right-hand behaving as $c_N m < m$ from Assumption 4.1.

Remark 4.2. *Similar to Remark 4.1, note that Lemma 4.2 can be further extended to*

$$\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{\tau_i\leq M_\eta,i\neq j}\tau_i v(\tau_i\gamma_N^\eta)z_iz_i^*\right)^{-1}z_j-\gamma_N^\eta\right|\xrightarrow{\text{a.s.}}0$$

for some η small enough, with M_η and γ_N^η defined in Remark 4.1.

Proof. One shows boundedness of $\lambda_1(\frac{1}{n}\sum_{\tau_i\leq M_\eta,i\neq j}\tau_i v(\tau_i\gamma_N^\eta)z_iz_i^*)$ simply by taking η for which $\tilde{\nu}_n((m,M_\eta))>1-\varepsilon$ for all large n a.s. in the proof of Lemma 4.2. Then it suffices to adapt all derivations by substituting τ_i by zero if $\tau_i>M_\eta$. The result follows straightforwardly. $\square$

The two lemmas above are standard random matrix results on $x_1,\ldots,x_n$, independent of the structure of $\hat{C}_N$. The next lemma introduces a first result on the matrix $\hat{C}_N$ which will be fundamental in what follows. Recall that we denoted $d_i=\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i$, with $\hat{C}_{(i)}=\hat{C}_N-\frac{1}{n}v(\tau_id_i)\tau_iz_iz_i^*$.

Lemma 4.3 (Boundedness of the d_i)**.** *There exist $d_+>d_->0$ such that, for all large n a.s.,*

$$d_-<\liminf_n\min_{1\leq i\leq n}d_i\leq\limsup_n\max_{1\leq i\leq n}d_i<d_+.$$

Proof. Let us denote $d_{\max}=\max_{1\leq i\leq n}d_i$ and $d_{\min}=\min_{1\leq i\leq n}d_i$. Take $j\in\{1,\ldots,n\}$ arbitrary and, for $0<\varepsilon<1-\phi_\infty^{-1}<1-c_+$, take $m>0$ such that for all large n a.s. $\tilde{\nu}_n([m,\infty))>1-\varepsilon$ (Assumption 4.2–2). Then, using the fact that v is non-increasing while ψ is non-decreasing,

$$\begin{aligned}\hat{C}_{(j)}&\succeq\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}\tau_iv(\tau_id_i)z_iz_i^*=\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}\frac{\psi(\tau_id_i)}{d_i}z_iz_i^*\succeq\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}\frac{\psi(md_i)}{d_i}z_iz_i^*\\&=\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}mv(md_i)z_iz_i^*\succeq mv(md_{\max})\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}z_iz_i^*.\end{aligned}\tag{4.14}$$

The right-hand side matrix is invertible for n large since $|\{\tau_i\geq m\}|>nc_+>N$ for all large n a.s. Therefore, choosing j to be such that $d_{\max}=\frac{1}{N}z_j^*\hat{C}_{(j)}^{-1}z_j$, and using $A\succeq B\succ 0\Rightarrow B^{-1}\succeq A^{-1}$ for Hermitian A,B matrices,

$$d_{\max}\leq\frac{1}{mv(md_{\max})}\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{\tau_i\geq m,i\neq j}z_iz_i^*\right)^{-1}z_j.$$

This implies

$$\psi(md_{\max})\leq\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{\tau_i\geq m,i\neq j}z_iz_i^*\right)^{-1}z_j$$

which can be rewritten, from the definition of ψ,

$$\phi(g^{-1}(md_{\max})) \leq \frac{\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j}{1 + c_N \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j}. \tag{4.15}$$

From Lemma A.4 in the Appendix and the fact that $\tilde{\nu}_n([m,\infty) = n^{-1}|\{\tau_i \geq m\}| > 1 - \varepsilon$ for all large n a.s., we then have for all large n a.s.

$$\begin{aligned} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j &= \frac{1}{\tilde{\nu}_n([m,\infty))} \frac{1}{N} z_j^* \left(\frac{1}{|\{\tau_i \geq m\}|} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j \\ &< \frac{1}{1-\varepsilon} \frac{1}{1 - \frac{c_N}{1-\varepsilon}} = \frac{1}{1 - c_N - \varepsilon}. \end{aligned}$$

Now, since $t \mapsto t/(1 + c_N t)$ is increasing, for all large n a.s.

$$\frac{\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j}{1 + c_N \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j} < \frac{1}{1 - c_N - \varepsilon} \frac{1}{1 + c_N \frac{1}{1 - c_N - \varepsilon}} = \frac{1}{1-\varepsilon}.$$

As $\varepsilon < 1 - \phi_\infty^{-1}$, $(1-\varepsilon)^{-1} < \phi_\infty$ so that, from the inequality above, we can apply ϕ^{-1} on both sides of (4.15) to obtain, for all large n a.s.

$$g^{-1}(md_{\max}) \leq \phi^{-1}\left(\frac{1}{1-\varepsilon} \right)$$

hence

$$d_{\max} \leq \frac{1}{m} g\left(\phi^{-1}\left(\frac{1}{1-\varepsilon} \right) \right)$$

from which $d_{\max}$ is uniformly bounded for all large n a.s. by say d_+.

To proceed to $d_{\min}$, note similarly that we can write

$$\begin{aligned} \hat{C}_{(j)} &\preceq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} Mv(Md_{\min}) z_i z_i^* + \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i v(\tau_i d_i) z_i z_i^* \\ &\preceq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} Mv(Md_{\min}) z_i z_i^* + v(0) \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \end{aligned}$$

for any $M > 0$. Selecting j meeting the minimum for d_j, we then have

$$d_{\min} \geq \frac{1}{Mv(Md_{\min})} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{Mv(Md_{\min})} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j$$

which, for all large n a.s., satisfies

$$d_{\min} \geq \frac{1}{Mv(Md_{\min})} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{Mv(Md_+)} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j$$

or equivalently

$$\psi(Md_{\min}) \geq \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{Mv(Md_+)} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j.$$

With the same arguments as in the proof of Lemma 4.2, note that, taking M large enough

$$\liminf_n \left\{ \inf_j \lambda_1 \left(\frac{1}{n} \sum_{i \neq j, \tau_i \leq M} z_i z_i^* \right) \right\} > 0$$

almost surely (from Lemma A.4 and since $\liminf_n \tilde{\nu}_n((M, \infty)) \to 1$ a.s. as $M \to \infty$). We can then apply Lemma A.3 to obtain, along with Lemma A.2, Markov inequality, and Borel-Cantelli lemma arguments,

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + E_M \right)^{-1} z_j - \frac{1}{N} \operatorname{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* + E_M \right)^{-1} \right| \xrightarrow{\text{a.s.}} 0 \tag{4.16}$$

where we defined $E_M = \frac{v(0)}{Mv(Md_+)} \frac{1}{n} \sum_{\tau_i > M} \tau_i z_i z_i^*$. Now, E_M is of maximum rank $|\{\tau_i > M\}|$. Taking M large enough to ensure $\tilde{\nu}_n((M, \infty)) = |\{\tau_i > M\}|/n < c_- \varepsilon'$ for some $\varepsilon' > 0$ arbitrary, we then have from $|\{\tau_i > M\}|$ applications of Lemma A.2

$$\left| \frac{1}{N} \operatorname{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* + E_M \right)^{-1} - \frac{1}{N} \operatorname{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* \right)^{-1} \right| \leq \varepsilon'.$$

This and (4.16) give for all large n a.s.

$$\psi(Md_{\min}) \geq \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* \right)^{-1} z_j + 2\varepsilon'$$

for all large n almost surely. From there, it suffices to proceed similar to the boundedness proof for $d_{\max}$ starting from (4.15) with inequality signs reverted and accounting for ε' arbitrarily small. This shows finally that $d_{\min}$ is uniformly bounded away from zero and this completes the proof. □

Equipped with Lemmas 4.1, 4.2, and 4.3, we are now in position to develop the core of the proof. For readability, we divide the proof in two parts. In the first part, we will assume that $\tau_1, \ldots, \tau_n$ have a uniformly bounded support. This will greatly simplify the calculus and will allow for a better understanding of the main arguments; in particular, the technical Assumption 4.3 will be irrelevant in this part. Then in a second part, we relax the boundedness assumption and fully exploit Assumption 4.3 in a more technical proof.

Bounded τ_i. First assume $\tau_1, \ldots, \tau_n \leq M$ a.s. for some $M > 0$. Define

$$e_i \triangleq \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N)} > 0 \tag{4.17}$$

with γ_N the value given by Lemma 4.1 and with d_i still defined as $d_i = \frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$. Up to labeling change, we reorder the e_i's as $e_1 \leq \ldots \leq e_n$. Our goal is to show that $e_1 \xrightarrow{\text{a.s.}} 1$ and $e_n \xrightarrow{\text{a.s.}} 1$ (hence $\max_{1\leq i\leq n} |e_i - 1| \xrightarrow{\text{a.s.}} 0$), which we will prove by a contradiction argument.

For any $j = 1, \ldots, n$, we have

$$\begin{aligned}
e_j &= \frac{v\left(\tau_j \frac{1}{N} z_j^* \hat{C}_{(j)}^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\
&= \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i d_i) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\
&= \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i \gamma_N) e_i z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \qquad (4.18) \\
&\leq \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i \gamma_N) e_n z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\
&= \frac{v\left(\frac{\tau_j}{e_n} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \qquad (4.19)
\end{aligned}$$

where the inequality arises from v being non-increasing and from (Horn and Johnson, 1985, Corollary 7.7.4). Similarly, for each j,

$$e_j \geq \frac{v\left(\frac{\tau_j}{e_1} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)}. \tag{4.20}$$

From Lemma 4.2, let now $0 < \varepsilon_n < \gamma_N$, $\varepsilon_n \downarrow 0$, be such that, for all large n a.s. and for all $j \leq n$,

$$\gamma_N - \varepsilon_n < \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i\neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j < \gamma_N + \varepsilon_n.$$

In particular, since v is non-increasing, taking $j = n$ in (4.19) and applying the left-hand inequality,

$$e_n < \frac{v\left(e_n^{-1}\tau_n(\gamma_N - \varepsilon_n)\right)}{v(\tau_n\gamma_N)}$$

or equivalently

$$\frac{e_n v(\tau_n\gamma_N)}{v\left(e_n^{-1}\tau_n(\gamma_N - \varepsilon_n)\right)} < 1. \tag{4.21}$$

By the definition of ψ, this can be further rewritten

$$\left(1 - \varepsilon_n{\gamma_N}^{-1}\right)\frac{\psi(\tau_n\gamma_N)}{\psi\left(e_n^{-1}\tau_n\gamma_N(1 - \varepsilon_n\gamma_N^{-1})\right)} < 1. \tag{4.22}$$

Assume now that, for some $\ell > 0$, $e_n > 1 + \ell$ infinitely often and let us restrict the sequence e_n to those indexes for which $e_n > 1 + \ell$.

We distinguish two scenarios. First, assume that $\liminf_n \tau_n = 0$. Then, by the definition (4.17) and since both d_n and γ_N are uniformly bounded (Lemma 4.1 and Lemma 4.3), on some subsequence $\{n_j\}$ satisfying $\lim_j \tau_{n_j} = 0$, $e_{n_j} \xrightarrow{\text{a.s.}} 1$, in contradiction with $e_n > 1 + \ell$.

We must then have $\liminf_n \tau_n > \tau_-$ for some $\tau_- > 0$ along with $\tau_n \leq M$ a.s. for some $M > 0$ (bounded τ_i assumption). Then, since γ_N is bounded and bounded away from zero for all large n a.s., so is $\tau_n\gamma_N$. Considering and restricting ourselves to a further subsequence over which $\tau_n\gamma_N \to x > 0$ and $c_N \to c$, we then have, with $\psi_c(x) = \lim_{c_N\to c}\psi(x)$ (recall that ψ depends on c_N through g),

$$\lim_n \left(1 - \varepsilon_n{\gamma_N}^{-1}\right)\frac{\psi(\tau_n\gamma_N)}{\psi\left(e_n^{-1}\tau_n\gamma_N(1 - \varepsilon_n\gamma_N^{-1})\right)} \geq \frac{\psi_c(x)}{\psi_c((1+\ell)^{-1}x)} > 1 \tag{4.23}$$

which contradicts (4.22). Gathering the results and reconsidering the initial sequence e_n (i.e., not a subsequence) we then have, for each $\ell > 0$, $e_n \leq 1 + \ell$ for all large n a.s.

Symmetrically, we obtain that, for some $\varepsilon_n \downarrow 0$ and for all large n a.s.

$$\frac{e_1 v(\tau_1\gamma_N)}{v\left(e_1^{-1}\tau_1(\gamma_N + \varepsilon_n)\right)} > 1.$$

From this, we conclude similar to above that, for each $\ell > 0$ small, $e_1 \geq 1 - \ell$, for all large n a.s. so that, finally

$$\max_{1\leq i\leq n} |e_i - 1| \xrightarrow{\text{a.s.}} 0$$

or, by uniform boundedness of the τ_i and γ_N,

$$\max_{1\leq i\leq n} |v(\tau_i d_i) - v(\tau_i\gamma_N)| \xrightarrow{\text{a.s.}} 0.$$

Hence, letting $\ell > 0$ and recalling that $\tau_i v(\tau_i \gamma_N) = \psi(\tau_i \gamma_N)/\gamma_N$, for all large n a.s.

$$(1-\ell)\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma_N)}{\gamma_N}z_iz_i^* \preceq \frac{1}{n}\sum_{i=1}^{n}v(\tau_id_i)\tau_iz_iz_i^* \preceq (1+\ell)\frac{1}{n}\sum_{i=1}^{n}\frac{\psi(\tau_i\gamma_N)}{\gamma_N}z_iz_i^*. \tag{4.24}$$

Therefore, since $\gamma_N > \gamma_-$ and $\left\|\frac{1}{n}\sum_{i=1}^{n} z_iz_i^*\right\| < (1+\sqrt{c_+})^2$ for all large n a.s. (Bai and Silverstein, 1998),

$$\left\|\hat{C}_N - \hat{S}_N\right\| \leq \ell(1+\sqrt{c_+})^2\frac{\psi_\infty}{\gamma_-}$$

where $\hat{S}_N = \gamma_N^{-1}\frac{1}{n}\sum_{i=1}^{n}\psi(\tau_i\gamma_N)z_iz_i^*$. Since ℓ is arbitrary, the difference tends to zero a.s. as $n \to \infty$, which concludes the proof for $\tau_i < M$ a.s. and for $C_N = I_N$.

If $C_N \neq I_N$ is positive definite, remark simply that neither d_i nor γ_N are affected in their values, so that the effect of C_N first appears in (4.24) with z_i having $C_N \neq I_N$ as a covariance matrix. But then, in this case, since $\|\frac{1}{n}\sum_{i=1}^{n} z_iz_i^*\| < (1+\sqrt{c_+})^2 \limsup_N \|C_N\| < \infty$ (Assumption 4.2), the last arguments still hold true and the result is also proved for these C_N.

Note the importance of the assumption on ϕ being increasing and not simply non-decreasing (as in (Maronna, 1976)) to ensure that (4.23) is a strict inequality. If this were to be replaced by "≥ 1", no contradiction with (4.22) could be evoked. There does not seem to be any easy way to work this limitation around. Similar reasons explain why Tyler robust estimator discussed in Section 2 cannot be analyzed in the same way as Maronna estimator. All the same, when $\tau_1, \ldots, \tau_n$ have unbounded support with growing n, the left-hand side of (4.23) may equal one provided $\limsup_n \tau_n = \infty$, which is not excluded. For this reason, a specific treatment is necessary where the set of $\{\tau_i\}_{i=1}^n$ is split into a large bounded set of τ_i and a small set of large τ_i. This is the approach followed in the second part of the proof below.

Unbounded τ_i. We now relax the boundedness assumption on the support of the distribution of τ_1 and use Assumption 4.3 instead.

Since $\{\tilde{\nu}_n\}_{n=1}^{\infty}$ is tight, we can exhibit pairs (η, M_η) with $\eta \downarrow 0$ as $M_\eta \uparrow \infty$ such that, for all large n a.s. $\tilde{\nu}_n((M_\eta, \infty)) < \eta$. Let us fix such a pair (η, M_η) with η small and restrict ourselves to a subsequence where $\tilde{\nu}_n((M_\eta, \infty)) < \eta$ for all n. Denote $\mathcal{C}_\eta = \{i, \tau_i \leq M_\eta\}$ with cardinality $|\mathcal{C}_\eta|/n = 1 - \tilde{\nu}_n((M_\eta, \infty))$.

We follow the same steps as in the previous proof but differentiating between indices in $\mathcal{C}_\eta$ and indices in $\mathcal{C}_\eta^c$. Also we denote

$$e_i^\eta \triangleq \frac{v(\tau_id_i)}{v(\tau_i\gamma_N^\eta)}$$

where γ_N^η is the unique positive solution to the equation in γ

$$1 = \frac{1}{n}\sum_{i\in\mathcal{C}_\eta}\frac{\psi(\tau_i\gamma)}{1+c_N\psi(\tau_i\gamma)}.$$

Recall first from Remark 4.1 that the conclusions of Lemma 4.1 are still valid and importantly in what follows, that $\gamma_- < \gamma_N^\eta < \gamma_+$ for some $\gamma_-, \gamma_+ > 0$, for all large N irrespective of $\eta < \eta_0$ for some η_0 small. This uniform control of γ_N^η with respect to η plays a key role here. For the moment, we do not make explicit the sufficiently small value of η_0 that is needed in the following; all what will matter if that we can always choose η arbitrarily small from here.

Let $j \in \mathcal{C}_\eta$ and denote ψ_∞ any upper bound on ψ_∞^N for all N. Then, similar to (4.18), with $e_{\underline{1}}^\eta = \min_{i\in\mathcal{C}_\eta}\{e_i^\eta\}$ and $e_{\bar{n}}^\eta = \max_{i\in\mathcal{C}_\eta}\{e_i^\eta\}$,

$$\begin{aligned}
e_j^\eta &= \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) e_i^\eta z_i z_i^* + \frac{1}{n}\sum_{i\in\mathcal{C}_\eta^c} \tau_i v(\tau_i d_i) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j\gamma_N^\eta)} \\
&\leq \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) e_{\bar{n}}^\eta z_i z_i^* + \frac{1}{n}\frac{\psi_\infty}{d_-}\sum_{i\in\mathcal{C}_\eta^c} z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j\gamma_N^\eta)} \\
&= \frac{v\left(\frac{\tau_j}{e_{\bar{n}}^\eta} \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) z_i z_i^* + \frac{1}{n}\frac{\psi_\infty}{e_{\bar{n}}^\eta d_-}\sum_{i\in\mathcal{C}_\eta^c} z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j\gamma_N^\eta)}
\end{aligned}$$

where the first inequality uses $d_i > d_-$ for all large n a.s (Lemma 4.3). Since $e_{\bar{n}}^\eta = \frac{v(\tau_{\bar{n}} d_{\bar{n}})}{v(\tau_{\bar{n}}\gamma_N^\eta)} = \frac{\psi(\tau_{\bar{n}} d_{\bar{n}})}{\psi(\tau_{\bar{n}}\gamma_N^\eta)}\frac{\gamma_N^\eta}{d_{\bar{n}}}$, with the bounds derived previously (Remark 4.1 and Lemma 4.3), $e_{\bar{n}}^\eta$ is almost surely bounded and bounded away from zero for all large n a.s., irrespective of η small enough (if $\liminf_n \tau_{\bar{n}} = 0$, the first equality ensures $\liminf_n e_{\bar{n}}^\eta > 0$ while if $\limsup_n \tau_{\bar{n}} = \infty$, the second equality ensures $\limsup_n e_{\bar{n}}^\eta < \infty$). Thus, in particular, $e_{\bar{n}}^\eta > e_-$ for some $e_- > 0$ for all large n a.s. From this observation, for all large n a.s.

$$\begin{aligned}
e_j^\eta &\leq \frac{v\left(\frac{\tau_j}{e_{\bar{n}}^\eta} \frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) z_i z_i^* + \frac{1}{n}\frac{\psi_\infty}{d_- e_-}\sum_{i\in\mathcal{C}_\eta^c} z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j\gamma_N^\eta)} \\
&= \frac{v\left(\frac{\tau_j}{e_{\bar{n}}^\eta}\left[\frac{1}{N} z_j^* \left(\frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) z_i z_i^*\right)^{-1} z_j + \zeta_{j,n}\right]\right)}{v(\tau_j\gamma_N^\eta)} \qquad (4.25)
\end{aligned}$$

where we defined

$$\zeta_{j,n} \triangleq \frac{1}{N} z_j^* \left(A_{\eta,(j)} + B_\eta\right)^{-1} z_j - \frac{1}{N} z_j^* A_{\eta,(j)}^{-1} z_j$$

with

$$A_{\eta,(j)} \triangleq \frac{1}{n}\sum_{\substack{i\in\mathcal{C}_\eta\\ i\neq j}} \tau_i v(\tau_i\gamma_N^\eta) z_i z_i^*, \quad B_\eta \triangleq \frac{1}{n}\frac{\psi_\infty}{d_- e_-}\sum_{i\in\mathcal{C}_\eta^c} z_i z_i^*.$$

Note that $A_{\eta,(j)}^{-1}$ is well defined as $A_{\eta,(j)}$ is invertible for all large n a.s. provided η is small enough. Similar to the proof of Lemma 4.3, note first that, for some $\kappa > 0$ and for all $j \in \mathcal{C}_\eta$, $\lambda_1(A_{\eta,(j)}) > \kappa > 0$ for all large n a.s. Indeed, with the same derivation as (4.14), for any $m > 0$ satisfying $\tilde{\nu}_n([m, M_\eta]) > c_+$ for all n a.s. (this may require M_η large enough), $\lambda_1(A_{\eta,(j)}) \geq mv(m\gamma_+)\lambda_1(\frac{1}{n}\sum_{\tau_i \in [m,M_\eta], i\neq j} z_i z_i^*)$ away from zero for all large n a.s., independently of η small enough (Lemma A.4). Then, since B_η is of maximum rank $|C_\eta^c| = \tilde{\nu}_n((M_\eta, \infty))$, the successive applications of Lemma A.3 and Lemma A.2 (see the similar steps in the proof of Lemma 4.3) lead to

$$\max_{j \in \mathcal{C}_\eta} |\zeta_{nj}| \leq K\tilde{\nu}_n((M_\eta, \infty)) \tag{4.26}$$

for some $K > 0$ constant, independent of η.

Now that $\zeta_{j,n}$ is controlled for all $j \in \mathcal{C}_\eta$, we can proceed similar to the proof in the bounded τ_i case. First, for any fixed $\eta > 0$ small enough, Remark 4.2 ensures that there exists a sequence $\varepsilon_n^\eta \downarrow 0$, such that a.s.

$$\max_{j \in \mathcal{C}_\eta} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta, i \neq j} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j - \gamma_N^\eta \right| \leq \varepsilon_n^\eta. \tag{4.27}$$

Combining (4.25), (4.26), and (4.28), we then have for all large n a.s. and for all $j \in \mathcal{C}_\eta$

$$e_j^\eta \leq \frac{v\left(\frac{\tau_j}{e_{\bar{n}}^\eta}\left(\gamma_N^\eta - \varepsilon_n^\eta - K\tilde{\nu}_n((M_\eta, \infty))\right)\right)}{v(\tau_j \gamma_N^\eta)} \tag{4.28}$$

which, for $j = \bar{n}$, is

$$e_{\bar{n}}^\eta \leq \frac{v\left(\frac{\tau_{\bar{n}}}{e_{\bar{n}}^\eta}\left(\gamma_N^\eta - \varepsilon_n^\eta - K\tilde{\nu}_n((M_\eta, \infty))\right)\right)}{v(\tau_{\bar{n}} \gamma_N^\eta)}.$$

Using the definition of ψ, this reads equivalently

$$\left(1 - \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}}\gamma_N^\eta)}{\psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\left(1 - \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta,\infty))}{\gamma_N^\eta}\right)\right)} < 1$$

which implies, from the growth of ψ,

$$\left(1 - \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}}\gamma_N^\eta)}{\psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\right)} < 1.$$

Adding $\frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta,\infty))}{\gamma_N^\eta} - 1$ on both sides, this further reads

$$\left(1 - \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}}\gamma_N^\eta) - \psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\right)}{\psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\right)} < \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta, \infty))}{\gamma_N^\eta}.$$

or equivalently, if η is taken small enough (recalling that $\gamma_N^\eta > \gamma_-$ uniformly on η small),

$$\frac{\psi(\tau_{\bar{n}}\gamma_N^\eta) - \psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\right)}{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta,\infty))} < \frac{\psi\left((e_{\bar{n}}^\eta)^{-1}\tau_{\bar{n}}\gamma_N^\eta\right)}{\gamma_N^\eta\left(1 - \frac{\varepsilon_n^\eta + K\tilde{\nu}_n((M_\eta,\infty))}{\gamma_N^\eta}\right)} < \frac{2\psi_\infty}{\gamma_-} \tag{4.29}$$

where the right-most bound holds for all large n a.s. provided η is chosen small enough.

Assume $\limsup_n e_{\bar{n}}^\eta > 1 + \ell$ for some $\ell > 0$. Then one must have $\liminf_n \tau_{\bar{n}} > \tau_-$ for (4.28) to remain valid, with $\tau_- > 0$ independent of η small since $\gamma_- < \gamma_N^\eta < \gamma_+$ for all n large a.s., both bounds being independent of η. Since $\tau_{\bar{n}}\gamma_N^\eta$ belongs to $[\tau_-\gamma_-, M_\eta\gamma_+]$ for all large N a.s., taking the limit of (4.29) over some converging subsequence over which $\tau_{\bar{n}}\gamma_N^\eta \to x^\eta \in [\tau_-\gamma_-, M_\eta\gamma_+]$, $c_N \to c$, and $\tilde{\nu}_n((M_\eta,\infty))$ converges, ensures that

$$\frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right)}{\lim_n \tilde{\nu}_n((M_\eta,\infty))} \leq K' \tag{4.30}$$

for $K' > 0$ independent of η, with $\psi_c = \lim_{c_N \to c}\psi$.

We now operate on η. If $\limsup_{\eta\to 0} x^\eta < \infty$, the left-hand side in (4.30) diverges to ∞ as $\eta \to 0$ so that, starting with an η sufficiently small and taking the limit over n on the subsequence under consideration raises a contradiction. If instead $\limsup_{\eta\to 0} x^\eta = \infty$, then, since $x^\eta \leq M_\eta\gamma_+$,

$$\frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right)}{\lim_n \tilde{\nu}_n((M_\eta,\infty))} \geq \frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right)}{\lim_n \tilde{\nu}_n((\frac{x^\eta}{\gamma_+},\infty))}.$$

Call $y^\eta = g^{-1}(x^\eta)$. Recalling that $\psi_c(t) = \phi(g^{-1}(t))(1 - c\phi(g^{-1}(t)))^{-1}$, we get

$$\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right) = \frac{\phi(y^\eta) - \phi\left(g^{-1}\left[\frac{1}{1+\ell}g(y^\eta)\right]\right)}{(1 - c\phi(y^\eta))(1 - c\phi(g^{-1}[\frac{1}{1+\ell}g(y^\eta)]))}.$$

Now, letting $\kappa > 0$ small, for all large t, $g(t) < t(1 - c\phi_\infty)^{-1}(1+\kappa)$ and similarly $g^{-1}(t) < t(1 - c\phi_\infty)(1+\kappa)$. Hence, letting κ small enough, for all large y^η, we have, say,

$$\phi\left(g^{-1}\left[\frac{1}{1+\ell}g(y^\eta)\right]\right) < \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right).$$

Moreover, using $0 < 1 - c\phi(t) < 1$, we have $(1 - c\phi(t))^{-1} > 1$. Using these results now gives, for all large y^η,

$$\begin{aligned}\frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right)}{\lim_n \tilde{\nu}_n((\frac{x^\eta}{\gamma_+},\infty))} &> \frac{\phi(y^\eta) - \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right)}{\lim_n \tilde{\nu}_n((\frac{y^\eta}{\gamma_+(1-c\phi(y^\eta))},\infty))}\\ &> \frac{\phi(y^\eta) - \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right)}{\lim_n \tilde{\nu}_n((\frac{y^\eta}{\gamma_+},\infty))}.\end{aligned}$$

Since $y^\eta \to \infty$ as $x^\eta \to \infty$, from Assumption 4.3, the right-hand side must go to ∞ as $x^\eta \to \infty$, or equivalently as $\eta \to 0$. Therefore, taking η sufficiently small from the beginning and then bringing n large on the subsequence under study leads to a contradiction. Consequently, we must have $\limsup_n e_{\bar{n}}^\eta \leq 1 + \ell$ a.s. A similar reasoning shows that $\liminf_n e_{\bar{1}}^\eta \geq 1 - \ell$ a.s., for any given $\ell > 0$. We conclude that

$$\max_{j \in \mathcal{C}_\eta} \left| e_j^\eta - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

We now have to deal with e_j^η for $j \in \mathcal{C}_\eta^c$. For such a j,

$$d_j = \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \tau_i v(\tau_i \gamma_N^\eta) e_i^\eta z_i z_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\eta^c, i \neq j} \frac{\psi(\tau_i d_i)}{d_i} z_i z_i^* \right)^{-1} z_j.$$

But then, from the same reasoning as with the $\zeta_{j,n}$ above (using in particular the uniform boundedness of d_i) and from $\max_{i \in \mathcal{C}_\eta} |e_i^\eta - 1| \xrightarrow{\text{a.s.}} 0$, we have

$$\max_{j \in \mathcal{C}_\eta^c} \left| d_j - \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j \right| < K \tilde{\nu}_n((M_\eta, \infty)) < K\eta$$

for some $K > 0$ independent of η, which further implies from Remark 4.2 that for all large n a.s. and for all $j \in \mathcal{C}_\eta^c$,

$$\gamma_N^\eta - K\eta \leq d_j \leq \gamma_N^\eta + K\eta.$$

Using the definition $e_j^\eta = \frac{\psi(\tau_j d_j)}{\psi(\tau_j \gamma_N^\eta)} \frac{\gamma_N^\eta}{d_j}$, the uniform bounds on γ_n^η, and the continuous growth of ψ shows finally that, a.s.

$$\limsup_n \max_{j \in \mathcal{C}_\eta^c} \left\{ \left| e_j^\eta - 1 \right| \right\} \leq \eta'$$

for some $\eta' > 0$ with $\eta' \to 0$ as $\eta \to 0$.

Gathering the results for $j \in \mathcal{C}_\eta$ and $j \in \mathcal{C}_\eta^c$, we therefore conclude that, for each $\ell > 0$, there exists $\eta > 0$ small enough such that a.s.

$$1 - \ell < \liminf_n \min_{1 \leq i \leq n} e_i^\eta \leq \limsup_n \max_{1 \leq i \leq n} e_i^\eta < 1 + \ell.$$

For such η small, we then have, by definition of e_i^η and from $\tau_i v(\tau_i \gamma_N^\eta) = \psi(\tau_i \gamma_N^\eta)/\gamma_N^\eta$,

$$(1-\ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N^\eta)}{\gamma_N^\eta} z_i z_i^* \preceq \frac{1}{n} \sum_{i=1}^n v(\tau_i d_i) \tau_i z_i z_i^* \preceq (1+\ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N^\eta)}{\gamma_N^\eta} z_i z_i^*. \tag{4.31}$$

It now remains to show that, for each $\varepsilon > 0$, there exists $\eta > 0$ for which $|\gamma_N^\eta - \gamma_N| < \varepsilon$ for all n large a.s. For this, observe that, by definition of γ_N and γ_N^η,

$$1 = \frac{1}{n}\sum_{i\in\mathcal{C}_\eta} \frac{\psi(\tau_i\gamma_N^\eta)}{1+c_N\psi(\tau_i\gamma_N^\eta)} = \frac{1}{n}\sum_{i=1}^n \frac{\psi(\tau_i\gamma_N)}{1+c_N\psi(\tau_i\gamma_N)}$$

so that, since $\psi/(1+c_N\psi)$ is increasing, we obtain $\gamma_N \leq \gamma_N^\eta$ and

$$\frac{1}{n}\sum_{i\in\mathcal{C}_\eta^c} \frac{\psi(\tau_i\gamma_N)}{1+c_N\psi(\tau_i\gamma_N)} = \frac{1}{n}\sum_{i\in\mathcal{C}_\eta} \frac{\psi(\tau_i\gamma_N^\eta)-\psi(\tau_i\gamma_N)}{(1+c_N\psi(\tau_i\gamma_N))(1+c_N\psi(\tau_i\gamma_N^\eta))} \geq 0.$$

Take an interval $[m,M]$, $M < M_\eta$ (chosen once for all, independently of M_η large), with $\tilde{\nu}_n([m,M]) > \kappa > 0$ for all large n a.s. (possible from Assumption 4.2–2). Then we can further write

$$\begin{aligned}\frac{1}{n}\sum_{i\in\mathcal{C}_\eta^c} \frac{\psi(\tau_i\gamma_N)}{1+c_N\psi(\tau_i\gamma_N)} &\geq \frac{1}{(1+c_+\psi_\infty)^2}\frac{1}{n}\sum_{\tau_i\in[m,M]}\big(\psi(\tau_i\gamma_N^\eta)-\psi(\tau_i\gamma_N)\big)\\ &\geq \frac{\kappa}{2(1+c_+\psi_\infty)^2}\min_{x\in[m,M]}\big(\psi(x\gamma_N^\eta)-\psi(x\gamma_N)\big)\end{aligned}$$

with the second inequality valid for all large n a.s. Now, for sufficiently small η, the left-hand side can be made arbitrarily small. Since γ_N and γ_N^η are uniformly bounded and bounded away from zero (irrespective of η small), if $|\gamma_N^\eta - \gamma_N|$ were uniformly away from zero for all η small, so would be the right-hand side, which is in contradiction with our previous statement. Therefore, for each $\varepsilon > 0$, one can choose η so that $|\gamma_N - \gamma_N^\eta| < \varepsilon$ for all n large a.s.

Now, by uniform continuity of ψ on bounded intervals along with the fact that $\psi(x) \uparrow \psi_\infty$, from (4.31), taking η small enough, for all large n a.s.

$$(1-\ell)^2\frac{1}{n}\sum_{i=1}^n \frac{\psi\left(\tau_i\gamma_N\right)}{\gamma_N}z_iz_i^* \preceq \hat{C}_N \preceq (1+\ell)^2\frac{1}{n}\sum_{i=1}^n \frac{\psi\left(\tau_i\gamma_N\right)}{\gamma_N}z_iz_i^* \tag{4.32}$$

which therefore implies, with the same arguments as in the case τ_i bounded, that $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$, when $C_N = I_N$. The arguments of the case τ_i bounded still hold for $C_N \neq I_N$ satisfying Assumption 4.2-3). This completes the proof.

4.2 Application: Robust G-MUSIC

With the results of Section 4.1, we fulfilled the first step in providing improved robust estimators for functionals $f(C_N)$ of population covariance matrices in the large N, n regime. As discussed in Chapter 2, our hope is now that $\hat{S}_N$, the random equivalent for $\hat{C}_N$, may be used in a plug-and-play manner to provide such estimates of $f(C_N)$.

This chapter provides two classes of estimators, for eigenvalues and eigenvectors, in a specific array processing context. For the sake of coherence in modelling though, the system model will be extended from that of Section 4.1 in such a way that the estimators will not be exactly of the form $f(C_N)$. In technical terms, we shall consider a spiked information-plus-noise random matrix generalization of the model for the x_i's and we shall retrieve information from the deterministic information part; we may have instead considered a multiplicative spiked model extension of the x_i's, which would have precisely led to retrieving information of the form $f(C_N)$ although the modelling (of random sources) would have been less general in practice.

Before delving into the technical part of the section, let us provide some reminders on spiked models and more importantly on what one should expect from spiked model extensions of sample covariance matrix versus robust covariance matrix in the presence of impulsive data. Since robust or non-robust covariance matrices following or not following a spiked model will need be discussed, we shall take some different notations in this section than in Section 4.1 for the sake of readability. To start with, the robust covariance matrix for zero mean elliptical signals as studied in Section 4.1, will now be denoted $\hat{C}_N^\circ$, while the same robust covariance matrix for non-zero mean elliptical signals (of interest here) will be denoted $\hat{C}_N$ and is the object of central interest. All the same we shall denote $\hat{S}_N^\circ$ the random equivalent for $\hat{C}_N^\circ$ and shall see that $\hat{C}_N$ has similarly a random equivalent that will be denoted $\hat{S}_N$.

To help understanding, in the first lines of the next section, hinging on the fact that $\hat{S}_N^\circ$ and $\hat{S}_N$ are merely random matrices of the separate covariance model and its spiked extension, we shall reuse these two notations in this wider sense.

4.2.1 Introduction

Spiked models are small rank perturbations of classical simple random matrix models (such as models having i.i.d. entries). The initial study of such models (Baik and Silverstein, 2006) for matrices of the type $\hat{S}_N = \frac{1}{n}(I_N + A)WW^*(I_N + A^*)$, where $W \in \mathbb{C}^{N\times n}$ has independent and identically distributed (i.i.d.) zero mean, unit variance, and finite fourth moment entries and A has fixed rank L, has shown that, as $N, n \to \infty$ with $N/n \to c \in (0, \infty)$, $\hat{S}_N$ may exhibit up to L isolated eigenvalues strictly away from the *bounded* support of the limiting empirical distribution μ of $\hat{S}_N^\circ = \frac{1}{n}WW^*$, while the other eigenvalues of $\hat{S}_N$ get densely compacted in the support of μ. This result has triggered multiple works on various low rank perturbation models for Gram, Wigner, or general square random matrices (Benaych-Georges and Rao, 2011; Paul, 2007; Benaych-Georges et al., 2010) with similar conclusions. Of particular interest to us here is the information-plus-noise model $\hat{S}_N = \frac{1}{n}(W + A)(W + A)^*$ introduced in (Benaych-Georges and Rao, 2011) which is closer to our present model. Other generalizations explored the direction of turning W into the more general $WT^{\frac{1}{2}}$ model for $T = \mathrm{diag}(\tau_1, \ldots, \tau_n) \succeq 0$, such that $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i} \to \tilde{\nu}$ weakly, where $\tilde{\nu}$ has bounded support $\mathrm{supp}(\tilde{\nu})$ and $\max_i\{\mathrm{dist}(\tau_i, \mathrm{supp}(\tilde{\nu}))\} \to 0$ (Chapon et al., 2014). In this scenario again, thanks to the fundamental assumption that no τ_i can escape $\mathrm{supp}(\tilde{\nu})$ asymptotically, only finitely many eigenvalues of $\hat{S}_N$ can be found away from the support of the limiting spectral distribution of $\frac{1}{n}WTW^*$, and these eigenvalues are intimately linked to A.

The major interest of the spiked models in practice is twofold. First, if the (non observable) perturbation matrix A constitutes the relevant information to the system observer, then the observable isolated eigenvalues and associated eigenvectors of $\hat{S}_N$ contain information about A. These isolated eigenvalues and eigenvectors are therefore important objects to characterize. Moreover, since $\hat{S}_N$ has the same limiting spectrum as that of simple random matrix models, this characterization is usually quite easy and leads to tractable expressions and computationally efficient algorithms. This led to notable contributions to statistical inference and in particular to detection and estimation techniques for signal processing (Mestre, 2008b; Nadler, 2010; Hachem et al., 2013; Couillet and Hachem, 2013).

However, from the discussion of the first paragraph, these works have a few severe practical limitations in that: (i) the support of the limiting spectral distribution of $\hat{S}_N$ must be bounded for isolated eigenvalues to be detectable and exploitable and (ii) no eigenvalue of $\hat{S}_N^\circ$ (the unperturbed model) can be isolated, to avoid risking a confusion between isolated eigenvalues of $\hat{S}_N$ arising from A and isolated eigenvalues of $\hat{S}_N$ intrinsically linked to $\hat{S}_N^\circ$. This therefore rules out the possibility to straightforwardly extend these techniques in practice to impulsive noise models $WT^{\frac{1}{2}}$ where $T = \operatorname{diag}(\tau_1, \ldots, \tau_n)$ with either τ_i i.i.d. arising from a distribution with unbounded support or $\tau_i = 1$ for all but a few indices i. In the former case, the support of the limiting spectrum of $\hat{S}_N^\circ$ is unbounded from Proposition 3.3.4 in Chapter 3, therefore precluding information detection, while in the latter spurious eigenvalues in the spectrum of $\hat{S}_N$ may arise that are also found in $\hat{S}_N^\circ$ and therefore constitute false information (note that this case can be seen as one where low rank perturbations are present *both in the population and in the sample directions* which cannot be discriminated). Such impulsive models are nonetheless fundamental in many applications ranging from statistical finance to radar array processing, where impulsive samples are classically met.

As already discussed, the natural way to handle impulsive data is by means of robust estimators. In particular, from the results of Section 4.1, it importantly appears that the limiting spectrum distribution of $\hat{C}_N^\circ$ always has bounded support, irrespective of the impulsiveness of the samples. In particular, it is clear that, asymptotically, isolated eigenvalues of $\hat{C}_N^\circ$ (arising from isolated τ_i) can be found away from the support but that none of the eigenvalues can exceed a fixed finite value dictated by the boundedness of the function ψ.

In the present section, we extend the model of the data x_i's by adding to it a deterministic part, intervening in the end as a finite rank perturbation A to the robust estimator of scale $\hat{C}_N^\circ$, the resulting matrix being denoted $\hat{C}_N$. As opposed to non-robust models, it shall appear (quite surprisingly on the onset) that $\hat{C}_N$ now allows for finitely many isolated eigenvalues to appear beyond the aforementioned fixed finite value (referred from now on to as the detection threshold), these eigenvalues being related to A. This holds even if $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i}$ has unbounded support in the large n regime. As such, any isolated eigenvalue of $\hat{C}_N$ found below the detection threshold may carry information about A or may merely be an outlier due to an isolated τ_i (as in the non-robust context) but any eigenvalue found beyond the detection threshold necessarily carries information about A. This has important consequences in practice as now low rank perturbations *in the sample direction* are appropriately harnessed by the robust estimator while the (more relevant) low rank perturbations *in the population direction* can be properly estimated. We shall introduce an application of these results to array processing by providing two novel estimators for the power and steering direction of signal sources captured by a large sensor array under impulsive noise.

The contribution of this section thus lies on both theoretical and practical grounds. We first introduce in Theorem 4.2.1 the generalization of Theorem 4.1.2 to the perturbed model $\hat{C}_N$ which we precisely define in Section 4.2.2. The main results are then contained in Section 4.2.3. In this section, Theorem 4.2.2 provides the localization of the eigenvalues of $\hat{C}_N$ in the large system regime along with associated population eigenvalue and eigenvector estimators when the limiting distribution for $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i}$ is known. This result is then extended in Theorem 4.2.3 thanks to a two-step estimator where the τ_i are directly estimated. A practical application of these novel methods to the context of steering angle estimation for array processing is then provided, leading to an improved algorithm referred to as robust G-MUSIC. Simulation results in this context are then displayed that confirm the improved performance of using robust schemes versus traditional sample covariance matrix-based techniques.

4.2.2 Model and Motivation

Let $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$, we consider the following statistical model

$$x_i = \sum_{l=1}^{L} \sqrt{p_l} a_l s_{li} + \sqrt{\tau_i} w_i \tag{4.33}$$

with $x_i \in \mathbb{C}^N$ satisfying the following hypotheses.

Assumption 4.4. *The vectors $x_1, \ldots, x_n \in \mathbb{C}^N$ satisfy the following conditions:*

1. *$\tau_1, \ldots, \tau_n \in (0, \infty)$ are random scalars such that $\tilde{\nu}_n \triangleq \frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i} \to \tilde{\nu}$ weakly, almost surely, where $\int t\tilde{\nu}(dt) = 1$;*
2. *$w_1, \ldots, w_n \in \mathbb{C}^N$ are random independent unitarily invariant $\sqrt{N}$-norm vectors, independent of $\tau_1, \ldots, \tau_n$;*
3. *$L \in \mathbb{N}$, $p_1 \geq \ldots \geq p_L \geq 0$ are deterministic and independent of N*
4. *$a_1, \ldots, a_L \in \mathbb{C}^N$ are deterministic or random and such that*
$$A^* A \xrightarrow{\text{a.s.}} \operatorname{diag}(p_1, \ldots, p_L)$$
as $N \to \infty$, with $A \triangleq [\sqrt{p_1} a_1, \ldots, \sqrt{p_L} a_L] \in \mathbb{C}^{N \times L}$
5. *$s_{1,1}, \ldots, s_{Ln} \in \mathbb{C}$ are independent with zero mean, unit variance, and uniformly bounded moments of all orders.*

For further use, we shall define

$$A_i \triangleq \begin{bmatrix} \sqrt{p_1} a_1 & \ldots & \sqrt{p_L} a_L & \sqrt{\tau_i} I_N \end{bmatrix} \in \mathbb{C}^{N \times (N+L)}.$$

In particular, $A_i A_i^* = AA^* + \tau_i I_N$.

Remark 4.3 (Application contexts)**.** *The system (4.33) can be adapted to multiple scenarios in which the s_{li} model scalar signals or data originated from L sources of respective powers $p_1, \ldots, p_L$ carried by the vectors $a_1, \ldots, a_L$, while the $\sqrt{\tau_i} w_i$ model additive impulsive noise. Two examples are:*

- *wireless communication channels in which signals s_{li} originating from L transmitters are captured by an N-antenna receiver. The vectors a_l are here random independent channels for which it is natural to assume that $a_l^* a_{l'} \to \boldsymbol{\delta}_{l-l'}$ (e.g., for independent $a_l \sim \mathcal{CN}(0, I_N/N)$);*
- *array processing in which L sources emit signals s_{li} captured by an antenna array through steering vectors $a_l = a(\theta_l)$ for a given $a(\theta)$ function and angles of arrival $\theta_1, \ldots, \theta_L \in [0, 2\pi)$. In the case of uniform linear arrays with inter-antenna distance d, $[a(\theta)]_j = N^{-\frac{1}{2}} \exp(2\pi \imath d j \sin(\theta))$.*

The noise impulsiveness is translated by the τ_i coefficients. The vectors $\sqrt{\tau_i} w_i$ are for instance i.i.d. elliptic random vectors if the τ_i are i.i.d. with absolutely continuous measure $\tilde{\tilde{\nu}}_n$ having a limit $\tilde{\tilde{\nu}}$ (in which case, we easily verify that $\tilde{\nu}_n \to \tilde{\nu} = \tilde{\tilde{\nu}}$ a.s. This particularizes to additive white Gaussian noise if $2N\tau_i$ is chi-square with $2N$ degrees of freedom (in this case, $\tilde{\nu} = \boldsymbol{\delta}_1$). Of interest in this section is however the scenarios where $\tilde{\nu}$ has unbounded support, e.g., when the τ_i are either random i.i.d. and heavy-tailed or contain a few arbitrarily large outliers, which both correspond to impulsive noise scenarios.

Remark 4.4 (Technical comments)**.** *From a purely technical perspective, it is easily seen from the proofs of our main results in Section 4.2.4 that some of the items of Assumption 4.4 could have been relaxed. In particular, Item (4) could have been relaxed into "all accumulation points of A^*A are similar to $\operatorname{diag}(q_1, \ldots, q_L)$ for given $q_1 \geq \ldots \geq q_L$" as in e.g., (Chapon et al., 2014). Also, similar to Section 4.1, the convergence of $\tilde{\nu}_n$ in Item (1) could be relaxed to the cost of introducing a tightness condition on the sequence $\{\tilde{\nu}_n\}_{n=1}^{\infty}$ and to loose the convergence of measure in the discussion following Theorem 4.2.1. For readability and since Assumption 4.4 gathers most of the scenarios of interest, we restrict ourselves to those (already quite general) hypotheses.*

We now define the robust estimate of scatter $\hat{C}_N$. We start by denoting $u : [0, \infty) \to (0, \infty)$ any function satisfying the following hypotheses.

Assumption 4.5. *The function u is characterized by*

1. *u is continuous, nonnegative, and non-increasing from $[0, \infty)$ onto $(0, u(0)] \subset (0, \infty)$;*
2. *for $x \geq 0$, $\phi(x) \triangleq x u(x)$ is increasing and bounded with*

$$\phi_\infty \triangleq \lim_{x \to \infty} \phi(x) > 1$$

3. *there exists $m > 0$ such that $\tilde{\nu}([0, m)) < 1 - \phi_\infty^{-1}$;*

4. for all $a > b > 0$,

$$\limsup_{t\to\infty} \frac{\tilde{\nu}((t,\infty))}{\phi(at)-\phi(bt)} = 0.$$

These assumptions are the same as in Section 4.1 which are therefore not altered by the updated model (4.33).

The function u being given, we now define $\hat{C}_N$, when it exists, as the unique solution to the fixed-point matrix-valued equation in Z:

$$Z = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{1}{N}x_i^* Z^{-1} x_i\right) x_i x_i^*.$$

For $i \in \{1,\dots,N\}$, we shall denote $\hat{\lambda}_i \triangleq \lambda_i(\hat{C}_N)$ and $\hat{u}_i \in \mathbb{C}^N$ the i-th largest eigenvalue of $\hat{C}_N$ and its associated eigenvector.

We shall assume the following system growth regime, which we take simpler than in the previous section as complications on this point are not necessary.

Assumption 4.6. *The integer $N = N(n)$ is such that $c_n \triangleq N/n$ satisfies*

$$\lim_{n\to\infty} c_n = c \in (0, \phi_\infty^{-1}).$$

Meanwhile, L remains constant independently of N, n.

As in Section 4.1, under Assumptions 4.4–4.6, $\hat{C}_N$ is easily shown to be almost surely well defined for all large n a.s. Also, we recall that $\hat{C}_N$ can be written (at least for all large n) in the technically more convenient form

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{n} v\left(\frac{1}{N}x_i^* \hat{C}_{(i)}^{-1} x_i\right) x_i x_i^*$$

where $v : x \mapsto u \circ g^{-1}$, $g : x \mapsto x/(1 - c_n\phi(x))$, and $\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n}u\left(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i\right)x_i x_i^*$. Similar to Section 4.1, we shall further denote $\psi(x) = xv(x)$. Recall that v is non-increasing while ψ is increasing with limit $\psi_\infty = \phi_\infty/(1 - c_n\phi_\infty)$.

With these definitions in place, we are now in position to present our main results.

4.2.3 Main Results

The first objective of the section is to study the spectrum of $\hat{C}_N$ and in particular its largest eigenvalues $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_L$ and associated eigenvectors $\hat{u}_1,\dots,\hat{u}_L$, in the large N, n regime. This study will in turn allow us to retrieve information on $p_1,\dots,p_L$ and $a_1,\dots,a_L$. As an application, a novel improved angle estimator for array processing will then be provided.

4.2.3.1 Localisation and estimation

Our first result is an extension of Theorem 4.1.2.

Theorem 4.2.1 (Asymptotic model equivalence)**.** *Let Assumptions 4.4, 4.5, and 4.6 hold. Then*

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n}\sum_{i=1}^{n} v_c(\tau_i\gamma)A_i\bar{w}_i\bar{w}_i^* A_i^*$$

with γ the unique solution to

$$1 = \int \frac{\psi_c(t\gamma)}{1 + c\psi_c(t\gamma)}\tilde{\nu}(dt)$$

v_c and ψ_c the limits of v and ψ as $c_n \to c$,[6] *and $\bar{w}_i = [s_{1i}, \ldots, s_{Li}, w_i r_i/\sqrt{N}]^\mathsf{T}$, with $r_i \geq 0$ such that $2Nr_i^2$ is a chi-square random variable with $2N$ degrees of freedom, independent of w_i.*[7]

Remark 4.5 (From robust estimator to sample covariance matrix)**.** *Note that, if the function v_c in the expression of $\hat{S}_N$ were replaced by the constant 1 (and $r_i/\sqrt{N}$ set to one), $\hat{S}_N$ would be the classical sample covariance matrix of $x_1, \ldots, x_n$. Although it is here highly non rigorous to let v_c tend to 1 uniformly in Theorem 4.2.1, this remark somewhat reveals the classical robust estimation intuition according to which the larger ϕ_∞ (as a consequence of u and v_c being close to 1) the less robust $\hat{C}_N$.*

As a corollary of Theorem 4.2.1, we have

$$\max_{1\leq i\leq N}\left|\hat{\lambda}_i - \lambda_i(\hat{S}_N)\right| \xrightarrow{\text{a.s.}} 0 \tag{4.34}$$

(which unfolds from applying (Horn and Johnson, 1985, Theorem 4.3.7)) and therefore all eigenvalues of $\hat{C}_N$ can be accurately controlled through the eigenvalues of $\hat{S}_N$.

Let us assume for a moment that $p_1 = \ldots = p_L = 0$. Then, from Theorem 4.2.1, Assumption 4.4, and the results of Chapter 3 (in fact from the earlier results of (Silverstein and Choi, 1995)), $\mu_n \triangleq \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\delta}_{\hat{\lambda}_i} \to \mu$ weakly, a.s., where μ has a density on $\mathbb{R}$ with bounded support $\operatorname{supp}(\mu) \subset \mathbb{R}_+$. Denote

$$\begin{aligned}
S_\mu^- &\triangleq \inf(\operatorname{supp}(\mu)) \\
S_\mu^+ &\triangleq \sup(\operatorname{supp}(\mu)) \\
S^+ &\triangleq \frac{\phi_\infty(1+\sqrt{c})^2}{\gamma(1-c\phi_\infty)}.
\end{aligned}$$

[6] Note that this notation diverges slightly from that of Theorem 4.1.2, where v and not v_c was considered. This makes little practical difference but for the simplicity of v_c not depending on n.

[7] Note that $w_i r_i/\sqrt{N}$ as defined above is a standard Gaussian vector and therefore $\bar{w}_i$ has independent entries of zero mean and unit variance. In fact, the result can be equivalently formulated with $\bar{w}_i$ replaced by $\bar{\bar{w}}_i \triangleq [s_{1i}, \ldots, s_{Li}, w_i]^\mathsf{T}$, but the former vector, having independent entries, is of more interest statistically.

Since $\tau_i v_c(\tau_i\gamma) = \gamma^{-1}\psi_c(\tau_i\gamma) < \gamma^{-1}\psi_{c,\infty}$ with $\psi_{c,\infty} = \phi_\infty/(1-c\phi_\infty)$, we have

$$\hat{S}_N \preceq \frac{\phi_\infty}{\gamma(1-c\phi_\infty)}\frac{1}{n}\sum_{i=1}^n \tilde{w}_i\tilde{w}_i^*$$

with $\tilde{w}_i = w_i r_i/\sqrt{N}$, so that, according to (Marčenko and Pastur, 1967; Bai and Silverstein, 1998) and (4.34), for each $\varepsilon > 0$, $\hat{\lambda}_1 < S^+ + \varepsilon$ for all large n a.s. Of course, $S^+ \geq S_\mu^+$. If in addition $\max_{1\leq i\leq n}\{\mathrm{dist}(\tau_i, \mathrm{supp}(\tilde{\nu}))\} \xrightarrow{\text{a.s.}} 0$, then from (Bai and Silverstein, 1998), we even have $\hat{\lambda}_1 \xrightarrow{\text{a.s.}} S_\mu^+$; but this constraint is of little practical interest so that in general one may have $S_\mu^+ < \hat{\lambda}_1 < S^+$ infinitely often.

Coming back to generic values for $p_1,\ldots,p_L$, the idea of the results below is that, for sufficiently large $p_1,\ldots,p_L$, the eigenvalues $\hat{\lambda}_1,\ldots,\hat{\lambda}_L$ may exceed $S^+ + \varepsilon$ and contain information to estimate $p_1,\ldots,p_L$ as well as bilinear forms involving $a_1,\ldots,a_L$. The exact location of the eigenvalues and the value of these estimates shall be expressed as a function of the fundamental object $\delta(x)$, defined for $x \in \mathbb{R}^* \setminus [S_\mu^-, S_\mu^+]$ as the unique real solution to

$$\delta(x) = c\left(-x + \int \frac{tv_c(t\gamma)}{1+\delta(x)tv_c(t\gamma)}\tilde{\nu}(dt)\right)^{-1}.$$

The function $\delta(x)$ is the restriction to $\mathbb{R}^* \setminus [S_\mu^-, S_\mu^+]$ of the Stieltjes transform of $c\mu + (1-c)\boldsymbol{\delta}_0$ and is, as such, increasing on $(S^+,\infty) \subset (S_\mu^+,\infty)$; see Chapter 3 and Section 4.2.4 for details. Therefore, the following definition of p_-, which will be referred to as the detectability threshold, is licit

$$p_- \triangleq \lim_{x\downarrow S^+} -c\left(\int \frac{\delta(x)v_c(t\gamma)}{1+\delta(x)tv_c(t\gamma)}\tilde{\nu}(dt)\right)^{-1}.$$

We shall further denote $\mathcal{L} \triangleq \{j, p_j > p_-\}$.

We are now in position to provide our main results.

Theorem 4.2.2 (Robust estimation under known $\tilde{\nu}$)**.** *Let Assumptions 4.4, 4.5, and 4.6 hold. Denote u_k the eigenvector associated with the k-th largest eigenvalue of AA^* (in case of multiplicity, take any vector in the eigenspace with $u_1,\ldots,u_L$ orthogonal) and $\hat{u}_1,\ldots,\hat{u}_N$ the eigenvectors of $\hat{C}_N$ respectively associated with the eigenvalues $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_N$. Then, we have the following three results.*

0. Extreme eigenvalues. *For each $j \in \mathcal{L}$,*

$$\hat{\lambda}_j \xrightarrow{\text{a.s.}} \Lambda_j > S^+$$

while $\limsup_n \hat{\lambda}_{|\mathcal{L}|+1} \leq S^+$ *a.s., where Λ_j is the unique positive solution to*

$$-c\left(\delta(\Lambda_j)\int \frac{v_c(\tau\gamma)}{1+\delta(\Lambda_j)\tau v_c(\tau\gamma)}\tilde{\nu}(d\tau)\right)^{-1} = p_j.$$

1. Power estimation. *For each $j \in \mathcal{L}$,*

$$-c\left(\delta(\hat{\lambda}_j)\int \frac{v_c(\tau\gamma)}{1+\delta(\hat{\lambda}_j)\tau v_c(\tau\gamma)}\tilde{\nu}(d\tau)\right)^{-1} \xrightarrow{\text{a.s.}} p_j.$$

2. Bilinear form estimation. *For each $a, b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$, and $j \in \mathcal{L}$*

$$\sum_{k, p_k = p_j} a^* u_k u_k^* b - \sum_{k, p_k = p_j} w_k a^* \hat{u}_k \hat{u}_k^* b \xrightarrow{\text{a.s.}} 0$$

where

$$w_k = \frac{\displaystyle\int \frac{v_c(t\gamma)}{\left(1+\delta(\hat{\lambda}_k)tv_c(t\gamma)\right)^2}\tilde{\nu}(dt)}{\displaystyle\int \frac{v_c(t\gamma)}{1+\delta(\hat{\lambda}_k)tv_c(t\gamma)}\tilde{\nu}(dt)\left(1-\frac{1}{c}\int \frac{\delta(\hat{\lambda}_k)^2 t^2 v_c(t\gamma)^2}{\left(1+\delta(\hat{\lambda}_k)tv_c(t\gamma)\right)^2}\tilde{\nu}(dt)\right)}.$$

Item 0. in Theorem 4.2.2 provides a necessary and sufficient condition, i.e., $p_j > p_-$, for the existence of outlying eigenvalues in the spectrum of $\hat{C}_N$. In turn, this provides a means to estimate each p_j, $j \in \mathcal{L}$, along with bilinear forms involving a_j, from $\hat{\lambda}_j$ and $\hat{u}_j$. It is important here to note that, although the right-edge of the spectrum of μ is S_μ^+, due to the little control on τ_i in practice (in particular some of the τ_i may freely be arbitrarily large), isolated eigenvalues may be found infinitely often beyond S_μ^+ which do not carry information. This is why the (possibly pessimistic) choice of S^+ as an eigenvalue discrimination threshold was made. The major potency of the robust estimator $\hat{C}_N$ is indeed to be able to maintain these non informative eigenvalues below the known value S^+. As such, eigenvalues found above S^+ must contain information about A (at least with high probability) and this information can be retrieved, while isolated eigenvalues found below S^+ may arise from spurious values of τ_i, therefore containing no relevant information, or may contain relevant information but that cannot be trusted.

Figure 4.4 and Figure 4.5 provide the histogram and limiting spectral distribution of $\hat{C}_N$ and $\frac{1}{n}XX^*$, $X = [x_1, \ldots, x_n]$, respectively, for $u(x) = (1+\alpha)/(\alpha+x)$, $\alpha = 0.2$, $N = 200$, $n = 1000$, τ_i i.i.d. equal in distribution to $t^2(\beta-2)\beta^{-1}$ with t a Student-t random scalar of parameter $\beta = 100$, and $L = 2$ with $p_1 = p_2 = 1$, $a_1 = a(\theta_1)$, $a_2 = a(\theta_2)$, $\theta_1 = 10°$, $\theta_2 = 12°$, $a(\theta)$ being defined in Remark 4.3 (as well as in Assumption 4.7 below). These curves confirm that, while the limiting spectral measure of $\frac{1}{n}XX^*$ is unbounded, that of $\hat{C}_N$ is bounded. The numerically evaluated values of S_μ^+ and S^+ are reported in Figure 4.4. They reveal a rather close proximity between both values. In terms of empirical eigenvalues, note the particularly large gap between the isolated eigenvalues of $\hat{C}_N$ and the $N-2$ smallest ones, which may seem at first somewhat surprising for $p_1 = p_2 = 1$ since this setting induces a ratio 1 between the power carried by information versus noise (indeed, $A^*A \simeq I_2$ while $\mathrm{E}[\tau_i w_i w_i^*] = I_N$); this in fact results from the function u which, in attenuating the rare samples of large amplitudes, significantly reduces the noise power but only weakly affects the information part which has roughly constant amplitude

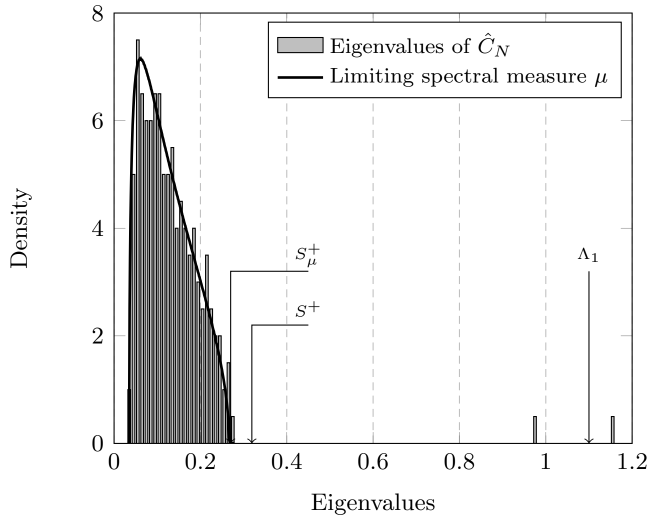


Figure 4.4: Histogram of the eigenvalues of $\hat{C}_N$ against the limiting spectral measure, for $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Student-t impulsions.

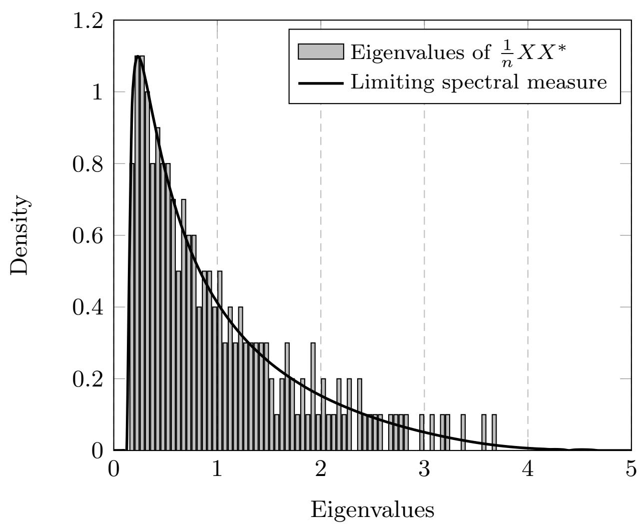


Figure 4.5: Histogram of the eigenvalues of $\frac{1}{n}XX^*$ against the limiting spectral measure, $L = 2$, $p_1 = p_2 = 1$, $N = 200$, $n = 1000$, Sudent-t impulsions.

across the samples. Also observe from Figure 4.5 that, as predicted, the largest two eigenvalues of $\frac{1}{n}XX^*$ do not isolate from the majority of the eigenvalues.

Items 1. and 2. in Theorem 4.2.2 then provide a means to estimate $p_1, \ldots, p_{|\mathcal{L}|}$ and bilinear forms involving the eigenvectors of AA^*. In particular, if p_k has multiplicity one in $\operatorname{diag}(p_1, \ldots, p_L)$, the summations in Item 2. are irrelevant and we obtain an estimator for $a^* u_k u_k^* b$. These however explicitly rely on $\tilde{\nu}$ which, for practical purposes, might be of limited interest if the τ_i are statistically unknown. It turns out, from a careful understanding of γ, that

$$\gamma - \hat{\gamma}_n \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{\gamma}_n \triangleq \frac{1}{n}\sum_{i=1}^n \frac{1}{N} x_i^* \hat{C}_{(i)}^{-1} x_i \tag{4.35}$$

and $\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n}u(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i)x_ix_i^*$. Also, for any $M>0$,

$$\max_{\substack{1\leq j\leq n\\ \tau_j\leq M}} |\tau_j - \hat{\tau}_j| \xrightarrow{\text{a.s.}} 0, \quad \max_{\substack{1\leq j\leq n\\ \tau_j> M}} \left|1 - \tau_j^{-1}\hat{\tau}_j\right| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{\tau}_i \triangleq \frac{1}{\hat{\gamma}_n}\frac{1}{N} x_i^* \hat{C}_{(i)}^{-1} x_i. \tag{4.36}$$

Details of these results are provided in Section 4.2.4. Letting $\varepsilon > 0$ small, for $x \in (S^+ + \varepsilon, \infty)$ and for all large n a.s., we then denote $\hat{\delta}(x)$ the unique negative solution to[8]

$$\hat{\delta}(x) = c_n \left(-x + \frac{1}{n}\sum_{i=1}^n \frac{\hat{\tau}_i v_c(\hat{\tau}_i\hat{\gamma}_n)}{1+\hat{\delta}(x)\hat{\tau}_i v_c(\hat{\tau}_i\hat{\gamma}_n)}\right)^{-1}. \tag{4.37}$$

From this, we then deduce the following alternative set of power and bilinear form estimators.

Theorem 4.2.3 (Robust estimation for unknown $\tilde{\nu}$)**.** *With the same notations as in Theorem 4.2.2, and with $\hat{\gamma}_n$, $\hat{\tau}_i$, and $\hat{\delta}$ defined in* (4.35)–(4.37)*, we have the following results.*

1. Purely empirical power estimation. *For each $j \in \mathcal{L}$,*

$$-\left(\hat{\delta}(\hat{\lambda}_j)\frac{1}{N}\sum_{i=1}^n \frac{v(\hat{\tau}_i\hat{\gamma}_n)}{1+\hat{\delta}(\hat{\lambda}_j)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma}_n)}\right)^{-1} \xrightarrow{\text{a.s.}} p_j.$$

2. Purely empirical bilinear form estimation. *For each $a, b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$, and each $j \in \mathcal{L}$,*

$$\sum_{k,p_k=p_j} a^* u_k u_k^* b - \sum_{k,p_k=p_j} \hat{w}_k a^* \hat{u}_k \hat{u}_k^* b \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{w}_k = \frac{\frac{1}{n}\sum_{i=1}^n \frac{v(\hat{\tau}_i\hat{\gamma}_n)}{\left(1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma}_n)\right)^2}}{\frac{1}{n}\sum_{i=1}^n \frac{v(\hat{\tau}_i\hat{\gamma}_n)}{1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma}_n)}\left(1 - \frac{1}{N}\sum_{i=1}^n \frac{\hat{\delta}(\hat{\lambda}_k)^2\hat{\tau}_i^2 v(\hat{\tau}_i\hat{\gamma}_n)^2}{\left(1+\hat{\delta}(\hat{\lambda}_k)\hat{\tau}_i v(\hat{\tau}_i\hat{\gamma}_n)\right)^2}\right)}.$$

[8] Remark here that, since $\hat{\tau}_i$, similar to τ_i, may be found away from $\operatorname{supp}(\tilde{\nu})$, $\hat{\delta}(x)$ may not be defined everywhere in (S_μ^+, S^+) but is defined beyond $S^+ + \varepsilon$ for n large a.s.

Theorem 4.2.3 provides a means to estimate powers and bilinear forms without any statistical knowledge on the τ_i, which are individually estimated. It is interesting to note that, since $\tilde{\nu}$ is only a limiting distribution, for practical systems, there is a priori no advantage in using the knowledge of $\tilde{\nu}$ or not. In particular, if n is not too large in practice or if $\tilde{\nu}$ has heavy tails, it is highly probable that $\tilde{\nu}_n$ be quite distinct from $\tilde{\nu}$, leading the estimators in Theorem 4.2.1 to be likely less accurate than the estimators in Theorem 4.2.2. Conversely, if N is not too large, $\hat{\tau}_i$ may be a weak estimate for τ_i so that, if $\tilde{\nu}$ has much lighter tails, the estimators of Theorem 4.2.1 may have a better advantage. Theoretical performance comparison between both schemes would require to exhibit central limit theorems for these quantities, which we discuss in Chapter 6, however for another application.

4.2.3.2 Application to angle estimation

An important application of Theorem 4.2.1 and Theorem 4.2.2 is found in the context of array processing, briefly evoked in the second item of Remark 4.3, in which $a_i = a(\theta_i)$ for some $\theta_i \in [0, 2\pi)$. For theoretical convenience, we use the classical linear array representation for a_i as follows.

Assumption 4.7. *For $i \in \{1, \ldots, L\}$, $a_i = a(\theta_i)$ with $\theta_1, \ldots, \theta_L$ distinct and, for $d > 0$ and $\theta \in [0, 2\pi)$,*

$$a(\theta) = N^{-\frac{1}{2}} [\exp(2\pi \imath d j \sin(\theta))]_{j=0}^{N-1}.$$

The objective in this specific model is to estimate $\theta_1, \ldots, \theta_L$ from the observations $x_1, \ldots, x_n$. In the regime $n \gg N$ with non-impulsive noise, this is efficiently performed by the traditional multiple signal classification (MUSIC) algorithm from (Schmidt, 1986). Using the fact that the vectors $a(\theta_i)$, $i \in \{1, \ldots, L\}$, are orthogonal to the subspace spanned by the eigenvectors with eigenvalue 1 of $\mathrm{E}[x_1 x_1^*] = AA^* + I_N$, the algorithm consists in retrieving the deepest minima of the nonnegative localization function $\hat{\eta}$ defined for $\theta \in [0, 2\pi)$ by

$$\hat{\eta}(\theta) = a(\theta)^* \Pi_{\frac{1}{n}XX^*} a(\theta)$$

where $\Pi_{\frac{1}{n}XX^*}$ is a projection matrix on the subspace associated with the $N - L$ smallest eigenvalues of $\frac{1}{n}XX^*$. Indeed, as $\frac{1}{n}XX^*$ is an almost surely consistent estimate for $\mathrm{E}[x_1 x_1^*]$ in the large n regime, $\hat{\eta}(\theta) \xrightarrow{\text{a.s.}} \eta(\theta)$ where

$$\eta(\theta) = a(\theta)^* \Pi_{\mathrm{E}[x_1 x_1^*]} a(\theta)$$

with here $\Pi_{\mathrm{E}[x_1 x_1^*]}$ a projection matrix on the subspace associated with the eigenvalue 1 in $\mathrm{E}[x_1 x_1^*]$; as such, $\hat{\eta}(\theta) \xrightarrow{\text{a.s.}} 0$ for $\theta \in \{\theta_1, \ldots, \theta_L\}$ and to a positive quantity otherwise. In (Mestre, 2008a), Mestre proved that this algorithm is however inconsistent in the regime of Assumption 4.6. This led to (Mestre, 2008b) in which an improved estimator (the G-MUSIC estimator) for $\theta_1, \ldots, \theta_L$ was designed, however for a more involved model than the spiked model (i.e., L is assumed commensurable with N). In (Hachem et al., 2013), a spiked model hypothesis was then assumed (i.e., with L small compared to N, n) which unfolded into a more practical and more theoretically tractable spiked G-MUSIC estimator. Similar to MUSIC, the latter consists in determining the deepest minima of an alternative localization function $\hat{\eta}_{\mathrm{G}}(\theta)$, which we shall define in a moment.

Although improved with respect to MUSIC, both algorithms still rely on exploiting the largest isolated eigenvalues of $\frac{1}{n}XX^*$ and the asymptotic boundedness of the noise spectrum. From the discussions in Section 4.2.1 and after Theorem 4.2.2, under the generic Assumption 4.4 with τ_i allowed to grow unbounded, these methods are now unreliable and in fact inefficient. From Item 2. in both Theorem 4.2.2 and Theorem 4.2.3, it is now possible to provide a consistent estimation method based on two novel localization functions $\hat{\eta}_{\mathrm{RG}}$ and $\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}$. The resulting algorithms are from now on referred to as robust G-MUSIC and empirical robust G-MUSIC, respectively.

Corollary 4.3 (Robust G-MUSIC)**.** *Let Assumptions 4.4–4.7 hold. Let* $0 < \kappa < \min_{i,j} |\theta_i - \theta_j|$ *and denote* $\mathcal{R}_i^\kappa = [\theta_i - \kappa/2, \theta_i + \kappa/2]$*. Also define* $\hat{\eta}_{\mathrm{RG}}(\theta)$ *and* $\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta)$ *as*

$$
\begin{aligned}
\hat{\eta}_{\mathrm{RG}}(\theta) &= 1 - \sum_{k=1}^{|\mathcal{L}|} w_k a(\theta)^* \hat{u}_k \hat{u}_k a(\theta) \\
\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta) &= 1 - \sum_{k=1}^{|\mathcal{L}|} \hat{w}_k a(\theta)^* \hat{u}_k \hat{u}_k a(\theta)
\end{aligned}
$$

where we used the notations from Theorems 4.2.2 and 4.2.3. Then, for each $j \in \mathcal{L}$*,*

$$
\begin{aligned}
\hat{\theta}_j &\xrightarrow{\text{a.s.}} \theta_j \\
\hat{\theta}_j^{\mathrm{emp}} &\xrightarrow{\text{a.s.}} \theta_j
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\theta}_j &\triangleq \mathrm{argmin}_{\theta \in \mathcal{R}_j^\kappa} \left\{ \hat{\eta}_{\mathrm{RG}}(\theta) \right\} \\
\hat{\theta}_j^{\mathrm{emp}} &\triangleq \mathrm{argmin}_{\theta \in \mathcal{R}_j^\kappa} \left\{ \hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta) \right\}.
\end{aligned}
$$

With the same reasoning as in Remark 4.5, it is now easy to check that, letting the v_c or v functions be replaced by the constant 1 in the expressions of w_k and $\hat{w}_k$, respectively, we fall back on G-MUSIC schemes devised in e.g., (Hachem et al., 2013). In what follows, we then define $\hat{\eta}_{\mathrm{G}}(\theta)$ and $\hat{\eta}_{\mathrm{G}}^{\mathrm{emp}}(\theta)$ similarly to $\hat{\eta}_{\mathrm{RG}}(\theta)$ and $\hat{\eta}_{\mathrm{RG}}^{\mathrm{emp}}(\theta)$ but with the functions v_c and v replaced by the constant 1 and with the couples $(\hat{\lambda}_k, \hat{u}_k)$ replaced by the k-th largest eigenvalue and associated eigenvectors of $\frac{1}{n}XX^*$. For a further comparison of the various methods, we also denote by $\hat{\eta}_{\mathrm{R}}(\theta)$ the robust counterpart to $\hat{\eta}(\theta)$ defined by $\hat{\eta}_{\mathrm{R}}(\theta) = a(\theta)^* \Pi_{\hat{C}_N} a(\theta)$ with $\Pi_{\hat{C}_N}$ a projection matrix on the subspace associated with the $N - L$ smallest eigenvalues of $\hat{C}_N$.

Simulation curves are provided below which compare the performance of the various improved MUSIC techniques. Since the methods based on the extraction of $\delta(\hat{\lambda}_i)$ may be void when this value does not exist, we blindly proceed by solving the fixed-point equation defining $\delta(\hat{\lambda}_i)$ thanks to the standard fixed-point algorithm until convergence or until a maximum number of iterations is reached. This effect is in fact marginal as it is theoretically highly probable that eigenvalues be found beyond S_μ^+ for each finite N, n. We also assume $\mathcal{L} = \{1, \ldots, L\}$ even if this does not hold, which in practice one cannot anticipate. Voluntarily disrupting from the theoretical claims of Theorems 4.2.1–4.2.3 will allow for an observation of problems arising when the assumptions are not fully satisfied. In all simulation figures, we consider $u(x) = (1+\alpha)(\alpha + x)^{-1}$ with $\alpha = 0.2$, $N = 20$, $n = 100$, $L = 2$, $\theta_1 = 10°$, $\theta_2 = 12°$. The noise impulsions are of two types: (i) single outlier impulsion for which $\tau_i = 1$, $i \in \{1, \ldots, n-1\}$ and $\tau_n = 100$, or (ii) Student impulsions for which $\tau_i = t^2(\beta - 2)\beta^{-1}$ with t a Student-t random variable with parameter $\beta = 100$ (the normalization ensures $\mathrm{E}[\tau_1] = 1$).

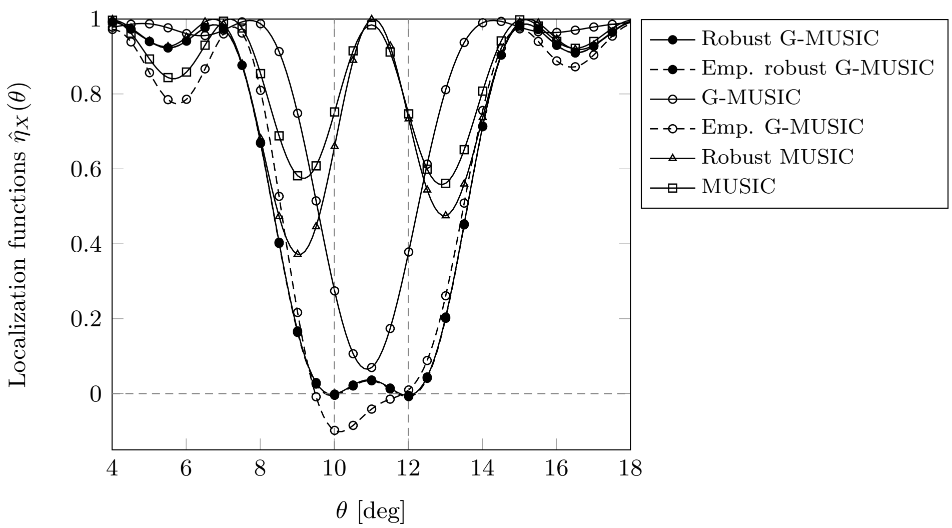


Figure 4.6: Random realization of the localization functions for the various MUSIC estimators, with $N = 20$, $n = 100$, two sources at 10° and 12°, Student-t impulsions with parameter $\beta = 100$, $u(x) = (1+\alpha)/(\alpha + x)$ with $\alpha = 0.2$. Powers $p_1 = p_2 = 10^{0.5} = 5$ dB.

Figure 4.6 provides a single realization (but representative of the multiple realizations we simulated) of the various localization functions $\hat{\eta}_X$ and $\hat{\eta}_X^{\mathrm{emp}}$ for θ in the vicinity of θ_1, θ_2, X being void, R, or RG. The scenario considered is that of a Student-t noise and $p_1 = p_2 = 1$. The figure confirms the advantage of the methods based on $\hat{C}_N$ over $\frac{1}{n}XX^*$ which unfolds from the proper extreme eigenvalue isolation observed under the same setting in Figure 4.4 against Figure 4.5. Due to N/n being non trivial, while the robust G-MUSIC methods accurately discriminate both angles at their precise locations and with appropriate localization function amplitude, the robust MUSIC approach discriminates the two angles at erroneous locations and erroneous localization function amplitude. Benefiting from the random matrix advantage, G-MUSIC in turn behaves better in amplitude than MUSIC but cannot discriminate angles. Observe also here that both empirical and non-empirical robust G-MUSIC approaches behave extremely similar (both curves are visually superimposed), suggesting that with $\beta = 100$ the samples from the Student-t distribution represent sufficiently well the actual distribution of $\tau_1 v(\tau_1 \gamma)$. This no longer holds for G-MUSIC versus empirical G-MUSIC, in which case the approximation of $\tilde{\nu}_n$ by the distribution $\tilde{\nu}$ of τ_1 is not appropriate.

Figure 4.7 and Figure 4.8 provide the mean square error performance for the first angle estimation $\mathrm{E}[|\hat{\theta}_1 - \theta_1|^2]$ as a function of the source powers $p_1 = p_2$; the estimates are based for each estimator on retrieving the local minima of $\hat{\eta}_X$. For fair comparison, the two deepest minima of the localization functions are extracted and $\hat{\theta}_1$ is declared to be the estimated angle closest to θ_1 (in particular, if a unique minimum is found close to any θ_i, $\hat{\theta}_1$ is attached to this minimum). Figure 4.7 assumes the Student-t impulsion scenario of Figure 4.6, while Figure 4.8 is concerned with the outlier impulsion model previously described. Both figures further confirm the advantage brought by the robust G-MUSIC scheme with asymptotic equivalence between empirical or non-empirical in the large source power regime. We observe in particular the outstanding advantage of (robust or not) G-MUSIC methods which perform well at high source power, while standard methods saturate. Interestingly, from Figure 4.7, the G-MUSIC schemes perform well in the high source power regime, which corresponds to scenarios in which the noise impulsion amplitudes are often small enough compared to source power to be assumed bounded and G-MUSIC is then consistent. Nonetheless, G-MUSIC never closes the gap with robust G-MUSIC which is likely explained by the much larger spacing between noise and information eigenvalues in the spectrum of $\hat{C}_N$. The situation is different in Figure 4.8 where G-MUSIC almost meets the performance of robust G-MUSIC at very high power, while performing poorly below 20 dB. This is explained by the presence of a single additional eigenvalue of amplitude around 100 (i.e., 20 dB) in the spectrum of $\frac{1}{n}XX^*$ which corrupts the G-MUSIC algorithm as long as this amplitude is larger than these of the two informative eigenvalues due to the steering vectors (about p_1).

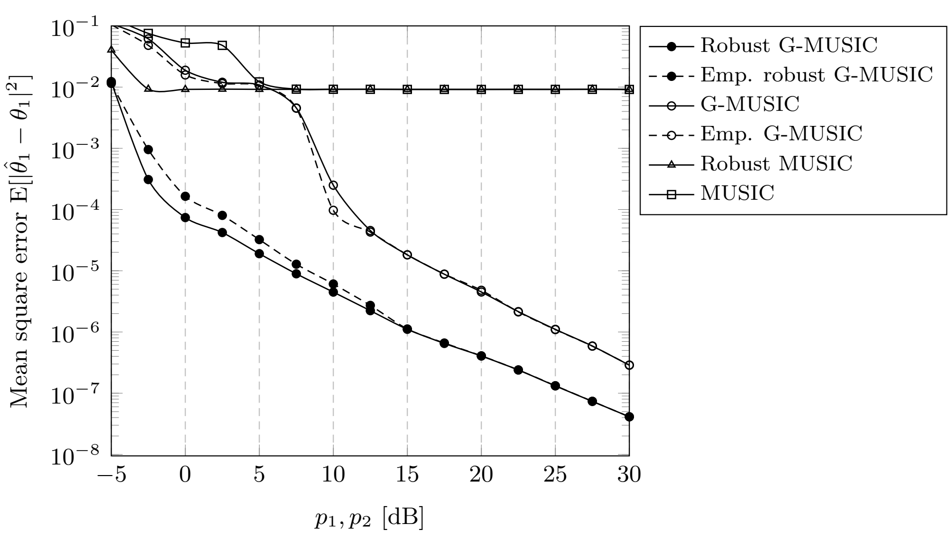


Figure 4.7: Means square error performance of the estimation of $\theta_1 = 10°$, with $N = 20$, $n = 100$, two sources at $10°$ and $12°$, Student-t impulsions with parameter $\beta = 10$, $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $p_1 = p_2$.

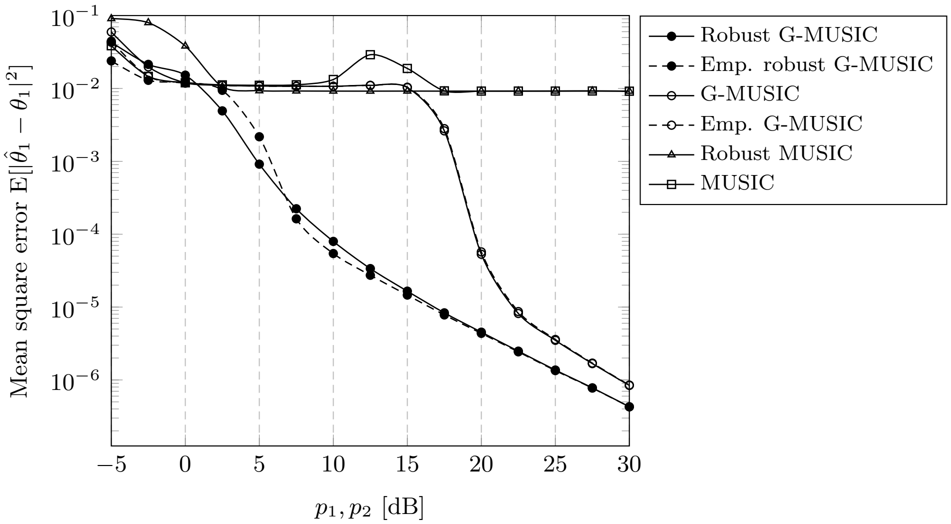


Figure 4.8: Means square error performance of the estimation of $\theta_1 = 10°$, with $N = 20$, $n = 100$, two sources at $10°$ and $12°$, sample outlier scenario $\tau_i = 1$, $i < n$, $\tau_n = 100$, $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.2$, $p_1 = p_2$.

4.2.4 Proof of the main results

4.2.4.1 Notations

Throughout the proof, we shall use the following shortcut notations:

$$\begin{aligned} T &= \mathrm{diag}(\{\tau_i\}_{i=1}^n) \in \mathbb{C}^{n\times n} \\ V &= \mathrm{diag}(\{v_c(\tau_i\gamma\}_{i=1}^n) \in \mathbb{C}^{n\times n} \\ S &= [\{s_{ij}\}_{1\leq i\leq L, 1\leq j\leq n}] \in \mathbb{C}^{L\times n} \\ W &= [w_1, \ldots, w_n] \in \mathbb{C}^{N\times n} \\ \tilde{W} &= [\tilde{w}_1, \ldots, \tilde{w}_n] \in \mathbb{C}^{N\times n} \end{aligned}$$

with $\tilde{w}_i = w_i r_i/\sqrt{N}$ as in the statement of Theorem 4.2.1. We shall expand A as the singular value decomposition $A = U\Omega\bar{U}^*$ with $U \in \mathbb{C}^{N\times L}$ isometric, $\Omega = \mathrm{diag}(\sigma_1, \ldots, \sigma_L)$, $\sigma_1 \geq \ldots \geq \lambda_L \geq 0$, and $\bar{U} \in \mathbb{C}^{L\times L}$ unitary.

We also define

$$\hat{S}_N^\circ = \frac{1}{n}\sum_{i=1}^n \tau_i v_c(\tau_i\gamma)\tilde{w}_i\tilde{w}_i^* = \frac{1}{n}\tilde{W}TV\tilde{W}^*$$

which corresponds to $\hat{S}_N$ with $p_1 = \ldots = p_L = 0$, i.e., with no perturbation, and

$$Q_z^\circ = (\hat{S}_N^\circ - zI_N)^{-1} = \left(\frac{1}{n}\tilde{W}TV\tilde{W}^* - zI_N\right)^{-1}$$

the resolvent of $\hat{S}_N^\circ$.

For couples (η, M_η), $\eta < 1$, such that $\tilde{\nu}((0, M_\eta)) > 0$ and $\tilde{\nu}((M_\eta, \infty)) < \eta$, it will be necessary to define T_η the matrix T in which all values of τ_i greater or equal to M_η are replaced by zeros, and similarly for V_η. Denote also γ^η the unique solution to

$$1 = \int_{\tau < M_\eta} \frac{\psi_c(\tau\gamma^\eta)\tilde{\nu}(d\tau)}{1 + c\psi_c(\tau\gamma^\eta)}. \tag{4.38}$$

and $\hat{S}_{N,\eta}$ the resulting $\hat{S}_N$ matrix with all τ_i greater than M_η discarded and γ replaced by γ^η.

Finally, we further define $T_{(j)} = \operatorname{diag}(\{\tau_i\}_{i \neq j})$ and similarly for $V_{(j)}$, $S_{(j)}$, $\tilde{W}_{(j)}$, $\hat{S}_{(j)} = \hat{S}_{N,(j)}$ the matrices with column or component j discarded, as well as $T_{(j),\eta}$ the matrix T_η with row-and-column j discarded, and similarly $V_{(j),\eta}$, $S_{(j),\eta}$, $\tilde{W}_{(j),\eta}$, $\hat{S}_{(j),\eta}$ the corresponding matrices with column or component j discarded.

4.2.4.2 Overall proof strategy

The existence and uniqueness of $\hat{C}_N$ as defined in the statement of Theorem 4.2.1 can be proved along the same lines as in Section 4.1 and will not be discussed here. One of the key elements of the proof of convergence in Theorem 4.2.1 is to ensure that there exists $\varepsilon > 0$ such that, for all large n a.s., all eigenvalues of $\{\hat{S}_{(j)}, 1 \leq j \leq n\}$ (and also of $\{\hat{S}_{(j),\eta}, 1 \leq j \leq n\}$ for given η small) are greater than ε. This is an important condition to ensure that the quadratic forms $\frac{1}{N}\tilde{w}_j^*\hat{S}_{(j)}^{-1}\tilde{w}_j$, which play a central role in the proof, are jointly controllable. In Section 4.1, where the convergence $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ is obtained for $p_1 = \ldots = p_L = 0$, this unfolded readily from the fact that the matrices $\frac{1}{n}\tilde{W}_{(j)}\tilde{W}_{(j)}^*$ have their smallest eigenvalue uniformly away from zero. Here, due to the existence of a small rank matrix A, this approach no longer holds as $\hat{S}_{(j)}$ may a priori exhibit finitely many isolated eigenvalues getting close to zero as $n \to \infty$. We shall show that this is not possible. Precisely, we shall prove that the large n spectrum of $\hat{S}_N$ is similar to that of $\hat{S}_N^\circ$ but possibly for finitely many isolated eigenvalues, none of which can be asymptotically found close to zero. We shall however characterize those eigenvalues of $\hat{S}_N$ found beyond the right-edge of the limiting spectrum of $\hat{S}_N^\circ$. Once this result is obtained, to complete the proof of Theorem 4.2.1, it will then suffice to check that most spectral statistics involved in the proof of Theorem 4.1.2 are not affected by the presence of the additional small rank matrix AS in the model. Since most results need be proved jointly for the matrix sets $\{\hat{S}_{(j)}, 1 \leq j \leq n\}$ (or $\{\hat{S}_{(j),\eta}, 1 \leq j \leq n\}$), high order moment bounds will be required to then apply union bound along with Markov inequality techniques. To avoid repeating all the arguments of the proof of Theorem 4.1.2, we only discuss in what follows the main new technical elements that differ from Theorem 4.1.2.

When Theorem 4.2.1 is obtained, the proofs of Theorems 4.2.2 and 4.2.3 unfolds from classical techniques for spiked random matrix models, using the approximation $\hat{S}_N$ for $\hat{C}_N$. The model $\hat{S}_N$ considered here is closely related to the scenario of (Chapon et al., 2014), but for the random non-Gaussian structure of the matrix S; also, (Chapon et al., 2014) imposes $\max_i \operatorname{dist}(\tau_i, \operatorname{supp}(\tilde{\nu})) \to 0$ which we do not enforce here. Because of these important differences, in order to keep track of the specificities of the model, a complete proof will be proposed below.

4.2.4.3 Localization of the eigenvalues of $\hat{S}_N$ and $\hat{S}_{(i)}$

We first study the localization of the eigenvalues of $\hat{S}_N$ and $\{\hat{S}_{(j),\eta}, 1 \leq j \leq n\}$. The strategy being the same, we concentrate mostly on the study of $\hat{S}_N$ and then briefly generalize the approach to $\{\hat{S}_{(j),\eta}, 1 \leq j \leq n\}$. Our approach is based on the original derivation in (Benaych-Georges and Rao, 2011; Benaych-Georges and Nadakuditi, 2012).

By isolating the small rank perturbation terms, we first develop $\hat{S}_N$ as

$$\hat{S}_N = \frac{1}{n}\sum_{i=1}^{n} v_c(\tau_i\gamma) A_i \bar{w}_i \bar{w}_i^* A_i^* \tag{4.39}$$

$$= \hat{S}_N^\circ + \frac{1}{n} ASVS^*A^* + \frac{1}{n} AST^{\frac{1}{2}} V\tilde{W}^* + \frac{1}{n}\tilde{W}T^{\frac{1}{2}}VS^*A^*. \tag{4.40}$$

Let $\lambda \in \mathbb{R}\backslash[\varepsilon, S^+ + \varepsilon]$ for some $\varepsilon > 0$ small be an eigenvalue of $\hat{S}_N$. Note that such a λ may not exist. However, from (Bai and Silverstein, 1998) and since in particular $\limsup_n \|AA^*\| < \infty$ and $\limsup_n \|T^{\frac{1}{2}}V\| < \infty$, the spectral norm of each matrix above is asymptotically bounded almost surely and thus $\limsup_n \lambda < \infty$ a.s. Also, from Section 4.1 and from the discussion prior to the statement of Theorem 4.2.1, for all large n a.s., λ is not an eigenvalue of $\hat{S}_N^\circ$ (for ε chosen small enough). Thus, by definition, λ is a solution of $\det(\hat{S}_N - \lambda I_N) = 0$ while $\|(\hat{S}_N^\circ - \lambda I_N)^{-1}\| < M$ for some $M > 0$ independent of n but increasing as $\varepsilon \to 0$. As such, from the development above, for all large n a.s.,

$$0 = \det\left(\hat{S}_N^\circ - \lambda I_N + \Gamma\right) = \det\left(Q_\lambda^\circ\right)^{-1} \det\left(I_N + (Q_\lambda^\circ)^{\frac{1}{2}}\Gamma(Q_\lambda^\circ)^{\frac{1}{2}}\right)$$

where $\Gamma = \frac{1}{n}ASVS^*A^* + \frac{1}{n}AST^{\frac{1}{2}}V\tilde{W}^* + \frac{1}{n}\tilde{W}T^{\frac{1}{2}}VS^*A^*$ can be further written

$$\Gamma = \begin{bmatrix} U\Omega^{\frac{1}{2}} & \frac{1}{n}\tilde{W}T^{\frac{1}{2}}VS^*\bar{U}\Omega^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \Omega^{\frac{1}{2}}\bar{U}^*\frac{1}{n}SVS^*\bar{U}\Omega^{\frac{1}{2}} & I_L \\ I_L & 0 \end{bmatrix} \begin{bmatrix} \Omega^{\frac{1}{2}}U^* \\ \Omega^{\frac{1}{2}}\frac{1}{n}\bar{U}^*ST^{\frac{1}{2}}V\tilde{W} \end{bmatrix}. \tag{4.41}$$

Exploiting the small rank of S and A, and the formula $\det(I + AB) = \det(I + BA)$ for properly sized A, B matrices, this induces

$$0 = \det\left(I_{2L} + \Gamma_L(\lambda)\right)$$

where

$$\Gamma_L(\lambda) \triangleq \begin{bmatrix} \Omega^{\frac{1}{2}}\bar{U}^*\frac{1}{n}SVS^*\bar{U}\Omega^{\frac{1}{2}} & I_L \\ I_L & 0 \end{bmatrix} \begin{bmatrix} \Omega^{\frac{1}{2}}U^* \\ \Omega^{\frac{1}{2}}\frac{1}{n}\bar{U}^*ST^{\frac{1}{2}}V\tilde{W} \end{bmatrix} Q_\lambda^\circ \begin{bmatrix} U\Omega^{\frac{1}{2}} & \frac{1}{n}\tilde{W}T^{\frac{1}{2}}VS^*\bar{U}\Omega^{\frac{1}{2}} \end{bmatrix}.$$

We now need the following central lemmas.

Lemma 4.4. *Let $\varepsilon > 0$ and $\mathcal{A}_\varepsilon$ be the event $\varepsilon < \lambda_N(\hat{S}_N^\circ) < \lambda_1(\hat{S}_N^\circ) < S^+ + \varepsilon$. Let also $a, b \in \mathbb{C}^N$*

be two vectors of unit norm. Then, for every $z \in \mathcal{C} \subset \mathbb{C} \setminus [\varepsilon, S^+ + \varepsilon]$ with $\mathcal{C}$ compact,

$$\mathrm{E}\left[\left|\frac{1}{n}S^*VS - \frac{1}{n}\operatorname{tr} V\right|^p\right] \leq KN^{-\frac{p}{2}}$$

$$\mathrm{E}\left[1_{\mathcal{A}_\varepsilon}\left|\frac{1}{n}ST^{\frac{1}{2}}V\frac{1}{n}\tilde{W}^*Q_z^\circ\tilde{W}VT^{\frac{1}{2}}S^* - \left[\frac{1}{n}\operatorname{tr} V + z\frac{1}{n}\operatorname{tr} V\tilde{Q}_z^\circ\right]\right|^p\right] \leq KN^{-\frac{p}{2}}$$

$$\mathrm{E}\left[1_{\mathcal{A}_\varepsilon}\left|a^*Q_z^\circ b - a^*b\frac{1}{N}\operatorname{tr} Q_z^\circ\right|^p\right] \leq KN^{-\frac{p}{2}}$$

$$\mathrm{E}\left[1_{\mathcal{A}_\varepsilon}\left\|\frac{1}{n}a^*Q_z^\circ\tilde{W}T^{\frac{1}{2}}VS^*\right\|^p\right] \leq KN^{-\frac{p}{2}}$$

where $\tilde{Q}_z^\circ = (\frac{1}{n}T^{\frac{1}{2}}V^{\frac{1}{2}}\tilde{W}^*\tilde{W}V^{\frac{1}{2}}T^{\frac{1}{2}} - zI_N)^{-1}$ and $K > 0$ does not depend on z.

Proof. The first convergence is a mere application of (Bai and Silverstein, 2009, Lemma B.26). Similarly, noticing that

$$\begin{aligned}\frac{1}{n}ST^{\frac{1}{2}}V\frac{1}{n}\tilde{W}^*Q_z^\circ\tilde{W}VT^{\frac{1}{2}}S^* &= \frac{1}{n}SV^{\frac{1}{2}}\left[T^{\frac{1}{2}}V^{\frac{1}{2}}\frac{1}{n}\tilde{W}^*\tilde{W}V^{\frac{1}{2}}T^{\frac{1}{2}}\tilde{Q}_z^\circ\right]V^{\frac{1}{2}}S^* \\ &= \frac{1}{n}SVS^* + z\frac{1}{n}S\tilde{Q}_z^\circ V^{\frac{1}{2}}S^*\end{aligned}$$

the second result follows again by (Bai and Silverstein, 2009, Lemma B.26) and the fact that $\limsup_n \|\tilde{Q}_z^\circ\| < 1/\operatorname{dist}(\mathcal{C}, [\varepsilon, S^+ + \varepsilon])$. Using the fact that $\tilde{W}$ is Gaussian, the third result follows from the same proof as in (Hachem et al., 2013, Lemma 3) using additionally $[VT]_{ii} < \psi_\infty$. Similarly, conditioning first on S, which is independent of $\tilde{W}$, we obtain by the same proof as in (Hachem et al., 2013, Lemma 4) that

$$\mathrm{E}_{\tilde{W}}\left[1_{\mathcal{A}_\varepsilon}\left|\frac{1}{n}a^*Q_z^\circ\tilde{W}T^{\frac{1}{2}}Vs_i\right|^p\right] \leq K\|n^{-\frac{1}{2}}s_i\|^pN^{-\frac{p}{2}}$$

where we denoted $S^* = [s_1, \ldots, s_L]$ (the proof follows from exploiting the left-unitary invariance of $\tilde{W}$ and applying the integration by parts and Poincaré–Nash inequality method for unitary Haar matrices described in (Pastur and Ŝerbina, 2011, Chapter 8)). Now, $\mathrm{E}[\|n^{-\frac{1}{2}}s_i\|^p] = O(1)$ by Hölder's inequality, and we obtain the last inequality. ☐

Lemma 4.5. *For $z \in \mathbb{C} \setminus [S_\mu^-, S_\mu^+]$, let $\delta(z)$ be the unique solution to the equation*

$$\delta(z) = c\left(-z + \int \frac{tv_c(t\gamma)}{1 + \delta(z)tv_c(t\gamma)}d\tilde{\nu}(t)\right)^{-1}$$

where we recall that γ is the unique positive solution to

$$1 = \int \frac{\psi_c(t\gamma)}{1 + c\psi_c(t\gamma)}d\tilde{\nu}(t).$$

Let now $z \in \mathcal{C}$, with $\mathcal{C}$ a compact set of $\mathbb{C} \setminus [\varepsilon, S_\mu^+ + \varepsilon]$ for some $\varepsilon > 0$ small enough. Then, denoting $\Psi_z^\circ = (I_n + \delta(z)VT)^{-1}$,

$$\sup_{z\in\mathcal{C}} \left| \frac{1}{N} \operatorname{tr} Q_z^\circ - \frac{\delta(z)}{c} \right| \xrightarrow{\text{a.s.}} 0$$

$$\sup_{z\in\mathcal{C}} \left| \frac{1}{n} \operatorname{tr} V + z\frac{1}{n} \operatorname{tr} V\tilde{Q}_z^\circ - \delta(z)\frac{1}{n} \operatorname{tr} V^2 T\Psi_z^\circ \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. The almost sure convergences to zero of the terms inside the norms (i.e., for each $z \in \mathcal{C}$) are classical, see e.g., (Silverstein and Bai, 1995). Considering a countable sequence $z_1, z_2, \ldots$ of such $z \in \mathcal{C}$ having an accumulation point, by the union bound, there exists a probability one set on which the convergence is valid for each point of the sequence. Now, as in Section 4.1, for all large n a.s., Q_z° and $\tilde{Q}_z^\circ$ are analytic on $\mathcal{C}$. Since $\delta(z)$ is also analytic on $\mathcal{C}$, by Vitali's convergence theorem (Titchmarsh, 1939), the convergences are uniform on $\mathcal{C}$. □

Similar again to Section 4.1, for $\varepsilon > 0$ small enough, the set $\mathcal{A}_\varepsilon$ introduced in Lemma 4.4 satisfies $1_{\mathcal{A}_\varepsilon} \xrightarrow{\text{a.s.}} 1$. As such, using the Markov inequality and the Borel Cantelli lemma, Lemma 4.4 for $p > 2$ ensures that all quantities in absolute values in the statement of Lemma 4.4 converge to zero almost surely as $n \to \infty$. Since the quantities involved are analytic on compact $\mathcal{C} \subset \mathbb{C} \setminus [\varepsilon, S^+ + \varepsilon]$, considering a countable sequence of $z \in \mathcal{C}$ having a limit point, it is clear by Vitali's convergence theorem (Titchmarsh, 1939) that these convergences are uniform on $\mathcal{C}$. Applying successively Lemma 4.4 for $p > 2$ and Lemma 4.5, we then obtain, for $\mathcal{C} \subset \mathbb{C}\setminus[\varepsilon, S^+ + \varepsilon]$,

$$\sup_{z\in\mathcal{C}} \left\{ \left\| \Gamma_L(z) - \begin{bmatrix} \Omega\frac{1}{n}\operatorname{tr} V & I_L \\ I_L & 0 \end{bmatrix} \begin{bmatrix} \Omega\frac{\delta(z)}{c} & 0 \\ 0 & \Omega\delta(z)\frac{1}{n}\operatorname{tr} V^2 T\Psi_z^\circ \end{bmatrix} \right\| \right\} \xrightarrow{\text{a.s.}} 0$$

or equivalently

$$\sup_{z\in\mathcal{C}} \left\{ \left\| \Gamma_L(z) - \begin{bmatrix} \Omega^2\frac{\delta(z)}{c}\frac{1}{n}\operatorname{tr} V & \Omega\delta(z)\frac{1}{n}\operatorname{tr} V^2 T\Psi_z^\circ \\ \Omega\frac{\delta(z)}{c} & 0 \end{bmatrix} \right\| \right\} \xrightarrow{\text{a.s.}} 0. \tag{4.42}$$

We may then particularize this result to $z = \lambda$ which, for ε sufficiently small, remains bounded away from $[\varepsilon, S^+ + \varepsilon]$ as n grows (but of course depends on n) to obtain

$$\left\| \Gamma_L(\lambda) - \begin{bmatrix} \Omega^2\frac{\delta(\lambda)}{c}\frac{1}{n}\operatorname{tr} V & \Omega\delta(\lambda)\frac{1}{n}\operatorname{tr} V^2 T\Psi_\lambda^\circ \\ \Omega\frac{\delta(\lambda)}{c} & 0 \end{bmatrix} \right\| \xrightarrow{\text{a.s.}} 0. \tag{4.43}$$

For $\bar{\lambda} \in \mathbb{R} \setminus [\varepsilon, S^+ + \varepsilon]$, let us now study the equation

$$\det\left(I_{2L} + \begin{bmatrix} \Omega^2\frac{\delta(\bar{\lambda})}{c}\frac{1}{n}\operatorname{tr} V & \Omega\delta(\bar{\lambda})\frac{1}{n}\operatorname{tr} V^2 T\Psi_{\bar{\lambda}}^\circ \\ \Omega\frac{\delta(\bar{\lambda})}{c} & 0 \end{bmatrix} \right) = 0. \tag{4.44}$$

After development of the determinant, this equation is equivalent to

$$\sigma_\ell^2\frac{\delta(\bar{\lambda})}{c}\left(\frac{1}{n}\operatorname{tr} V - \delta(\bar{\lambda})\frac{1}{n}\operatorname{tr} V^2 T\Psi_{\bar{\lambda}}^\circ\right) + 1 = 0$$

for some $\ell \in \{1,\dots,L\}$, or equivalently, using $V - \delta(\bar\lambda)V^2T\Psi^\circ_{\bar\lambda} = V\Psi^\circ_{\bar\lambda}$

$$\sigma_\ell^2\delta(\bar\lambda)\frac{1}{N}\sum_{i=1}^n \frac{v_c(\tau_i\gamma)}{1+\tau_i v_c(\tau_i\gamma)\delta(\bar\lambda)} + 1 = 0.$$

In the limit $n\to\infty$, using $A^*A \xrightarrow{\text{a.s.}} \operatorname{diag}(p_1,\dots,p_L)$ and $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{\tau_i} \to \tilde\nu$ a.s., any accumulation point $\bar\Lambda \in (\mathbb{R}\setminus(\varepsilon, S^+ + \varepsilon)) \cup \{\infty\}$ of $\bar\lambda$ must satisfy

$$1 + p_\ell \frac{1}{c}\int \frac{\delta(\bar\Lambda)v_c(\tau\gamma)}{1+\delta(\bar\Lambda)\tau v_c(\tau\gamma)}\tilde\nu(dt) = 0. \tag{4.45}$$

This unfolds from dominated convergence, using $\delta((S^+,\infty)) \subset (-(\tau_+ v(\tau_+\gamma))^{-1}, 0)$ with $\tau_+ \in (0,\infty]$ the right-edge of the support of $\tilde\nu$; in particular, if $\operatorname{supp}(\tilde\nu)$ is unbounded, $\delta((S^+,\infty)) \subset (-\gamma/\psi_\infty, 0)$ (see Chapter 3 or (Silverstein and Choi, 1995) for details). Let us then consider the equation in the variable $\Lambda \in (S^+,\infty)$

$$-\left(\frac{1}{c}\int \frac{\delta(\Lambda)v_c(\tau\gamma)}{1+\delta(\Lambda)\tau v_c(\tau\gamma)}\tilde\nu(d\tau)\right)^{-1} = p_\ell. \tag{4.46}$$

We know from Section 4.1 that, since $\tilde\nu([0,m)) < 1-\phi_\infty^{-1}$ for some $m>0$ (by Assumption 4.5), $S_\mu^- > 0$. Also, as the Stieltjes transform of a measure with support included in $[S_\mu^-, S_\mu^+] \subset [S_\mu^-, S^+]$, δ is increasing on both $[0, S_\mu^-)$ and (S^+,∞). Moreover, $\delta([0,S_\mu^-)) \subset (0,\infty)$ and $\delta((S^+,\infty)) \subset (-(\tau_+ v(\tau_+\gamma))^{-1}, 0)$. Therefore, the left-hand side of (4.46) is negative for $\Lambda \in [0, S_\mu^-)$ and the equation has no solution in this set. It is now easily seen that the left-hand side of (4.46) is increasing with Λ with limits infinity as $\Lambda\to\infty$ and $p_- > 0$ as $\Lambda \downarrow S^+$. Therefore, if $p_- < p_\ell$, the above equation has a unique solution $\Lambda_\ell \in (S^+,\infty)$, distinct for each distinct p_ℓ. Hence, $\bar\lambda \to \bar\Lambda = \Lambda_\ell$.

By the argument principal, for all n large a.s., the number of eigenvalues of $\hat S_N$, i.e., the number of zeros of $\det(I_{2L}+\Gamma_L(\lambda))$, in any open set $\mathcal{V} \subset \mathbb{R}\setminus[\varepsilon, S^+ + \varepsilon]$ is

$$\frac{1}{2\pi\imath}\oint_{\mathcal{I}} \frac{[\det(I_{2L}+\Gamma_L(z))]'}{\det(I_{2L}+\Gamma_L(z))}dz$$

with $\mathcal{I}$ a contour enclosing $\mathcal{V}$. By the uniform convergence of (4.42) on $\mathcal{V}$, the analyticity of the quantities involved, and the fact that the involved determinant is a polynomial of order at most $2L$ of its entries, this value asymptotically corresponds to the number of solutions to (4.44) in $\mathcal{V}$ counted with multiplicity, which in the limit are the $\Lambda_k \in \mathcal{V}$. Particularizing $\mathcal{V}$ to $(-1, 2\varepsilon)$ for $\varepsilon > 0$ small enough and then to any small open ball around Λ_ℓ for each ℓ such that $p_\ell > p_-$, we then conclude that $\hat S_N$ has asymptotically no eigenvalue in $[0,\varepsilon]$ but that $\lambda_\ell(\hat S_N) \xrightarrow{\text{a.s.}} \Lambda_\ell$ for all $\ell \in \mathcal{L}$, which is the expected result.

The precise localization of the eigenvalues of $\hat S_N$ will be fundamental for the proof of Theorems 4.2.2 and 4.2.3. To prove Theorem 4.2.1 though, we need to generalize part of this result to the matrices $\hat S_{(j)}$ and $\hat S_{(j),\eta}$ defined at the beginning of the section. Precisely, we need to show that there exists $\varepsilon > 0$ such that $\min_{1\le j\le N}\{\lambda_N(\hat S_{(j)})\} > \varepsilon$ for all large n a.s., and similarly for $\hat S_{(j),\eta}$.

Take $j \in \{1,\ldots,n\}$. Replacing $\hat{S}_N$ by $\hat{S}_{(j)}$ in the proof above leads to the same conclusions. Indeed, by a rank-one perturbation argument (Silverstein and Bai, 1995, Lemma 2.6), for each $\varepsilon > 0$, for all large n a.s.

$$\frac{1}{n}\operatorname{tr}\tilde{Q}_z^\circ - \frac{1}{n}\operatorname{tr}\left(\frac{1}{n}\tilde{W}_{(j)}T_{(j)}V_{(j)}\tilde{W}_{(j)} - zI_N\right)^{-1} \leq \frac{1}{n}\frac{1}{\operatorname{dist}(z,[\varepsilon,S^+ + \varepsilon])}$$

and therefore, up to replacing all matrices X by $X_{(j)}$ in their statements, Lemmas 4.4 and 4.5 hold identically (with $\delta(z)$ unchanged). Exploiting $\frac{1}{n-1}\sum_{i\neq j}\boldsymbol{\delta}_{\tau_i} \to \tilde{\nu}$ a.s., the remainder of the proof unfolds all the same and we have in particular that for all large n a.s. $\hat{S}_{(j)}$ has no eigenvalue below some $\varepsilon > 0$.

We now prove that this result can be made uniform across j. Denote $\Gamma_{L,(j)}(z)$ the matrix $\Gamma_L(z)$ with all matrices X replaced by $X_{(j)}$. Also rename Lemmas 4.4 and 4.5 respectively Lemma 4.4-(j) and Lemma 4.5-(j), and rename $\mathcal{A}_\varepsilon$ by $\mathcal{A}_{\varepsilon,(j)}$ in the statement of Lemma 4.4-(j). Then, taking $p > 4$ in Lemma 4.4-(j), by the union bound and the Markov inequality, for $e > 0$,

$$\begin{aligned}
&P\left(\max_{1\leq j\leq n} 1_{A_{\varepsilon,(j)}}\left\|\Gamma_{L,(j)}(z) - \begin{bmatrix}\Omega^2\frac{\delta(z)}{c}\frac{1}{n}\operatorname{tr}V & \Omega\delta(z)\frac{1}{n}\operatorname{tr}V^2T\Psi_z^\circ \\ \Omega\frac{\delta(z)}{c} & 0\end{bmatrix}\right\| > e\right) \\
&\leq \frac{1}{e^p}\sum_{j=1}^n \mathrm{E}\left[1_{A_{\varepsilon,(j)}}\left\|\Gamma_{L,(j)}(z) - \begin{bmatrix}\Omega^2\frac{\delta(z)}{c}\frac{1}{n}\operatorname{tr}V & \Omega\delta(z)\frac{1}{n}\operatorname{tr}V^2T\Psi_z^\circ \\ \Omega\frac{\delta(z)}{c} & 0\end{bmatrix}\right\|^p\right] \\
&= O(N^{1-\frac{p}{2}})
\end{aligned}$$

which is summable. By the Borel Cantelli lemma, the event in the probability parentheses then converges a.s. to zero. Finally, from Lemma A.4, there exists $\varepsilon > 0$ such that $1_{\cap_{j=1}^n A_{\varepsilon,(j)}} \xrightarrow{\text{a.s.}} 1$. We then conclude that, for each $z \in \mathcal{C} \subset \mathbb{C}\setminus[\varepsilon, S_\mu^+ + \varepsilon]$ for some $\varepsilon > 0$,

$$\sup_{1\leq j\leq n}\left\|\Gamma_{L,(j)}(z) - \begin{bmatrix}\Omega^2\frac{\delta(z)}{c}\frac{1}{n}\operatorname{tr}V & \Omega\delta(z)\frac{1}{n}\operatorname{tr}V^2T\Psi_z^\circ \\ \Omega\frac{\delta(z)}{c} & 0\end{bmatrix}\right\| \xrightarrow{\text{a.s.}} 0$$

Let now $\mathcal{V} \subset \mathbb{C}\setminus[\varepsilon, S_\mu^+ + \varepsilon]$ be a bounded open set containing $[0,\varepsilon/2]$ and $\mathcal{I}$ be its smooth boundary. Taking the determinant of each matrix inside the norm and using again the analyticity of the functions involved, we now get that the quantity

$$\frac{1}{2\pi\imath}\oint_{\mathcal{I}}\frac{[\det(I_{2L} + \Gamma_{L,(j)}(z))]'}{\det(I_{2L} + \Gamma_{L,(j)}(z))}dz$$

converges almost surely uniformly across $j \in \{1,\ldots,n\}$ to the number of eigenvalues of any of the $\hat{S}_{(j)}$ within $[0,\varepsilon/2]$. But by the previous proof, this must be zero. Hence, for all large n a.s., none of the $\hat{S}_{N,(j)}$ has eigenvalues smaller than $\varepsilon/2$, which is what we wanted.

Let now (η, M_η) be such that $\tilde{\nu}((0,M_\eta)) > 0$ and $\tilde{\nu}((M_\eta,\infty)) < \eta$. We have now $\frac{1}{n}\sum_{i=1}^n 1_{\tau_i\leq M_\eta}\boldsymbol{\delta}_{\tau_i} \xrightarrow{\text{a.s.}} \tilde{\nu}\eta \triangleq c_\eta\tilde{\nu} + (1-c_\eta)\boldsymbol{\delta}_0$ with $c_\eta = \lim_n n^{-1}|\{\tau_i \leq M_\eta\}| = 1-\eta$ (which almost surely exists by the law of large numbers), so that $\tilde{\nu}_\eta([0,m)) < \eta + (1-\eta)(1-\phi_\infty^{-1})$ for some $m > 0$ (Assumption 4.5). Taking η small enough so that $\tilde{\nu}_\eta([0,m)) < 1-\phi_\infty^{-1}$, we are still under the assumptions of Theorem 4.1.2 and therefore we again have that for all large n a.s. none of the matrices $\hat{S}_{(j),\eta}$ has eigenvalues below a certain positive value $\varepsilon_\eta > 0$.

These elements are sufficient to now turn to the proof of the main theorems.

4.2.4.4 Proof of Theorem 4.2.1

When $p_1 = \ldots = p_L = 0$, Theorem 4.2.1 unfolds directly from Theorem 4.1.2. Indeed, in this scenario, the latter result states

$$\left\| \hat{C}_N - \frac{1}{n} \sum_{i=1}^n v(\tau_i \gamma_N) w_i w_i^* \right\| \xrightarrow{\text{a.s.}} 0 \tag{4.47}$$

with γ_N the unique positive solution to

$$1 = \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{1 + c_n \psi(\tau_i \gamma_N)}.$$

Using $\frac{1}{n} \sum_{i=1}^n \boldsymbol{\delta}_{\tau_i} \xrightarrow{\text{a.s.}} \tilde{\nu}$, $c_n \to c$, along with the boundedness of ψ, we have that any accumulation point $\gamma \in [0, \infty]$ of γ_N as $n \to \infty$ must satisfy

$$1 = \int \frac{\psi_c(\tau\gamma)\tilde{\nu}(d\tau)}{1 + c\psi_c(\tau\gamma)}$$

the solution of which is easily shown to be unique in $(0, \infty)$ as the right-hand side term is increasing in γ with limits zero as $\gamma \to 0$ and $\psi_\infty > 1$ as $\gamma \to \infty$ (unless $\tilde{\nu} = \boldsymbol{\delta}_0$ which is excluded). Using the continuity and boundedness of v, it then comes $\max_i |v(\tau_i \gamma_N) - v_c(\tau_i \gamma)| \xrightarrow{\text{a.s.}} 0$. Now, $w_i w_i^* = (w_i w_i^* r_i^2/N)/(r_i^2/N)$ where in the numerator $w_i r_i/\sqrt{N}$ is Gaussian and where the denominator satisfies $\max_i |r_i^2/N - 1| \xrightarrow{\text{a.s.}} 0$ (using classical probability bounds on the chi-square distribution). With these results, along with (Bai and Silverstein, 1998) which ensures that $\frac{1}{nN} \sum_i w_i w_i^* r_i^2$ has bounded spectral norm for all large n a.s., Theorem 4.1.2 implies

$$\left\| \hat{C}_N - \frac{1}{nN} \sum_{i=1}^n v_c(\tau_i \gamma) w_i w_i^* r_i^2 \right\| \xrightarrow{\text{a.s.}} 0$$

which is the desired result for $p_1 = \ldots = p_L = 0$.

The generalization to generic $p_1, \ldots, p_L$ follows from a careful control of the elements of proof of Theorem 4.1.2. We see that Lemma 4.1 and Remark 4.1 are not affected by $p_1, \ldots, p_L$ as these results only depend on $\tau_1, \ldots, \tau_n$. The fundamental Lemma 4.2 (and its extension Remark 4.2) as well as Lemma 4.3 however need be updated.

We shall not go into the details of every generalization which is painstaking and in fact similar for each lemma. Instead, we detail the generalization of the important remark Remark 4.2 and merely give elements for the other results. Remark 4.2 is now updated as follows.

Lemma 4.6. *Let (η, M_η) be couples indexed by $\eta \in (0,1)$ such that $\tilde{\nu}((0, M_\eta)) > 0$ and $\tilde{\nu}((M_\eta, \infty)) < \eta$ and define γ^η as the unique solution to (4.38). Also let $M > 0$ be arbitrary. Then, for all η small enough,*

$$\max_{\substack{1\leq j\leq n\\ \tau_j\leq M}} \left| \frac{1}{N} x_j^* \left(\frac{1}{n} \sum_{\tau_i \leq M_\eta, i\neq j} v\left(\tau_i\gamma^\eta\right) x_i x_i^* \right)^{-1} x_j - \tau_j \gamma^\eta \right| \xrightarrow{\text{a.s.}} 0$$
$$\max_{\substack{1\leq j\leq n\\ \tau_j> M}} \left| \frac{1}{\tau_j} \frac{1}{N} x_j^* \left(\frac{1}{n} \sum_{\tau_i \leq M_\eta, i\neq j} v\left(\tau_i\gamma^\eta\right) x_i x_i^* \right)^{-1} x_j - \gamma^\eta \right| \xrightarrow{\text{a.s.}} 0. \tag{4.48}$$

Proof. Note that, replacing the terms x_i by $\tau_i w_i$ in (4.48) gives exactly Remark 4.2. To ensure that the result holds, we then only need verify that the terms involving AS become negligible.

For η sufficiently small, define

$$\check{S}_{(j),\eta} = \frac{1}{n} \sum_{\tau_i \leq M_\eta, i\neq j} v\left(\tau_i\gamma^\eta\right) x_i x_i^* = \frac{1}{n}(AS_{(j)} + W_{(j)})V_{(j),\eta}(AS_{(j)} + W_{(j)})^*.$$

Using the fact that $\max_{1\leq i\leq n}\{|r_i/\sqrt{N} - 1|\} \xrightarrow{\text{a.s.}} 0$ and that all matrices in the equality above have bounded norm almost surely by (Bai and Silverstein, 1998), we then have $\sup_{1\leq j\leq n} \|\check{S}_{(j),\eta} - \hat{S}_{(j),\eta}\| \xrightarrow{\text{a.s.}} 0$. From the results in the previous section, we then conclude that there exists $\varepsilon > 0$ such that the eigenvalues of $\check{S}_{(j),\eta}$ for all j are all greater than ε for all large n almost surely. Now, recalling that $S = [s_1, \ldots, s_n]$,

$$\frac{1}{N} x_j^* \check{S}_{(j),\eta}^{-1} x_j = \frac{1}{N} s_j^* A^* \check{S}_{(j),\eta}^{-1} A s_j + 2\Re\left[\sqrt{\tau_j} \frac{1}{N} s_j^* A^* \check{S}_{(j),\eta}^{-1} w_j \right] + \tau_j \frac{1}{N} w_j^* \check{S}_{(j),\eta}^{-1} w_j.$$

By the trace lemma (Bai and Silverstein, 2009, Lemma B.26), denoting $\mathcal{A}$ the probability set over which the eigenvalues of $\check{S}_{(j),\eta}$ for all j are greater than ε, for each $p > 2$,

$$\mathrm{E}\left[1_{\mathcal{A}} \left| \frac{1}{N} \tilde{w}_j^* \check{S}_{(j),\eta}^{-1} \tilde{w}_j - \frac{1}{N} \operatorname{tr} \check{S}_{(j),\eta}^{-1} \right|^p \right] \leq K N^{-\frac{p}{2}}$$

where K only depends on ε (which is obtained by first conditioning on $\tilde{W}_{(j)}$ then averaging over it). Taking $p > 3$ and using the union bound on n events, the Markov inequality and the Borel Cantelli lemma, along with $1_{\mathcal{A}} \xrightarrow{\text{a.s.}} 1$ and $\max_j\{|r_j^2/N - 1|\} \xrightarrow{\text{a.s.}} 0$, leads to

$$\max_{1\leq j\leq n} \left| \frac{1}{N} w_j^* \check{S}_{(j),\eta}^{-1} w_j - \frac{1}{N} \operatorname{tr} \check{S}_{(j),\eta}^{-1} \right| \xrightarrow{\text{a.s.}} 0.$$

Using the same result and the fact that $\frac{1}{N} \operatorname{tr} A^* \check{S}_{(j),\eta}^{-1} A \leq K\varepsilon^{-1}/N$ for all large n a.s., we also have

$$\max_{1\leq j\leq n} \left| \frac{1}{N} s_j^* A^* \check{S}_{(j),\eta}^{-1} A s_j \right| \xrightarrow{\text{a.s.}} 0.$$

Using both results and $|\frac{1}{N}s_j^* A^* \check{S}_{(j),\eta}^{-1} w_j|^2 \leq \frac{1}{N}s_j^* A^* \check{S}_{(j),\eta}^{-1} A s_j \frac{1}{N} w_j^* \check{S}_{(j),\eta}^{-1} w_j$ (Cauchy–Schwarz inequality), we finally get

$$\max_{1\leq j\leq n} \left| \frac{1}{N} s_j^* A^* \check{S}_{(j),\eta}^{-1} w_j \right| \xrightarrow{\text{a.s.}} 0.$$

All this then ensures that

$$\max_{1\leq j\leq n, \tau_j \leq M} \left| \frac{1}{N} x_j^* \check{S}_{(j),\eta}^{-1} x_j - \tau_j \frac{1}{N} \operatorname{tr} \check{S}_{(j),\eta}^{-1} \right| \xrightarrow{\text{a.s.}} 0$$
$$\max_{1\leq j\leq n, \tau_j > M} \left| \frac{1}{\tau_j} \frac{1}{N} x_j^* \check{S}_{(j),\eta}^{-1} x_j - \frac{1}{N} \operatorname{tr} \check{S}_{(j),\eta}^{-1} \right| \xrightarrow{\text{a.s.}} 0.$$

Since A has rank at most L, $\check{S}_{(j),\eta}$ is an at most rank-$2L+1$ perturbation of $\frac{1}{n} W T_\eta V_\eta W^*$, i.e., the matrix obtained for $p_1 = \ldots = p_L = 0$, by an additive symmetric matrix. A $(2L+1)$-fold application of the rank-one perturbation lemma (Silverstein and Bai, 1995, Lemma 2.6) along with the facts that $\|W - \tilde{W}\| \xrightarrow{\text{a.s.}} 0$ and that all eigenvalues of the matrices involved are uniformly away from zero almost surely then ensures that

$$\max_{1\leq j\leq n} \left| \frac{1}{N} \operatorname{tr} \check{S}_{(j),\eta}^{-1} - \frac{1}{N} \operatorname{tr} \left(\frac{1}{n} W T_\eta V_\eta W^* \right)^{-1} \right| \xrightarrow{\text{a.s.}} 0.$$

But now, recalling Remark 4.2, $\frac{1}{N} \operatorname{tr} \left(\frac{1}{n} W T_\eta V_\eta W^* \right)^{-1} \xrightarrow{\text{a.s.}} \gamma^\eta$. Putting these results together finally leads to the requested result

$$\max_{1\leq j\leq n, \tau_j \leq M} \left| \frac{1}{N} x_j^* \check{S}_{(j),\eta}^{-1} x_j - \tau_j \gamma^\eta \right| \xrightarrow{\text{a.s.}} 0$$
$$\max_{1\leq j\leq n, \tau_j > M} \left| \frac{1}{\tau_j} \frac{1}{N} x_j^* \check{S}_{(j),\eta}^{-1} x_j - \gamma^\eta \right| \xrightarrow{\text{a.s.}} 0.$$

□

Note that the proof only exploits the boundedness away from zero of the various matrices involved and not their bounded spectral norm. Therefore, with the same derivations, we also generalize Lemma 4.2 as follows.

Lemma 4.7. *For every $M > 0$, we have*

$$\max_{\substack{1\leq j\leq n \\ \tau_j \leq M}} \left| \frac{1}{N} x_j^* \left(\frac{1}{n} \sum_{i\neq j} v\left(\tau_i \gamma^\eta\right) x_i x_i^* \right)^{-1} x_j - \tau_j \gamma \right| \xrightarrow{\text{a.s.}} 0$$
$$\max_{\substack{1\leq j\leq n \\ \tau_j > M}} \left| \frac{1}{\tau_j} \frac{1}{N} x_j^* \left(\frac{1}{n} \sum_{i\neq j} v\left(\tau_i \gamma^\eta\right) x_i x_i^* \right)^{-1} x_j - \gamma \right| \xrightarrow{\text{a.s.}} 0.$$

Define now $d_i = \frac{1}{\tau_i}\frac{1}{N}x_i^*\hat{C}_{(i)}^{-1}x_i$ with $\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n}u(\frac{1}{N}x_i^*\hat{C}_N^{-1}x_i)x_ix_i^*$, from which in passing we can write

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^n v(\tau_i d_i)x_ix_i^* = \frac{1}{n}\sum_{i=1}^n v(\tau_i d_i)A_i\bar{\bar{w}}_i\bar{\bar{w}}_i^*A_i^* \tag{4.49}$$

with $\bar{\bar{w}}_i = [s_{1i},\ldots,s_{Li},w_i]^\mathsf{T}$. Then Lemma 4.3 remains valid and reads

Lemma 4.8. *There exists $d_+ > d_- > 0$ such that, for all large n a.s.*

$$d_- < \liminf_n \min_{1\leq i\leq n} d_i \leq \limsup_n \max_{1\leq i\leq n} d_i < d_+.$$

Proof. Taking $m>0$ small enough and denoting $d_{\max} = \max_j d_j$, Equation (4.14) becomes here

$$\hat{C}_{(j)} \succeq mv(md_{\max})\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}\frac{1}{\tau_i}\left(As_is_i^*A^* + 2\sqrt{\tau_i}\Re\left[w_i^*As_i\right] + \tau_iw_iw_i^*\right)$$

so that, taking j such that $d_j = d_{\max}$,

$$d_{\max} \leq \frac{1}{mv(md_{\max})}\frac{1}{\tau_j}\frac{1}{N}x_j^*\left(\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\geq m}}\frac{As_is_i^*A^* + 2\sqrt{\tau_i}\Re\left[w_i^*As_i\right] + \tau_iw_iw_i^*}{\tau_i}\right)^{-1}x_j.$$

If $\liminf_n \tau_j > 0$ (with j always defined to be such that $d_j = d_{\max}$), with the same arguments as in the proof of Lemma 4.6 (here the boundedness from above of the τ_i is irrelevant) and recalling Lemma A.4, the right-hand side term can be bounded by $(mv_c(md_{\max})(1-c))^{-1}(1+\varepsilon)$ for arbitrarily small $\varepsilon>0$ by taking m small enough and n large enough. From there the proof of Lemma 4.3 for the boundedness of $d_{\max}$ remains valid. If instead $\liminf_n \tau_j = 0$, we restrict ourselves to a subsequence over which $\tau_j \to 0$. Multiplying both sides of the equation above by τ_j, we get by a similar result as Lemma 4.6 that $\tau_j d_{\max}$ can be bounded by $\tau_j(mv_c(md_{\max})(1-c))^{-1}(1+\varepsilon)$ for arbitrarily small $\varepsilon>0$ (again taking m small and n large), and the result unfolds again.

To obtain the lower bound, in the proof of Lemma 4.3, denoting $d_{\min} = \min_j d_j$, one needs now write

$$\begin{aligned}\hat{C}_{(j)} \preceq{}& Mv(Md_{\min})\frac{1}{n}\sum_{\substack{i\neq j\\ m\leq\tau_i\leq M}}\frac{1}{\tau_i}\left(As_is_i^*A^* + 2\sqrt{\tau_i}\Re\left[w_i^*As_i\right] + \tau_iw_iw_i^*\right)\\ &+ v(0)\frac{1}{n}\sum_{\substack{i\neq j\\ \tau_i\in\mathbb{R}\setminus[m,M]}}\left(As_is_i^*A^* + 2\sqrt{\tau_i}\Re\left[w_i^*As_i\right] + \tau_iw_iw_i^*\right).\end{aligned}$$

The controls established for the upper bound on $d_{\max}$ can be similarly used here for $d_{\min}$ and the proof of Lemma 4.3 for $d_{\min}$ unfolds then similarly. $\square$

Equipped with these lemmas, the proof of Theorem 4.2.1 unfolds similar to the proof of Theorem 4.1.2 but for a particular care to be taken for terms involving $\tau_j^{-1}x_j$ which need to be controlled if $\liminf_n \tau_j = 0$. This is easily performed as previously by either using approximations of d_j or of $\tau_j d_j$ depending on whether $\liminf_n \tau_j > 0$ or $\liminf_n \tau_j = 0$, respectively. Assumption 4.5 is precisely used here. In particular, by the end of the proof, we obtain similar to Section 4.1 the important convergence

$$\begin{aligned} \max_{1\leq j\leq n, \tau_j < M} |\tau_j d_j - \tau_j \gamma| &\xrightarrow{\text{a.s.}} 0 \\ \max_{1\leq j\leq n, \tau_j \geq M} |d_j - \gamma| &\xrightarrow{\text{a.s.}} 0 \end{aligned} \tag{4.50}$$

from which, expanding both $\hat{S}_N$ and $\hat{C}_N$ as in (4.40) (noting the similarity between (4.39) and (4.49)) and exploiting the almost sure asymptotic boundedness in norm of the various matrices then involved, we obtain $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ as desired.

4.2.4.5 Eigenvalues of $\hat{C}_N$ and power estimation

From Theorem 4.2.1, $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ so that in particular $\max_{1\leq i\leq n} |\lambda_i(\hat{C}_N) - \lambda_i(\hat{S}_N)| \xrightarrow{\text{a.s.}} 0$. This means that it suffices to study the individual eigenvalues of $\hat{S}_N$ in order to study the individual eigenvalues of $\hat{C}_N$. In particular, from the results of Section 4.2.4.3, we have that, for any small $\varepsilon > 0$, $\hat{C}_N$ has asymptotically no eigenvalue in $[0, \varepsilon]$ almost surely, that $\hat{\lambda}_{|\mathcal{L}|+i} < S^+ + \varepsilon$ for all large n a.s. for each $i \in \{1, \dots, N - |\mathcal{L}|\}$ and that $\hat{\lambda}_i \xrightarrow{\text{a.s.}} \Lambda_i > S^+$ for each $i \in \mathcal{L}$, where Λ_i is as in the statement of Theorem 4.2.2, Item 0. Along with the continuity of δ and $\delta((S^+, \infty)) \subset (-(\tau_+ v_c(\tau_+ \gamma)), 0)$, we then get Theorem 4.2.2, Item 1.

4.2.4.6 Localization function estimation

Let $a, b \in \mathbb{C}^N$ be two vectors of unit norm. Then, from the first part of Theorem 4.2.2 and from Cauchy's integral formula, for any $k \in \mathcal{L}$ and for all large N a.s.,

$$\sum_{\substack{1\leq i\leq L \\ p_i = p_\ell}} a^* \hat{u}_i \hat{u}_i^* b = -\frac{1}{2\pi\imath} \oint_{\mathcal{I}_\ell} a^* \left(\hat{C}_N - zI_N\right)^{-1} b dz \tag{4.51}$$

for $\mathcal{I}_\ell$ defined as above as a positively oriented contour around a sufficiently small neighborhood of Λ_ℓ, where Λ_ℓ is the unique positive solution of the equation in Λ (4.46) when $p_k = p_\ell$. Using $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ along with the uniform boundedness of $\|(\hat{S}_N - zI_N)^{-1}\|$ and $\|(\hat{C}_N - zI_N)^{-1}\|$ on $\mathcal{I}_\ell$ (for all n large), we then have

$$\sum_{\substack{1\leq i\leq L \\ p_i = p_\ell}} a^* \hat{u}_i \hat{u}_i^* b + \frac{1}{2\pi\imath} \oint_{\mathcal{I}_\ell} a^* \left(\hat{S}_N - zI_N\right)^{-1} b dz \xrightarrow{\text{a.s.}} 0$$

so that it suffices to determine the second left-hand side expression.

Let us develop the term $a^*(\hat{S}_N - zI_N)^{-1}b$. Proceeding similar to Section 4.2.4.3, we find

$$a^*\left(\hat{S}_N - zI_N\right)^{-1} b = a^*\left(\hat{S}_N^\circ - zI_N + \Gamma\right)^{-1} b$$

with Γ defined in (4.41). Using Woodbury's identity $(A+BCB^*)^{-1} = A^{-1} - A^{-1}B(C^{-1} + B^*A^{-1}B)^{-1}B^*A^{-1}$ for invertible A, B, this becomes, with the same notations as in the previous paragraph,

$$a^*\left(\hat{S}_N - zI_N\right)^{-1} b = a^*Q_z^\circ b - a^*Q_z^\circ G\left(H^{-1} + G^*Q_z^\circ G\right)^{-1} G^*Q_z^\circ b \tag{4.52}$$

where

$$\begin{aligned} G &= \begin{bmatrix} U\Omega^{\frac{1}{2}} & \frac{1}{n}\tilde{W}T^{\frac{1}{2}}VS^*\bar{U}\Omega^{\frac{1}{2}} \end{bmatrix} \\ H &= \begin{bmatrix} \Omega^{\frac{1}{2}}\bar{U}^*\frac{1}{n}\tilde{W}V\tilde{W}^*\bar{U}\Omega^{\frac{1}{2}} & I_L \\ I_L & 0 \end{bmatrix}. \end{aligned}$$

The matrix H is clearly invertible and we then find, using Lemma 4.4 and Lemma 4.5 that, uniformly on z in a small neighborhood of Λ_ℓ,

$$\left\| H^{-1} - \begin{bmatrix} 0 & I_L \\ I_L & -\Omega\frac{1}{n}\operatorname{tr} V \end{bmatrix} \right\| \xrightarrow{\text{a.s.}} 0$$

so that, again by Lemma 4.4 and Lemma 4.5,

$$\left\| H^{-1} + G^*Q_z^\circ G - \begin{bmatrix} \Omega\frac{\delta(z)}{c} & I_L \\ I_L & -\Omega\frac{1}{n}\operatorname{tr} V\Psi_z^\circ \end{bmatrix} \right\| \xrightarrow{\text{a.s.}} 0. \tag{4.53}$$

To ensure that $H^{-1} + G^*Q_z^\circ G$ is invertible for $z \in \mathcal{I}_\ell$, let us study the determinant of the rightmost matrix. We have easily

$$\det\left(\begin{bmatrix} \Omega\frac{\delta(z)}{c} & I_L \\ I_L & -\Omega\frac{1}{n}\operatorname{tr} V\Psi_z^\circ \end{bmatrix}\right) = \det\left(-\Omega^2\frac{\delta(z)}{c}\frac{1}{n}\operatorname{tr} V\Psi_z^\circ - I_L\right).$$

From the discussion around (4.46), the right-hand side term cancels exactly once in a neighborhood of $z = \Lambda_k$ for each $k \in \mathcal{L}$. Now, for $z \in \mathbb{C}\setminus\mathbb{R}$, it is easily seen that it has non-zero imaginary part. Therefore, since the convergence (4.53) is uniform on a small neighborhood of Λ_ℓ, for all large n a.s., the determinant of $H^{-1} + G^*Q_z^\circ G$ is uniformly away from zero on $\mathcal{I}_\ell$ (up to taking n larger). We can then freely take inverses in (4.53) and have, uniformly on $\mathcal{I}_\ell$,

$$\left\| (H^{-1} + G^*Q_z^\circ G)^{-1} - \begin{bmatrix} \Omega\frac{\delta(z)}{c} & I_L \\ I_L & -\Omega\frac{1}{n}\operatorname{tr} V\Psi_z^\circ \end{bmatrix}^{-1} \right\| \xrightarrow{\text{a.s.}} 0.$$

To compute the inverse of the rightmost matrix, it is convenient to write

$$\begin{bmatrix} \Omega\frac{\delta(z)}{c} & I_L \\ I_L & \Omega\frac{1}{n}\operatorname{tr} V\Psi_z^\circ \end{bmatrix} = P\left\{\begin{bmatrix} \sigma_k\frac{\delta(z)}{c} & 1 \\ 1 & -\sigma_k\frac{1}{n}\operatorname{tr} V\Psi_z^\circ \end{bmatrix}\right\}_{k=1}^{L} P^*$$

where $\{A_k\}_{k=1}^L$ is a block-diagonal matrix with diagonal blocks $A_1,\ldots,A_L$ in this order, and where $P\in\mathbb{C}^{2L\times 2L}$ is the symmetric permutation matrix with $[P]_{ij}=\boldsymbol{\delta}_{j-(L+i/2)}$ for even $i\leq L$ and $[P]_{ij}=\boldsymbol{\delta}_{j-(i+1)/2}$ for odd $i\leq L$. With this notation, we have

$$\begin{bmatrix}\Omega\frac{\delta(z)}{c} & I_L\\ I_L & \Omega\frac{1}{n}\operatorname{tr}V\Psi_z^\circ\end{bmatrix}^{-1}=P\left\{\frac{-1}{\frac{\delta(z)}{c}\sigma_k^2\frac{1}{n}\operatorname{tr}V\Psi_z^\circ+1}\begin{bmatrix}-\sigma_k\frac{1}{n}\operatorname{tr}V\Psi_z^\circ & -1\\ -1 & \sigma_k\frac{\delta(z)}{c}\end{bmatrix}\right\}_{k=1}^L P^*.$$

Denoting $U=[u_1,\ldots,u_L]\in\mathbb{C}^{N\times L}$ and $\bar U=[\bar u_1,\ldots,\bar{u_L}]\in\mathbb{C}^{L\times L}$, we have

$$GP=\begin{bmatrix}\sqrt{\sigma_1}u_1 & \sqrt{\sigma_1}\frac{1}{n}\tilde WT^{\frac12}VS^*\bar u_1 & \cdots & \sqrt{\sigma_L}u_L & \sqrt{\sigma_L}\frac{1}{n}\tilde WT^{\frac12}VS^*\bar u_L\end{bmatrix}.$$

From this remark, using again Lemma 4.4 and Lemma 4.5, we finally have

$$\sup_{z\in\mathcal{I}_\ell}\left|a^*Q_z^\circ G(H^{-1}+G^*Q_z^\circ G)^{-1}G^*Q_z^\circ b-\sum_{k=1}^L a^*u_ku_k^*b\frac{\frac{\delta(z)^2}{c^2}\sigma_k^2\frac{1}{n}\operatorname{tr}V\Psi_z^\circ}{\frac{\delta(z)}{c}\sigma_k^2\frac{1}{n}\operatorname{tr}V\Psi_z^\circ+1}\right|\xrightarrow{\text{a.s.}}0.$$

Putting things together, using the results above which we recall are uniform on $\mathcal{I}_\ell$, and also using the fact that Q_z° has no pole in $\mathcal{I}_\ell$, we finally have

$$\sum_{\substack{1\leq i\leq L\\ p_i=p_\ell}}a^*\hat u_i\hat u_i^*b-\sum_{k=1}^L\frac{1}{2\pi\imath}\oint_{\mathcal{I}_\ell}a^*u_ku_k^*b\frac{\frac{\delta(z)^2}{c^2}\sigma_k^2\frac{1}{n}\operatorname{tr}V\Psi_z^\circ}{\frac{\delta(z)}{c}\sigma_k^2\frac{1}{n}\operatorname{tr}V\Psi_z^\circ+1}dz\xrightarrow{\text{a.s.}}0$$

which, after taking the limits on the fraction in the integrand, gives

$$\sum_{\substack{1\leq i\leq L\\ p_i=p_\ell}}a^*\hat u_i\hat u_i^*b-\sum_{k=1}^L\frac{1}{2\pi\imath}\oint_{\mathcal{I}_\ell}a^*u_ku_k^*b\frac{\frac{\delta(z)^2}{c^2}p_k\int\frac{v(\tau\gamma)\tilde\nu(d\tau)}{1+\tau v(\tau\gamma)\delta(z)}}{\frac{\delta(z)}{c}p_k\int\frac{v(\tau\gamma)\tilde\nu(d\tau)}{1+\tau v(\tau\gamma)\delta(z)}+1}dz\xrightarrow{\text{a.s.}}0$$

For $z\in(S^+,\infty)$, we already saw that $\delta(z)$ is negative while $\int\frac{v(\tau\gamma)\tilde\nu(d\tau)}{1+\tau v(\tau\gamma)\delta(z)}$ is positive. For z non real, both quantities are non real, and therefore do no have poles in $\mathcal{I}_\ell$. The only pole is then obtained for $\frac{\delta(z)}{c}p_k\int\frac{v(\tau\gamma)\tilde\nu(d\tau)}{1+\tau v(\tau\gamma)\delta(z)}+1=0$, that is for $z=\Lambda_\ell$ as defined in the previous section. Using l'Hospital rule, the residue of the right complex integral is then evaluated to be

$$\begin{aligned}\operatorname{Res}(\Lambda_\ell)&=\lim_{z\to\Lambda_\ell}(z-\Lambda_\ell)a^*\Pi_\ell b\frac{\delta(z)}{c}\frac{\int\frac{p_\ell v(t\gamma)}{1+tv(t\gamma)\delta(z)}\tilde\nu(dt)}{\int\frac{p_\ell v(t\gamma)}{1+tv(t\gamma)\delta(z)}\tilde\nu(dt)+\frac{c}{\delta(z)}}\\&=\lim_{z\to\Lambda_\ell}a^*\Pi_\ell b\frac{\frac{\delta(z)}{c}\int\frac{v(t\gamma)p_\ell}{1+tv(t\gamma)\delta(z)}\tilde\nu(dt)+(z-\Lambda_\ell)\frac{d}{dz}\left(\frac{\delta(z)}{c}\int\frac{p_\ell v(t\gamma)}{1+tv(t\gamma)\delta(z)}\tilde\nu(dt)\right)}{-c\frac{\delta'(z)}{\delta(z)^2}-\int\frac{tv(t\gamma)^2p_\ell\delta'(z)}{(1+tv(t\gamma)\delta(z))^2}\tilde\nu(dt)}\\&=a^*\Pi_\ell b\left(c\frac{\delta'(\Lambda_\ell)}{\delta(\Lambda_\ell)^2}+p_\ell\delta'(\Lambda_\ell)\int\frac{\tau v_c(\tau\gamma)^2\tilde\nu(d\tau)}{(1+\tau v_c(\tau\gamma)\delta(\Lambda_\ell))^2}\right)^{-1}\end{aligned}\tag{4.54}$$

where $\Pi_\ell \triangleq \sum_{i,p_i=p_\ell} u_i u_i^*$ and the last equality uses $\frac{\delta(\Lambda_\ell)}{c} p_\ell \int \frac{v(\tau\gamma)\tilde{\nu}(d\tau)}{1+\tau v(\tau\gamma)\delta(\Lambda_\ell)} = -1$. Recall now that

$$\delta(\Lambda_\ell) = c\left(-\Lambda_\ell + \int \frac{\tau v_c(\tau\gamma)}{1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma)}\tilde{\nu}(d\tau)\right)^{-1}$$

from which

$$\delta'(\Lambda_\ell) = \frac{\delta(\Lambda_\ell)^2}{c}\left(1 - \frac{\delta(\Lambda_\ell)^2}{c}\int \frac{\tau^2 v_c(\tau\gamma)^2}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}\tilde{\nu}(d\tau)\right)^{-1} > 0.$$

From the expression of p_ℓ in the previous paragraph and these values, we then further find

$$\begin{aligned}
\operatorname{Res}(\Lambda_\ell) &= a^*\Pi_\ell b\left(1 - \frac{\int \frac{\delta(\Lambda_\ell)\tau v_c(\tau\gamma)^2\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}}{\int \frac{v_c(\tau\gamma)\tilde{\nu}(d\tau)}{1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma)}}\right)^{-1}\left(1 - \frac{\delta(\Lambda_\ell)^2}{c}\int \frac{t^2 v_c(\tau\gamma)^2\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}\right) \\
&= a^*\Pi_\ell b\frac{\int \frac{v_c(\tau\gamma)\tilde{\nu}(d\tau)}{1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma)}\left(1 - \frac{\delta(\Lambda_\ell)^2}{c}\int \frac{t^2 v_c(\tau\gamma)^2\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}\right)}{\int \frac{v_c(\tau\gamma)\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}}.
\end{aligned}$$

Inverting the relation

$$\sum_{\substack{1\leq i\leq L\\ p_i=p_\ell}} a^*\hat{u}_i\hat{u}_i^* b - a^*\Pi_\ell b\frac{\int \frac{v_c(\tau\gamma)\tilde{\nu}(d\tau)}{1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma)}\left(1 - \frac{\delta(\Lambda_\ell)^2}{c}\int \frac{t^2 v_c(\tau\gamma)^2\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}\right)}{\int \frac{v_c(\tau\gamma)\tilde{\nu}(d\tau)}{(1+\delta(\Lambda_\ell)\tau v_c(\tau\gamma))^2}} \xrightarrow{\text{a.s.}} 0$$

and using $\hat{\lambda}_\ell \xrightarrow{\text{a.s.}} \Lambda_\ell$ for all $\ell \in \mathcal{L}$ then completes the proof.

4.2.4.7 Empirical estimators

To prove Theorem 4.2.3, one needs to ensure that the empirical estimators introduced in the statement of the theorem are consistent with the estimators introduced in Theorem 4.2.2.

Note first that $\gamma - \hat{\gamma}_n \xrightarrow{\text{a.s.}} 0$ is a consequence of (4.50). Indeed, letting $M > 0$, from (4.50),

$$\frac{1}{n}\sum_{\tau_j<M} \tau_j d_j - \gamma\frac{1}{n}\sum_{\tau_j<M} \tau_j \xrightarrow{\text{a.s.}} 0.$$

Still from (4.50), we also have, a.s.

$$\frac{1}{n}\sum_{\tau_j\geq M} \tau_j d_j - \gamma\frac{1}{n}\sum_{\tau_j\geq M} \tau_j = o\left(\frac{1}{n}\sum_{\tau_j\geq M} \tau_j\right).$$

But $\frac{1}{n}\sum_{\tau_j\geq M} \tau_j \xrightarrow{\text{a.s.}} \int_{(M,\infty)} t\tilde{\nu}(dt) \leq 1$ (say M is a continuity point of $\tilde{\nu}$). Also, $\frac{1}{n}\sum_j \tau_j \xrightarrow{\text{a.s.}} 1$. Putting the results together then gives $\gamma - \hat{\gamma}_n \xrightarrow{\text{a.s.}} 0$. From this, we now get, again with (4.50),

$$\max_{1\leq j\leq n, \tau_j\leq M}\left|\frac{\tau_j d_j}{\hat{\gamma}_n} - \tau_j\right| \xrightarrow{\text{a.s.}} 0$$

$$\max_{1\leq j\leq n, \tau_j> M}\left|\frac{d_j}{\hat{\gamma}_n} - 1\right| \xrightarrow{\text{a.s.}} 0$$

which is $\max_{\tau_j \leq M} |\tau_j - \hat{\tau}_j| \xrightarrow{\text{a.s.}} 0$ and $\max_{\tau_j > M} |\tau_j^{-1}\hat{\tau}_j - 1| \xrightarrow{\text{a.s.}} 0$, as desired.

We now need to prove that $\hat{\delta}(x) - \delta(x) \xrightarrow{\text{a.s.}} 0$ uniformly on any bounded set of $(S^+ + \varepsilon, \infty)$. For this, recall first that both $\hat{\delta}$ and δ are Stieltjes transforms of distributions with support contained in $[0, S^+]$ and, as such, are analytic in $(S^+ + \varepsilon, \infty)$ and uniformly bounded in any compact of $(S^+ + \varepsilon, \infty)$. Taking the difference and denoting $\hat{\tilde{\nu}}_n = \frac{1}{n} \sum_{i=1}^n \boldsymbol{\delta}_{\hat{\tau}_i}$, we have

$$
\begin{aligned}
&\hat{\delta}(x) - \delta(x) \\
&= \left(1 - \frac{c}{c_n}\right) \hat{\delta}(x) + \frac{\hat{\delta}(x)\delta(x)}{c_n} \left(\int \frac{tv_c(t\gamma)\tilde{\nu}(dt)}{1 + \delta(x)tv_c(t\gamma)} - \int \frac{tv_c(t\hat{\gamma}_n)\hat{\tilde{\nu}}_n(dt)}{1 + \hat{\delta}(x)tv_c(t\hat{\gamma}_n)} \right) \\
&= \left(1 - \frac{c}{c_n}\right) \hat{\delta}(x) + \frac{\hat{\delta}(x)\delta(x)}{c_n} \left((\hat{\delta}(x) - \delta(x)) \int \frac{t^2 v_c(t\gamma) v_c(t\hat{\gamma}_n)\tilde{\nu}(dt)}{(1 + \delta(x)tv_c(t\gamma))(1 + \hat{\delta}(x)tv_c(t\hat{\gamma}_n))} \right. \\
&\left. + \int \frac{t(v_c(t\gamma) - v_c(t\hat{\gamma}_n))\tilde{\nu}(dt)}{(1 + \delta(x)tv_c(t\gamma))(1 + \hat{\delta}(x)tv_c(t\hat{\gamma}_n))} + \int \frac{tv_c(t\hat{\gamma}_n)(\hat{\tilde{\nu}}_n(dt) - \tilde{\nu}(dt))}{1 + \hat{\delta}(x)tv_c(t\hat{\gamma}_n)} \right).
\end{aligned}
$$

From uniform boundedness of $tv_c(t\hat{\gamma}_n)$ and $tv_c(t\gamma)$, and $\hat{\tilde{\nu}}_n((t, M)) \xrightarrow{\text{a.s.}} \tilde{\nu}((t, M))$ weakly and $\hat{\gamma}_n \xrightarrow{\text{a.s.}} \gamma$, it is easily seen that the last two integrals on the right-hand side can be made arbitrarily small (e.g., by isolating $\tau_i \leq M$ and $\tau_i > M$ and letting M large enough in the previous convergence). Also, the first integral on the right hand side is clearly bounded. Gathering the terms $\hat{\delta}(x) - \delta(x)$ on the left-hand side and taking x large enough so to ensure $\hat{\delta}(x)\delta(x)$ is uniformly smaller than one (recall that their limit is zero as $x \to \infty$), we finally get that $\hat{\delta}(x) - \delta(x)$ can be made arbitrarily small. This is valid for any given large x and therefore on some sequence $\{x^{(i)}\}$ of $(S^+ + \varepsilon, \infty)$ having an accumulation point, $\hat{\delta}(x^{(i)})\delta(x^{(i)}) \xrightarrow{\text{a.s.}} 0$. Since $\hat{\delta}(x) - \delta(x)$ is complex analytic in $(S^+ + \varepsilon, \infty)$, by Vitali's convergence theorem, we therefore get that the convergence is uniform over any bounded set of $(S^+ + \varepsilon, \infty)$, which is what we wanted.

Since, for $i \in \mathcal{L}$ and for some $\varepsilon, M > 0$, $\hat{\lambda}_i \in [S^+ + \varepsilon, M]$ for all large n a.s., we therefore have that $\hat{\delta}(\hat{\lambda}_i) - \delta(\lambda_i) \xrightarrow{\text{a.s.}} 0$ for each $i \in \mathcal{L}$. Using all these convergence results, we then obtain, with the same line of arguments the asymptotic consistence between the estimates in Item 1. and Item 2. of both Theorems 4.2.2 and 4.2.3. This concludes the proof of Theorem 4.2.3.

4.2.4.8 Proof of Corollary 4.3

We are here in the same setting as (Hachem et al., 2013, Theorem 3), only for our improved model. The proof is the same as in (Hachem et al., 2013) and relies on showing the uniform convergence of $\hat{\eta}_{\text{RG}}(\theta) - \eta(\theta)$ across θ, from which the result unfolds. In our setting, the point-wise convergence easily follows from Items 3. in both Theorem 4.2.2 and Theorem 4.2.3. Uniform convergence then hinges on a regular discretization of the set $[0, 2\pi)$ into N^2 subsets and on (i) a Lipschitz control of the differences $\hat{\eta}_{\text{RG}}(\theta) - \hat{\eta}_{\text{RG}}(\theta')$ for $|\theta - \theta'| = O(N^{-2})$ and (ii) a joint convergence of $\hat{\eta}_{\text{RG}}(\theta) - \eta(\theta)$ over the $N^2 + 1$ edges of the subsets. Point (i) uses the defining properties of $a(\theta)$ from Assumption 4.7 similar to (Hachem et al., 2013), while Point (ii) is obtained thanks to a classical union bound on N^2 events, the validity of which follows from considering sufficiently high order moment bounds on the vanishing random quantities involved in $\hat{\eta}_{\text{RG}}(\theta) - \eta(\theta)$. In our setting, the latter moment bounds are obtained by selecting p large enough in Lemma 4.4 of Section 4.2.4 (in a similar fashion as is performed for the technical proof that $\min_j \lambda_N(\hat{S}_{(j)}) > \varepsilon$ for all large n a.s. in Section 4.2.4). It is easily seen that, this being ensured, the proof of Corollary 4.3 unfolds similar to that of (Hachem et al., 2013, Theorem 3), which as a consequence we do not further detail.

Chapter 5

Robust shrinkage estimates of scatter

The previous chapter laid down the theoretical mechanism for studying the large dimensional behavior of robust estimators of the Maronna type. As mentioned in the first paragraph of Chapter 4, the approach cannot be used for Tyler's estimator and this all has to do with the fact that the constant function $x \mapsto 1/x$ used as a weighing function does not meet the property of Maronna's u function that $xu(x)$ is increasing. However, the approach carried out in Chapter 4 can be used, almost immediately, to handle robust shrinkage estimates defined in (2.4) and (2.5). The results are similar in spirit, that is for both cases one can exhibit a random approximating matrix with simple structure that is asymptotically equivalent to the robust model.

The interest of the chapter is therefore not so much in the theoretical tools developed for the proofs of the main results. Rather, the interest of this section is rooted in practical grounds, as the robust shrinkage estimators have multiple advantages over pure robust or pure shrinkage estimators which find many application interests. This being said, the theoretical proofs in themselves, at least as far as $\hat{C}_N(\rho)$ is concerned, are interesting in their being much simpler and clearer than the proofs exposed in Section 4.1. The reader not having reached an overall understanding of these proofs should be more at ease with the present chapter.

5.1 Theory

We study here the two hybrid robust shrinkage covariance matrix estimates $\hat{C}_N(\rho)$ (hereafter referred to as the Abramovich–Pascal estimate) and $\check{C}_N(\rho)$ (hereafter referred to as the Chen estimate) proposed in parallel in (Abramovich and Spencer, 2007; Pascal et al., 2013)[1] and in (Chen et al., 2011), respectively. Both matrices, whose definition is introduced in Section 5.1.1 below, are empirically built upon Tyler's M-estimate (Tyler, 1987) originally designed to cope with elliptical samples whose distribution is unknown to the experimenter and upon the Ledoit–

[1]To the author's knowledge, the first instance of the estimator dates back to (Abramovich and Spencer, 2007) although the non-obvious proof of $\hat{C}_N(\rho)$ being well-defined is only found later in (Pascal et al., 2013).

Wolf shrinkage estimator (Ledoit and Wolf, 2004). This allows for an improved degree of freedom for approximating the population covariance matrix and importantly allows for $N > n$, which Maronna's and Tyler's estimators do not. In (Pascal et al., 2013) and (Chen et al., 2011), $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ were proved to be well-defined as the unique solutions to their defining fixed-point matrices. However, little is known of their performance as estimators of C_N in the regime $N \simeq n$ of interest here. Some progress in this direction was made in (Chen et al., 2011) but this work does not manage to solve the optimal shrinkage problem consisting of finding ρ such that $\mathrm{E}[\mathrm{tr}((\check{C}_N(\rho) - C_N)^2)]$ is minimized and resorts to solving an approximate problem instead.

The present section studies the matrices $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ in the regime where $N, n \to \infty$ with $N/n \to c \in (0, \infty)$, and under the assumption of the absence of outliers. The main results in this section are as follows:

- as in the previous chapter, we show that, under the aforementioned setting, both $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ asymptotically behave similar to well-known random matrix models and prove in particular that both have a well-identified limiting spectral distribution;
- we prove that, up to a change in the variable ρ, the matrices $\check{C}_N(\rho)$ and $\hat{C}_N(\rho)/(\frac{1}{N}\,\mathrm{tr}\,\hat{C}_N(\rho))$ are essentially the same for N, n large, implying that both achieve the same optimal shrinkage performance;
- we determine the optimal shrinkage parameters $\hat{\rho}^\star$ and $\check{\rho}^\star$ that minimize the almost sure limits $\lim_N \frac{1}{N}\,\mathrm{tr}[(\hat{C}_N(\rho)/(\frac{1}{N}\,\mathrm{tr}\,\hat{C}_N(\rho)) - C_N)]^2$ and $\lim_N \frac{1}{N}\,\mathrm{tr}[(\check{C}_N(\rho) - C_N)]^2$, respectively, both limits being the same. We then propose consistent estimates $\hat{\rho}_N$ and $\check{\rho}_N$ for $\hat{\rho}^\star$ and $\check{\rho}^\star$ which achieve the same limiting performance. We finally show by simulations that a significant gain is obtained using $\hat{\rho}^\star$ (or $\hat{\rho}_N$) and $\check{\rho}^\star$ (or $\check{\rho}_N$) compared to the solution $\check{\rho}_O$ of the approximate problem developed in (Chen et al., 2011).

In practice, these results allow for a proper use of $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ in anticipation of the absence of outliers. In the presence of outliers, it is then expected that both Abramovich–Pascal and Chen estimates will exhibit robustness properties that their asymptotic random matrix equivalents will not. Note in particular that, although $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ are shown to be asymptotically equivalent in the absence of outliers, it is not clear at this point whether one of the two estimates will show better performance in the presence of outliers. The study of this scenario has not been precisely carried out but some hints will be provided in Chapter 7.

We start with our main results.

5.1.1 Main results

We start by introducing the main assumptions of the data model under study, which do not defer much from these of Section 4.1, but for the fact that now the eigenvalues of C_N but no longer the τ_i will play a central role. The n sample vectors $x_1, \ldots, x_n \in \mathbb{C}^N$ (or $\mathbb{R}^N$) have the following characteristics.

Assumption 5.1 (Growth rate). *Denoting $c_N = N/n$, $c_N \to c \in (0, \infty)$ as $N \to \infty$.*

Assumption 5.2 (Population model)**.** *The vectors $x_1, \ldots, x_n \in \mathbb{C}^N$ (or $\mathbb{R}^N$) are independent with*

a. *$x_i = \sqrt{\tau_i} A_N y_i$, where $y_i \in \mathbb{C}^{\bar{N}}$ (or $\mathbb{R}^{\bar{N}}$), $\bar{N} \geq N$, is a random zero mean unitarily (or orthogonally) invariant vector with norm $\|y_i\|^2 = \bar{N}$, $A_N \in \mathbb{C}^{N \times \bar{N}}$ is deterministic, and $\tau_1, \ldots, \tau_n$ is a collection of positive scalars. We shall denote $z_i = A_N y_i$.*

b. *$C_N \triangleq A_N A_N^*$ is nonnegative definite, with trace $\frac{1}{N} \operatorname{tr} C_N = 1$ and spectral norm satisfying $\limsup_N \|C_N\| < \infty$.*

c. *$\nu_N \triangleq \frac{1}{N} \sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i(C_N)}$ satisfies $\nu_N \to \nu$ weakly with $\nu \neq \boldsymbol{\delta}_0$ almost everywhere.*

Since all considerations to come are equally valid over $\mathbb{C}$ or $\mathbb{R}$, we will consider by default that $x_1, \ldots, x_n \in \mathbb{C}^N$. As the analysis will show, the positive scalars τ_i have no impact on the robust covariance estimates; with this definition, the distribution of the vectors x_i contains in particular the class of elliptical distributions. Note that the assumption that y_i is zero mean unitarily invariant with norm $\bar{N}$ is equivalent to saying that $y_i = \sqrt{\bar{N}} \frac{\tilde{y}_i}{\|\tilde{y}_i\|}$ with $\tilde{y}_i \in \mathbb{C}^{\bar{N}}$ standard Gaussian. This, along with $A_N \in \mathbb{C}^{N \times \bar{N}}$ and $\limsup_N \|C_N\| < \infty$, implies in particular that $\|x_i\|^2$ is of order N. The assumption that $\nu \neq \boldsymbol{\delta}_0$ almost everywhere avoids the degenerate scenario where an overwhelming majority of the eigenvalues of C_N tend to zero, whose practical interest is quite limited. Finally note that the constraint $\frac{1}{N} \operatorname{tr} C_N = 1$ is inconsequential and in fact defines uniquely both terms in the product $\tau_i C_N$.

The following two theorems introduce the robust shrinkage estimators $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$, and constitute the main technical results of this section that can be paralleled to Theorem 4.1.2 in the previous chapter.

Theorem 5.1.1 (Abramovich–Pascal Estimate)**.** *Let Assumptions 5.1 and 5.2 hold. For $\varepsilon \in (0, \min\{1, c^{-1}\})$, define $\hat{\mathfrak{R}}_\varepsilon = [\varepsilon + \max\{0, 1 - c^{-1}\}, 1]$. For each $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1]$, let $\hat{C}_N(\rho)$ be the unique solution to*

$$\hat{C}_N(\rho) = (1 - \rho) \frac{1}{n} \sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N(\rho)^{-1} x_i} + \rho I_N.$$

Then, as $N \to \infty$,

$$\sup_{\rho \in \hat{\mathfrak{R}}_\varepsilon} \left\| \hat{C}_N(\rho) - \hat{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N(\rho) = \frac{1}{\hat{\gamma}(\rho)} \frac{1 - \rho}{1 - (1 - \rho) c} \frac{1}{n} \sum_{i=1}^n z_i z_i^* + \rho I_N$$

and $\hat{\gamma}(\rho)$ is the unique positive solution to the equation in $\hat{\gamma}$

$$1 = \int \frac{t}{\hat{\gamma} \rho + (1 - \rho) t} \nu(dt).$$

Moreover, the function $\rho \mapsto \hat{\gamma}(\rho)$ thus defined is continuous on $(0, 1]$.

Proof. The proof is deferred to Section 5.1.3.1. ☐

Theorem 5.1.2 (Chen Estimate)**.** *Let Assumptions 5.1 and 5.2 hold. For* $\varepsilon \in (0,1)$*, define* $\check{\mathfrak{R}}_\varepsilon = [\varepsilon, 1]$*. For each* $\rho \in (0,1]$*, let* $\check{C}_N(\rho)$ *be the unique solution to*

$$\check{C}_N(\rho) = \frac{\check{B}_N(\rho)}{\frac{1}{N}\operatorname{tr}\check{B}_N(\rho)}$$

where

$$\check{B}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^{n}\frac{x_i x_i^*}{\frac{1}{N}x_i^*\check{C}_N(\rho)^{-1}x_i} + \rho I_N.$$

Then, as $N \to \infty$*,*

$$\sup_{\rho\in\check{\mathfrak{R}}_\varepsilon}\left\|\check{C}_N(\rho) - \check{S}_N(\rho)\right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\check{S}_N(\rho) = \frac{1-\rho}{1-\rho+T_\rho}\frac{1}{n}\sum_{i=1}^{n} z_i z_i^* + \frac{T_\rho}{1-\rho+T_\rho}I_N$$

in which $T_\rho = \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)$ *with, for all* $x > 0$*,*

$$F(x;\rho) = \frac{1}{2}\left(\rho - c(1-\rho)\right) + \sqrt{\frac{1}{4}\left(\rho - c(1-\rho)\right)^2 + (1-\rho)\frac{1}{x}}$$

and $\check{\gamma}(\rho)$ *is the unique positive solution to the equation in* $\check{\gamma}$

$$1 = \int \frac{t}{\check{\gamma}\rho + \frac{1-\rho}{(1-\rho)c+F(\check{\gamma};\rho)}t}\nu(dt).$$

Moreover, the function $\rho \mapsto \check{\gamma}(\rho)$ *thus defined is continuous on* $(0,1]$*.*

Proof. The proof is deferred to Section 5.1.3.2. ☐

Similar to Theorem 4.1.2, Theorem 5.1.1 and Theorem 5.1.2 show that, as $N, n \to \infty$ with $N/n \to c$, the matrices $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$, defined as the non-trivial solution of fixed-point equations, behave similar to matrices $\hat{S}_N(\rho)$ and $\check{S}_N(\rho)$, respectively, which are here random matrices of the sample covariance matrix type (not separable).

Technically speaking, the proof of both Theorem 5.1.1 and Theorem 5.1.2 unfold from the same technique as in Chapter 4. However, while the proof of Theorem 5.1.1 comes with no major additional difficulty compared to these works, due to the scale normalization imposed in the definition of $\check{C}_N(\rho)$, the proof of Theorem 5.1.2 requires a more elaborate approach than used previously. Another difference to previous works lies here in that, unlike Maronna's estimator that only attenuates the effect of the scale parameters τ_i, the proposed Tyler-based estimators discard this effect altogether. Also, the technical study of Maronna's estimator can be made under the assumption that $C_N = I_N$ (from a natural variable change) while here, because of the regularization term ρI_N, C_N does intervene in an intricate manner in the results.

As a side remark, it is shown in (Pascal et al., 2013) that for each N, n fixed with $n \geq N+1$, $\hat{C}_N(\rho) \to \hat{C}_N(0)$ as $\rho \to 0$ with $\hat{C}_N(0)$ defined (almost surely) as one of the (uncountably many) solutions to

$$\hat{C}_N(0) = \frac{1}{n}\sum_{i=1}^{n} \frac{x_i x_i^*}{\frac{1}{N} x_i^* \hat{C}_N(0)^{-1} x_i}. \tag{5.1}$$

In the regime where $N, n \to \infty$ and $N/n \to c$, this result is difficult to generalize as it is challenging to handle the limit $\|\hat{C}_N(\rho_N) - \hat{S}_N(\rho_N)\|$ for a sequence $\{\rho_N\}_{N=1}^{\infty}$ with $\rho_N \to 0$. The requirement that $\rho_N \to \rho_0 > 0$ on any such sequence is indeed at the core of the proof of Theorem 5.1.1 (see Equations (5.6) and (5.7) in Section 5.1.3.1 where $\rho_0 > 0$ is necessary to ensure $e^+ < 1$). This explains why the set $\hat{\mathcal{R}}_\varepsilon$ in Theorem 5.1.1 excludes the region $[0, \varepsilon)$. Similar arguments hold for $\check{C}_N(\rho)$. Although the behavior of any solution $\hat{C}_N(0)$ to (5.1) in the large N, n regime was recently discovered in (Zhang et al., 2014), this result remains difficult to handle with our proof technique.

As an immediate consequence of Theorem 5.1.1 and Theorem 5.1.2 we have the following result on the limiting eigenvalue distribution of the matrices are study.

Corollary 5.1 (Limiting spectral distribution)**.** *Under the settings of Theorem 5.1.1 and Theorem 5.1.2,*

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\delta}_{\lambda_i(\hat{C}_N(\rho))} \xrightarrow{\text{a.s.}} \hat{\mu}_\rho, \;\; \rho \in \hat{\mathcal{R}}_\varepsilon$$
$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\delta}_{\lambda_i(\check{C}_N(\rho))} \xrightarrow{\text{a.s.}} \check{\mu}_\rho, \;\; \rho \in \check{\mathcal{R}}_\varepsilon$$

where the convergence arrow is understood as the weak convergence of probability measures, for almost every sequence $\{x_1, \ldots, x_n\}_{n=1}^{\infty}$, and where

$$\hat{\mu}_\rho = \max\{0, 1 - c^{-1}\}\boldsymbol{\delta}_\rho + \underline{\hat{\mu}}_\rho$$
$$\check{\mu}_\rho = \max\{0, 1 - c^{-1}\}\boldsymbol{\delta}_{\frac{T_\rho}{1-\rho+T_\rho}} + \underline{\check{\mu}}_\rho$$

with $\underline{\hat{\mu}}_\rho$ and $\underline{\check{\mu}}_\rho$ continuous finite measures with compact support in $[\rho, \infty)$ and $[T_\rho(1-\rho+T_\rho)^{-1}, \infty)$ respectively, real analytic wherever their density is positive. The measure $\hat{\mu}_\rho$ is the only measure with Stieltjes transform $m_{\hat{\mu}_\rho}(z)$ defined, for $z \in \mathbb{C}$ with $\Im[z] > 0$, as

$$m_{\hat{\mu}_\rho}(z) = \hat{\gamma}\frac{1-(1-\rho)c}{1-\rho}\int \frac{1}{\hat{z}(\rho) + \frac{t}{1+c\hat{\delta}(z)}}\nu(dt)$$

where $\hat{z}(\rho) = (\rho - z)\hat{\gamma}(\rho)\frac{1-(1-\rho)c}{1-\rho}$ and $\hat{\delta}(z)$ is the unique solution with positive imaginary part of the equation in $\hat{\delta}$

$$\hat{\delta} = \int \frac{t}{\hat{z}(\rho) + \frac{t}{1+c\hat{\delta}}}\nu(dt).$$

The measure $\check{\mu}_\rho$ is the only measure with Stieltjes transform $m_{\check{\mu}_\rho}(z)$ defined, for $\Im[z] > 0$ as

$$m_{\check{\mu}_\rho}(z) = \frac{1-\rho+T_\rho}{1-\rho} \int \frac{1}{\check{z}(\rho) + \frac{t}{1+c\check{\delta}(z)}} \nu(dt)$$

with $\check{z}(\rho) = \frac{1}{1-\rho} T_\rho (1-z) - z$ and $\check{\delta}(z)$ the unique solution with positive imaginary part of the equation in $\check{\delta}$

$$\check{\delta} = \int \frac{t}{\check{z}(\rho) + \frac{t}{1+c\check{\delta}}} \nu(dt).$$

Proof. This is an immediate application of the results of Chapter 3 or more simply of (Silverstein and Bai, 1995; Silverstein and Choi, 1995) along with Theorems 5.1.1 and 5.1.2. □

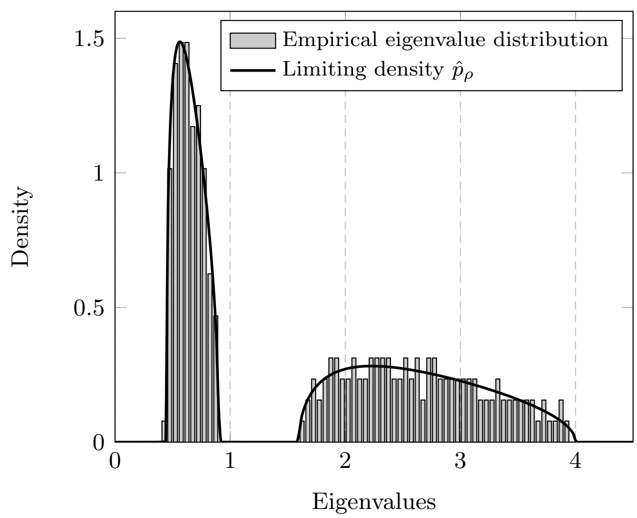


Figure 5.1: Histogram of the eigenvalues of $\hat{C}_N$ (Abramovich–Pascal type) for $n = 2048$, $N = 256$, $C_N = \frac{1}{3}\operatorname{diag}(I_{128}, 5I_{128})$, $\rho = 0.2$, versus limiting eigenvalue distribution.

From Corollary 5.1, $\hat{\mu}_\rho$ is continuous on (ρ, ∞) so that $\hat{\mu}_\rho(dx) = \hat{p}_\rho(x)dx$ where, from the inverse Stieltjes transform formula for all $x \in (\rho, \infty)$,

$$\hat{p}_\rho(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \Im\left[m_{\hat{\mu}_\rho}(x + \imath\varepsilon)\right].$$

Letting $\varepsilon > 0$ small and approximating $\hat{p}_\rho(x)$ by $\frac{1}{\pi}\Im[m_{\hat{\mu}_\rho}(x + \imath\varepsilon)]$ allows one to depict $\hat{p}_\rho$ approximately. Similarly, $\check{\mu}_\rho(dx) = \check{p}_\rho(x)dx$ for all $x \in (T_\rho(1-\rho+T_\rho)^{-1}, \infty)$ which can be obtained equivalently. This is performed in Figure 5.1 and Figure 5.2 which depict the histogram of the eigenvalues of $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ for $\rho = 0.2$, $N = 256$, $n = 2048$, $C_N = \operatorname{diag}(I_{128}, 5I_{128})$, versus their limiting distributions for $c = 1/8$. Figure 5.3 depicts $\check{C}_N(\rho)$ for $\rho = 0.8$, $N = 1024$, $n = 512$, $C_N = \operatorname{diag}(I_{128}, 5I_{128})$ versus its limiting distribution for $c = 2$. Note that, when $c = 1/8$, the eigenvalues of $\check{C}_N(\rho)$ concentrate in two bulks close to $1/3$ and $5/3$, as expected.

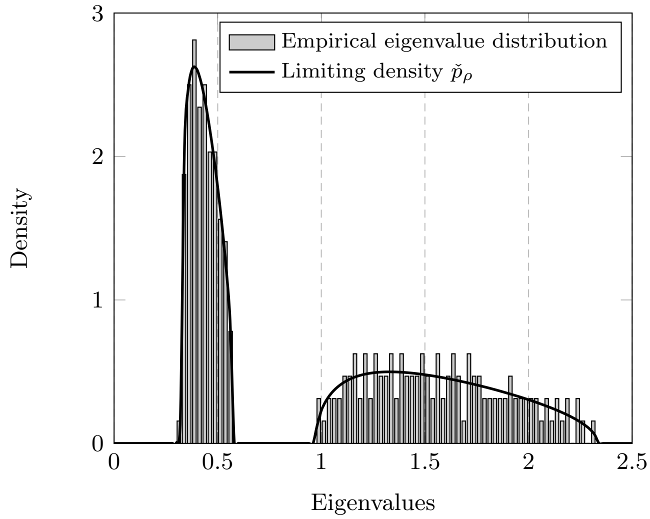


Figure 5.2: Histogram of the eigenvalues of $\check{C}_N$ (Chen type) for $n = 2048$, $N = 256$, $C_N = \frac{1}{3}\operatorname{diag}(I_{128}, 5I_{128})$, $\rho = 0.2$, versus limiting eigenvalue distribution.

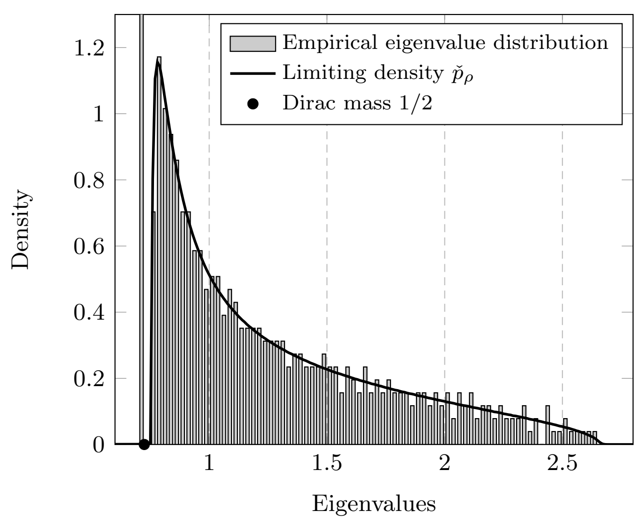


Figure 5.3: Histogram of the eigenvalues of $\check{C}_N$ (Chen type) for $n = 512$, $N = 1024$, $C_N = \frac{1}{3}\operatorname{diag}(I_{128}, 5I_{128})$, $\rho = 0.8$, versus limiting eigenvalue distribution.

Due to the different trace normalization of $\hat{C}_N(\rho)$, the same reasoning holds up to a multiplicative constant. However, when $c = 2$, the eigenvalues of $\check{C}_N(\rho)$ are quite remote from masses in $1/3$ and $5/3$, an observation known since (Marčenko and Pastur, 1967).

Another corollary of Theorem 5.1.1 and Theorem 5.1.2 is the joint convergence (over both ρ and the eigenvalue index) of the individual eigenvalues of $\hat{C}_N(\rho)$ to those of $\hat{S}_N(\rho)$ and of the individual eigenvalues of $\check{C}_N(\rho)$ to those of $\check{S}_N(\rho)$, as well as the joint convergence over ρ of the moments of the empirical spectral distributions of $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$. These joint convergence properties are fundamental in problems of optimization of the parameter ρ as discussed in Section 5.1.2.

Corollary 5.2 (Joint convergence properties)**.** *Under the settings of Theorem 5.1.1 and Theorem 5.1.2,*

$$\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \max_{1\leq i\leq n} \left|\lambda_i(\hat{C}_N(\rho)) - \lambda_i(\hat{S}_N(\rho))\right| \xrightarrow{\text{a.s.}} 0$$
$$\sup_{\rho\in\check{\mathcal{R}}_\varepsilon} \max_{1\leq i\leq n} \left|\lambda_i(\check{C}_N(\rho)) - \lambda_i(\check{S}_N(\rho))\right| \xrightarrow{\text{a.s.}} 0.$$

This result implies

$$\limsup_N \sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \|\hat{C}_N(\rho)\| < \infty$$
$$\limsup_N \sup_{\rho\in\check{\mathcal{R}}_\varepsilon} \|\check{C}_N(\rho)\| < \infty.$$

almost surely. This, and the weak convergence of Corollary 5.1, in turn induce that, for each $\ell \in \mathbb{N}$,

$$\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \left|\frac{1}{N}\operatorname{tr}\left(\hat{C}_N(\rho)^\ell\right) - M_{\hat{\mu}_\rho,\ell}\right| \xrightarrow{\text{a.s.}} 0$$
$$\sup_{\rho\in\check{\mathcal{R}}_\varepsilon} \left|\frac{1}{N}\operatorname{tr}\left(\check{C}_N(\rho)^\ell\right) - M_{\check{\mu}_\rho,\ell}\right| \xrightarrow{\text{a.s.}} 0$$

where we recall that $M_{\mu,\ell} = \int t^\ell \mu(dt) \in (0,\infty]$ *for any probability measure* μ *with support in* $\mathbb{R}_+$*; in particular,* $M_{\hat{\mu}_\rho,1} = \frac{1}{\hat{\gamma}(\rho)}\frac{1-\rho}{1-(1-\rho)c} + \rho$ *and* $M_{\check{\mu}_\rho,1} = 1$.

Proof. The proof is provided in Section 5.1.3.3. □

5.1.2 Optimal shrinkage

We now apply Theorems 5.1.1 and 5.1.2 to the problem of optimal linear shrinkage, originally considered in (Ledoit and Wolf, 2004) for the simpler sample covariance matrix model. The optimal linear shrinkage problem consists in choosing ρ to be such that a certain distance metric between $\hat{C}_N(\rho)$ (or $\check{C}_N(\rho)$) and C_N is minimized, therefore allowing for a more appropriate estimation of C_N via $\hat{C}_N(\rho)$ or $\check{C}_N(\rho)$. The metric selected here is the squared Frobenius norm of the difference between the (possibly scaled) robust estimators and C_N, which has the advantage of being a widespread matrix distance (e.g., as considered in (Ledoit and Wolf, 2004)) and a metric amenable to mathematical analysis.[2] In (Chen et al., 2011), the authors studied this problem in the specific case of $\check{C}_N(\rho)$ but did not find an expression for the optimal theoretical ρ due to the involved structure of $\check{C}_N(\rho)$ for all finite N, n and therefore resorted to solving an

[2] Alternative metrics (such as the geodesic distance on the cone of nonnegative definite matrices) can be similarly considered. The appropriate choice of such a metric heavily depends on the ultimate problem to optimize.

approximate problem, the solution of which is denoted here $\check{\rho}_O$. Instead, we show that for large N, n values the optimal ρ under study converges to a limiting value $\check{\rho}^\star$ that takes an extremely simple explicit expression and a similar result holds for $\hat{C}_N(\rho)$ for which an equivalent optimal $\hat{\rho}^\star$ is defined.

Our first result is a lemma of fundamental importance which demonstrates that, up to a change in the variable ρ, $\hat{S}_N(\rho)/M_{\hat{\mu}_\rho,1}$ and $\check{S}_N(\rho)$ (constructed from the samples $x_1, \ldots, x_n$) are completely equivalent to the original Ledoit–Wolf linear shrinkage model for the (non observable) samples $z_1, \ldots, z_n$.

Lemma 5.1 (Model Equivalence)**.** *For each $\rho \in (0,1]$, there exist unique $\hat{\rho} \in (\max\{0, 1-c^{-1}\}, 1]$ and $\check{\rho} \in (0,1]$ such that*

$$\frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}} = \check{S}_N(\check{\rho}) = (1-\rho)\frac{1}{n}\sum_{i=1}^n z_i z_i^* + \rho I_N.$$

Besides, the maps $(0,1] \to (\max\{0, 1-c^{-1}\}, 1]$, $\rho \mapsto \hat{\rho}$ and $(0,1] \to (0,1]$, $\rho \mapsto \check{\rho}$ thus defined are continuously increasing and onto.

Proof. The proof is provided in Section 5.1.3.4. □

Along with Theorem 5.1.1 and Theorem 5.1.2, Lemma 5.1 indicates that, up to a change in the variable ρ, $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$ can be somewhat viewed as asymptotically equivalent (but there is no saying whether they can be claimed equivalent for all finite N, n). As such, thanks to Lemma 5.1, we now show that the optimal shrinkage parameters ρ for both $\hat{C}_N(\rho)/(\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho))$ and $\check{C}_N(\rho)$ lead to the same asymptotic performance, which corresponds to the asymptotically optimal Ledoit–Wolf linear shrinkage performance but for the vectors $z_1, \ldots, z_n$.

Proposition 5.1.1 (Optimal Shrinkage)**.** *For each $\rho \in (0,1]$, define*[3]

$$\hat{D}_N(\rho) = \frac{1}{N}\operatorname{tr}\left(\left(\frac{\hat{C}_N(\rho)}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)} - C_N\right)^2\right)$$

$$\check{D}_N(\rho) = \frac{1}{N}\operatorname{tr}\left(\left(\check{C}_N(\rho) - C_N\right)^2\right).$$

Also denote $D^\star = c\frac{M_{\nu,2}-1}{c+M_{\nu,2}-1}$, $\rho^\star = \frac{c}{c+M_{\nu,2}-1}$, and $\hat{\rho}^\star \in (\max\{0, 1-c^{-1}\}, 1]$, $\check{\rho}^\star \in (0,1]$ the unique solutions to

$$\frac{\hat{\rho}^\star}{\frac{1}{\hat{\gamma}(\hat{\rho}^\star)}\frac{1-\hat{\rho}^\star}{1-(1-\hat{\rho}^\star)c} + \hat{\rho}^\star} = \frac{T_{\check{\rho}^\star}}{1-\check{\rho}^\star + T_{\check{\rho}^\star}} = \rho^\star.$$

Then, letting $\varepsilon < \min(\hat{\rho}^\star - \max\{0, 1-c^{-1}\}, \check{\rho}^\star)$, under the setting of Theorem 5.1.1 and Theorem 5.1.2,

$$\inf_{\rho \in \hat{\mathcal{R}}_\varepsilon} \hat{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star, \quad \inf_{\rho \in \check{\mathcal{R}}_\varepsilon} \check{D}_N(\rho) \xrightarrow{\text{a.s.}} D^\star$$

[3]Recall that, for A Hermitian, $\frac{1}{N}\operatorname{tr}(A^2) = \frac{1}{N}\operatorname{tr}(AA^*) = \frac{1}{N}\|A\|_F^2$ with $\|\cdot\|_F$ the Frobenius norm for matrices.

and

$$\hat{D}_N(\hat{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star, \quad \check{D}_N(\check{\rho}^\star) \xrightarrow{\text{a.s.}} D^\star.$$

Moreover, letting $\hat{\rho}_N$ *and* $\check{\rho}_N$ *be random variables such that* $\hat{\rho}_N \xrightarrow{\text{a.s.}} \hat{\rho}^\star$ *and* $\check{\rho}_N \xrightarrow{\text{a.s.}} \check{\rho}^\star$*,*

$$\hat{D}_N(\hat{\rho}_N) \xrightarrow{\text{a.s.}} D^\star, \quad \check{D}_N(\check{\rho}_N) \xrightarrow{\text{a.s.}} D^\star.$$

Proof. The proof is provided in Section 5.1.3.5. □

The last part of Proposition 5.1.1 states that, if consistent estimates $\hat{\rho}_N$ and $\check{\rho}_N$ of $\hat{\rho}^\star$ and $\check{\rho}^\star$ exist, then they have optimal shrinkage performance in the large N, n limit. Such estimates may of course be defined in multiple ways. We present below a simple example based on $\hat{C}_N(\rho)$ and $\check{C}_N(\rho)$.

Proposition 5.1.2 (Optimal Shrinkage Estimate)**.** *Under the setting of Proposition 5.1.1, let* $\hat{\rho}_N \in (\max\{0, 1 - c^{-1}\}, 1]$ *and* $\check{\rho}_N \in (0, 1]$ *be solutions (not necessarily unique) to*

$$\frac{\hat{\rho}_N}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho}_N)} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}$$

$$\frac{\check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}}{1 - \check{\rho}_N + \check{\rho}_N \frac{1}{n}\sum_{i=1}^n \frac{x_i^* \check{C}_N(\check{\rho}_N)^{-1} x_i}{\|x_i\|^2}} = \frac{c_N}{\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2}\right)^2\right] - 1}$$

defined arbitrarily when no such solutions exist. Then $\hat{\rho}_N \xrightarrow{\text{a.s.}} \hat{\rho}^\star$ *and* $\check{\rho}_N \xrightarrow{\text{a.s.}} \check{\rho}^\star$*, so that* $\hat{D}_N(\hat{\rho}_N) \xrightarrow{\text{a.s.}} D^\star$ *and* $\check{D}_N(\check{\rho}_N) \xrightarrow{\text{a.s.}} D^\star$*.*

Proof. The proof is deferred to Section 5.1.3.6. □

Figure 5.4 illustrates the performance in terms of the metric $\check{D}_N$ of the empirical shrinkage coefficient $\check{\rho}_N$ introduced in Proposition 5.1.2 versus the optimal value $\inf_{\rho\in(0,1]}\{\check{D}_N(\rho)\}$, averaged over 10 000 Monte Carlo simulations. We also present in this graph the almost sure limiting value $D^\star$ of both $\check{D}_N(\check{\rho}_N)$ and $\inf_{\rho\in\check{\mathcal{R}}_\varepsilon}\{\check{D}_N(\rho)\}$ for some sufficiently small ε, as well as $\check{D}_N(\check{\rho}_O)$ of $\check{\rho}_O$ defined in (Chen et al., 2011, Equation (12)) as the minimizing solution of $\mathrm{E}[\frac{1}{N}\operatorname{tr}(\check{C}_O(\rho) - C_N)^2]$ with $\check{C}_O(\rho)$ the so-called "clairvoyant estimator"

$$\check{C}_O(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}x_i^* C_N^{-1} x_i} + \rho I_N.$$

We consider in this graph $N = 32$ constant, $n \in \{2^k, k = 1, \dots, 7\}$, and $C_N = [C_N]_{i,j=1}^N$ with $[C_N]_{ij} = r^{|i-j|}$, $r = 0.7$, which is the same setting as considered in (Chen et al., 2011, Section 4).

It appears in Figure 5.4 that a significant improvement is brought by $\check{\rho}_N$ over $\check{\rho}_O$, especially for small n, which translates the poor quality of $\check{C}_O(\rho)$ as an approximation of $\check{C}_N(\rho)$ for large values of c_N (obviously linked to $\frac{1}{N}x_i^* C_N^{-1} x_i$ being then a bad approximation for $\frac{1}{N}x_i^* \check{C}_N(\rho)^{-1} x_i$). Another important remark is that, even for so small values of N, n, $\inf_{\rho\in(0,1]} \check{D}_N(\rho)$ is extremely close to the limiting optimal, suggesting here that the limiting results of Proposition 5.1.1 are already met for small practical values. The approximation $\check{\rho}_N$ of $\check{\rho}^\star$, translated here through $\check{D}_N(\check{\rho}_N)$, also demonstrates good practical performance at small values of N, n.

We additionally mention that we produced similar curves for $\hat{C}_N(\rho)$ in place of $\check{C}_N(\rho)$ which happened to show virtually the same performance as the equivalents curves for $\check{C}_N(\rho)$. This is of course expected (with exact match) for $\inf_{\rho\in(0,1]} \hat{D}_N(\rho)$ which, up to the region $[0, \varepsilon)$, matches $\inf_{\rho\in(0,1]} \check{D}_N(\rho)$ for large enough N, n, and similarly for $\hat{D}_N(\hat{\rho}_N)$ since $\hat{\rho}_N$ was designed symmetrically to $\check{\rho}_N$.

Associated to Figure 5.4 is Figure 5.5 which provides the shrinkage parameter values, optimal and approximated, for both the Abramovich–Pascal and Chen estimates, along with the clairvoyant $\check{\rho}_O$ of (Chen et al., 2011). Recall that the $\hat{(\cdot)}$ values must only be compared to one another, and similarly for the $\check{(\cdot)}$ values (so in particular $\check{\rho}_O$ only compares against the $\check{(\cdot)}$ values). It appears here that $\check{\rho}_O$ is a rather poor estimate for $\operatorname{argmin}_{\rho\in(0,1]} \check{D}_N(\rho)$ for a large range of values of n. It tends in particular to systematically overestimate the weight to be put on the sample covariance matrix.

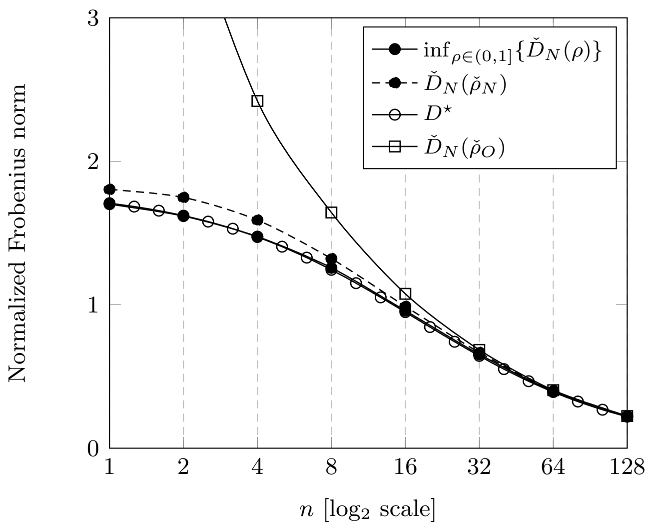


Figure 5.4: Performance of optimal shrinkage averaged over 10 000 Monte Carlo simulations, for $N = 32$, various values of n, $[C_N]_{ij} = r^{|i-j|}$ with $r = 0.7$; $\check{\rho}_N$ is given in Proposition 5.1.2; $\check{\rho}_O$ is the clairvoyant estimator proposed in (Chen et al., 2011, Equation (12)); $D^\star$ taken with $c = N/n$.

5.1.3 Proofs

This section successively introduces the proofs of Theorem 5.1.1, Theorem 5.1.2, Corollary 5.2, Lemma 5.1, Proposition 5.1.1, and Proposition 5.1.2. The methodology of proof of Theorem 5.1.1 closely follows that of Chapter 4. The proof of Theorem 5.1.2 also relies on the same ideas but

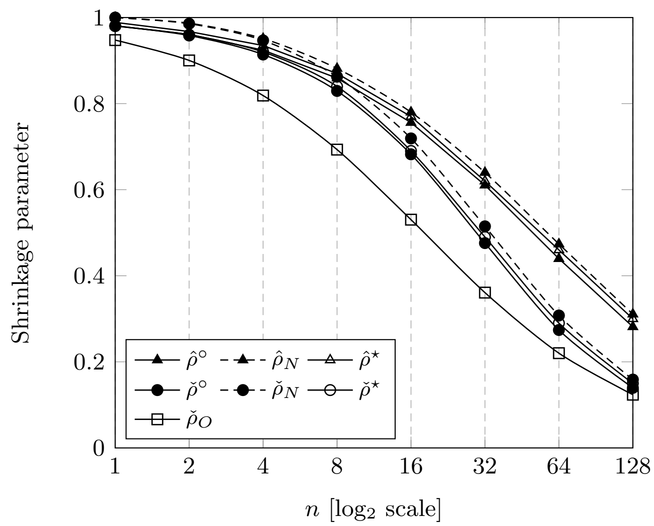


Figure 5.5: Shrinkage parameter ρ averaged over 10 000 Monte Carlo simulations, for $N = 32$, various values of n, $[C_N]_{ij} = r^{|i-j|}$ with $r = 0.7$; $\hat{\rho}_N$ and $\check{\rho}_N$ given in Proposition 5.1.2; $\check{\rho}_O$ is the clairvoyant estimator proposed in (Chen et al., 2011, Equation (12)); $\rho^\star$, $\hat{\rho}^\star$, and $\check{\rho}^\star$ taken with $c = N/n$; $\hat{\rho}^\circ = \operatorname{argmin}_{\{\rho\in(\max\{0,1-c_N^{-1}\},1]\}}\{\hat{D}_N(\rho)\}$ and $\check{\rho}^\circ = \operatorname{argmin}_{\{\rho\in(0,1]\}}\{\check{D}_N(\rho)\}$.

is more technical due to the imposed normalization of $\check{C}_N(\rho)$ to be of trace N. The proofs of the corollary, lemma, and propositions then rely mostly on the important joint convergence over ρ proved in Theorem 5.1.1 and Theorem 5.1.2, and on standard manipulations of random matrix theory and fixed-point equation analysis.

5.1.3.1 Proof of Theorem 5.1.1

The proof of existence and uniqueness of $\hat{C}_N(\rho)$ is given in (Pascal et al., 2013).

The existence and uniqueness of $\hat{\gamma}(\rho)$ is quite immediate as the right-hand side integral in the definition of $\hat{\gamma}(\rho)$ is a decreasing function of $\hat{\gamma}$ (since $\rho > 0$) with limits $1/(1-\rho) > 1$ as $\hat{\gamma} \to 0$ (since $\nu \neq \boldsymbol{\delta}_0$ almost everywhere) and zero as $\hat{\gamma} \to \infty$. We now prove the continuity of $\hat{\gamma}$ on $(0, 1]$. Let $\rho_0, \rho \in (0, 1]$ and $\hat{\gamma}_0 = \hat{\gamma}(\rho_0)$, $\hat{\gamma} = \hat{\gamma}(\rho)$. Then

$$\int \frac{t}{\hat{\gamma}\rho + (1-\rho)t}\nu(dt) - \int \frac{t}{\hat{\gamma}_0\rho_0 + (1-\rho_0)t}\nu(dt) = 0.$$

Setting the difference into a common integral and isolating the term $\hat{\gamma}_0 - \hat{\gamma}$, this becomes, after some calculus,

$$(\hat{\gamma}_0 - \hat{\gamma})\rho_0 = -\hat{\gamma}(\rho_0 - \rho) + (\rho - \rho_0)\frac{\int \frac{t^2}{(\hat{\gamma}\rho+(1-\rho)t)(\hat{\gamma}_0\rho_0+(1-\rho_0)t)}\nu(dt)}{\int \frac{t}{(\hat{\gamma}\rho+(1-\rho)t)(\hat{\gamma}_0\rho_0+(1-\rho_0)t)}\nu(dt)}.$$

Since the support of ν is bounded by $\limsup_N \|C_N\| < \infty$ and in particular $\hat{\gamma}(\rho) \le \rho^{-1}\limsup_N \|C_N\|$ by definition of $\hat{\gamma}$, the ratio of integrals above is uniformly bounded on ρ in a certain small neighborhood of $\rho_0 > 0$. Taking the limit $\rho \to \rho_0$ then brings $\hat{\gamma}_0 - \hat{\gamma} \to 0$, which proves the continuity.

From now on, for readability, we discard all unnecessary indices ρ when no confusion is possible.

Note first that x_i can be equivalently replaced by z_i from the definition of $\hat{C}_N(\rho)$ which is independent of $\tau_1, \ldots, \tau_n$. Consider $\rho \in \hat{\mathcal{R}}_\varepsilon$ fixed and assume $\hat{C}_N$ exists for all N on the realization $\{z_1, \ldots, z_n\}_{n=1}^\infty$ (a probability one event). We start by rewriting $\hat{C}_N$ in a more convenient form. Denoting $\hat{C}_{(i)} \triangleq \hat{C}_N - (1-\rho)\frac{1}{n}\frac{z_i z_i^*}{\frac{1}{N}z_i^*\hat{C}_N^{-1}z_i}$ and using $(A + tvv^*)^{-1}v = A^{-1}v/(1 + tv^*A^{-1}v)$ for positive definite Hermitian A, vector v, and scalar $t > 0$, we have

$$\frac{1}{N}z_i^*\hat{C}_N^{-1}z_i = \frac{\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i}{1 + (1-\rho)c\frac{\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i}{\frac{1}{N}z_i^*\hat{C}_N^{-1}z_i}}$$

so that

$$\frac{1}{N}z_i^*\hat{C}_N^{-1}z_i = (1 - (1-\rho)c_N)\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i$$

and we can rewrite $\hat{C}_N$ as

$$\hat{C}_N = \frac{1-\rho}{1-(1-\rho)c_N}\frac{1}{n}\sum_{i=1}^n \frac{z_i z_i^*}{\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i} + \rho I_N.$$

We recall that the interest of this rewriting, detailed in Chapter 4, mostly lies in the intuition that $\frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i$ should be close to $\frac{1}{N}\operatorname{tr}(\hat{C}_N^{-1})$ for all i, while $\frac{1}{N}z_i^*\hat{C}_N^{-1}z_i$ is a priori more involved.

To proceed with the proof, for $i \in \{1, \ldots, n\}$, denote $\hat{d}_i(\rho) \triangleq \frac{1}{N}z_i^*\hat{C}_{(i)}^{-1}z_i$ and, up to relabeling, assume $\hat{d}_1(\rho) \leq \ldots \leq \hat{d}_n(\rho)$. Then, using $A \succeq B \Rightarrow B^{-1} \succeq A^{-1}$ for positive Hermitian matrices A, B,

$$\begin{aligned}
\hat{d}_n(\rho) &= \frac{1}{N}z_n^*\left(\frac{1-\rho}{1-(1-\rho)c_N}\frac{1}{n}\sum_{i=1}^{n-1}\frac{z_i z_i^*}{\hat{d}_i(\rho)} + \rho I_N\right)^{-1} z_n \\
&\leq \frac{1}{N}z_n^*\left(\frac{1-\rho}{1-(1-\rho)c_N}\frac{1}{n}\sum_{i=1}^{n-1}\frac{z_i z_i^*}{\hat{d}_n(\rho)} + \rho I_N\right)^{-1} z_n.
\end{aligned}$$

Since $z_n \neq 0$, this implies

$$1 \leq \frac{1}{N}z_n^*\left(\frac{1-\rho}{1-(1-\rho)c_N}\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \hat{d}_n(\rho)\rho I_N\right)^{-1} z_n. \tag{5.2}$$

Similarly, with the same derivations, but with opposite inequalities

$$1 \geq \frac{1}{N}z_1^*\left(\frac{1-\rho}{1-(1-\rho)c_N}\frac{1}{n}\sum_{i=2}^{n} z_i z_i^* + \hat{d}_1(\rho)\rho I_N\right)^{-1} z_1.$$

Our objective is to show that $\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}\max_{1\leq i\leq n}|\hat{d}_i(\rho)-\hat{\gamma}(\rho)|\xrightarrow{\text{a.s.}}0$ where $\hat{\gamma}(\rho)$ is given in the statement of the theorem. This is proved via a contradiction argument.

For this, assume that there exists a sequence $\{\rho_n\}_{n=1}^\infty$ over which $\hat{d}_n(\rho_n)>\hat{\gamma}(\rho_n)+\ell$ infinitely often, for some $\ell>0$ fixed. Since $\{\rho_n\}_{n=1}^\infty$ is bounded, it has a limit point $\rho_0\in\hat{\mathcal{R}}_\varepsilon$. Let us restrict ourselves to such a subsequence on which $\rho_n\to\rho_0$ and $\hat{d}_n(\rho_n)>\hat{\gamma}(\rho_n)+\ell$. On this sequence, from (5.2)

$$1\leq\frac{1}{N}z_n^*\left(\frac{1-\rho_n}{1-(1-\rho_n)c_N}\frac{1}{n}\sum_{i=1}^{n-1}z_iz_i^*+(\hat{\gamma}(\rho_n)+\ell)\rho_nI_N\right)^{-1}z_n\triangleq\hat{e}_n. \tag{5.3}$$

Assume first $\rho_0\neq1$. From Chapter 3 and particularly (3.5) (up to a slight modification in the $\delta(x)$ notation), we have

$$\begin{aligned}\hat{e}_n&=\frac{1-(1-\rho_n)c_N}{1-\rho_n}\frac{1}{N}z_n^*\left(\frac{1}{n}\sum_{i=1}^{n-1}z_iz_i^*+(\hat{\gamma}(\rho_n)+\ell)\rho_n\frac{1-(1-\rho_n)c_N}{1-\rho_n}I_N\right)^{-1}z_n\\&\xrightarrow{\text{a.s.}}\frac{1-(1-\rho_0)c}{1-\rho_0}\delta\left(-(\hat{\gamma}(\rho_0)+\ell)\rho_0\frac{1-(1-\rho_0)c}{1-\rho_0}\right)\triangleq e^+\end{aligned} \tag{5.4}$$

where, for $x>0$, $\delta(x)$ is the unique positive solution to the equation

$$\delta(x)=\int\frac{t}{-x+\frac{t}{1+c\delta(x)}}\nu(dt).$$

The convergence (5.4) follows from several classical ingredients. For this, we first use the fact that, for each $p\geq2$, $w>0$, and $j\in\{1,\ldots,n\}$,

$$\mathrm{E}\left[\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+wI_N\right)^{-1}z_j-\delta(-w)\right|^p\right]=\mathcal{O}\left(N^{-p/2}\right) \tag{5.5}$$

which, taking $p\geq4$ along with Boole's inequality, Markov inequality, and Borel–Cantelli lemma, ensures that

$$\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+wI_N\right)^{-1}z_j-\delta(-w)\right|\xrightarrow{\text{a.s.}}0.$$

Using successively $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$ for invertible A,B matrices and the fact that $\|(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+wI_N)^{-1}\|<w^{-1}$ and $\limsup_n\max_{1\leq i\leq n}\frac{1}{N}\|z_i\|^2=M_{\nu,1}=1<\infty$ a.s., we then have, for any positive sequence $w_n \to w > 0$,

$$\begin{aligned}
&\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+w_nI_N\right)^{-1}z_j-\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+wI_N\right)^{-1}z_j\right|\\
&=|w_n-w|\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+w_nI_N\right)^{-1}\left(\frac{1}{n}\sum_{i\neq j}z_iz_i^*+wI_N\right)^{-1}z_j\right|\\
&\leq|w_n-w|\frac{1}{w_nw}\max_{1\leq j\leq n}\frac{1}{N}\|z_j\|^2\\
&\xrightarrow{\text{a.s.}}0
\end{aligned}$$

from which the convergence (5.4) unfolds.

Developing the expression of e^+ then leads to e^+ being the unique positive solution of the equation

$$e^+=\int\frac{t}{(\hat{\gamma}(\rho_0)+\ell)\rho_0+\frac{t}{\frac{1-(1-\rho_0)c}{1-\rho_0}+ce^+}}\nu(dt)$$

which we write equivalently

$$1=\int\frac{t}{(\hat{\gamma}(\rho_0)+\ell)\rho_0e^++\frac{te^+}{\frac{1-(1-\rho_0)c}{1-\rho_0}+ce^+}}\nu(dt). \tag{5.6}$$

Note that the right-hand side term is a decreasing function f of e^+. From the definition of $\hat{\gamma}(\rho_0)$, we can in parallel write

$$1=\int\frac{t}{\hat{\gamma}(\rho_0)\rho_0\times 1+\frac{t\times 1}{\frac{1-(1-\rho_0)c}{1-\rho_0}+c\times 1}}\nu(dt) \tag{5.7}$$

where we purposely made the terms 1 explicit. Now, since both integrals above equal 1, since $\ell>0$, and since f is decreasing, we must have $e^+<1$. But this is in contradiction with $\hat{e}_n\geq 1$ and the convergence (5.4).

If instead, $\rho_0=1$, then from the definition of $\hat{e}_n$ in (5.3), and since $\frac{1}{N}\|z_n\|^2\xrightarrow{\text{a.s.}}M_{\nu,1}=1$ (from $\lim_n\max_{1\leq i\leq n}|\frac{1}{N}\|z_i\|^2-M_{\nu,1}|\xrightarrow{\text{a.s.}}0$), $\limsup_n\|\frac{1}{n}\sum_{i=1}^n z_iz_i^*\|<\infty$ a.s. (from Assumption 5.2–b. and (Bai and Silverstein, 1998)), and $\hat{\gamma}(1)=M_{\nu,1}=1$, we have

$$\hat{e}_n\xrightarrow{\text{a.s.}}\frac{M_{\nu,1}}{M_{\nu,1}+\ell}=\frac{1}{1+\ell}<1$$

again contradicting $\hat{e}_n\geq 1$.

Hence, for all large n, there is no sequence of ρ_n for which $\hat{d}_n(\rho_n)>\hat{\gamma}(\rho_n)+\ell$ infinitely often and therefore $\hat{d}_n(\rho)\leq\hat{\gamma}(\rho)+\ell$ for all large n a.s., uniformly on $\rho\in\hat{\mathfrak{R}}_\varepsilon$.

The same reasoning holds for $\hat{d}_1(\rho)$ which can be proved greater than $\hat{\gamma}(\rho) - \ell$ for all large n uniformly on $\rho \in \hat{\mathcal{R}}_\varepsilon$. Consequently, since $\ell > 0$ is arbitrary, from the ordering of the $\hat{d}_i(\rho)$, we have proved that $\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \max_{1\leq i\leq n} |\hat{d}_i(\rho) - \hat{\gamma}(\rho)| \xrightarrow{\text{a.s.}} 0$.

From there, we then find that

$$\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \left\| \hat{S}_N(\rho) - \hat{C}_N(\rho) \right\| \leq \left\| \frac{1}{n}\sum_{i=1}^n z_i z_i^* \right\| \sup_{\rho\in\hat{\mathcal{R}}_\varepsilon} \max_{1\leq i\leq n} \frac{1-\rho}{1-(1-\rho)c_N} \left| \frac{\hat{d}_i(\rho) - \hat{\gamma}(\rho)}{\hat{\gamma}(\rho)\hat{d}_i(\rho)} \right|$$
$$\xrightarrow{\text{a.s.}} 0$$

where we used the fact that $\limsup_n \left\| \frac{1}{n}\sum_{i=1}^n z_i z_i^* \right\| < \infty$ a.s. from Assumption 5.2–b. and (Bai and Silverstein, 1998), and the fact that $0 < \varepsilon < c^{-1}$.

5.1.3.2 Proof of Theorem 5.1.2

The proof of existence and uniqueness is given in (Chen et al., 2011). The proof of Theorem 5.1.2 unfolds similarly as the proof of Theorem 5.1.1 but it slightly more involved due to the difficulty brought by the normalization of $\check{C}_N(\rho)$ by its trace. For this reason, we first introduce some preliminary results needed in the main core of the proof. Note also that, similar to the proof of Theorem 5.1.1, we may immediately consider z_i in place of x_i in the expression of $\check{C}_N(\rho)$ from the independence of $\check{C}_N(\rho)$ with respect to $\tau_1, \ldots, \tau_n$.

From now on, for the sake of readability, we discard the unnecessary indices ρ.

Some preliminaries We start by some considerations on $\check{\gamma}(\rho)$ and $F_N(x)$ defined as the unique positive solution to the equation in F_N

$$F_N = (1-\rho)\frac{1}{x}\frac{1}{F_N} + \rho - c_N(1-\rho). \tag{5.8}$$

Note first that, for $x > 0$, (5.8) can be written as a second order polynomial whose solutions have opposite signs, the positive one being explicitly given by

$$F_N(x) = \frac{1}{2}(\rho - c_N(1-\rho)) + \sqrt{\frac{1}{4}(\rho - c_N(1-\rho))^2 + (1-\rho)\frac{1}{x}}.$$

The function $F_N(x)$ is decreasing with $\lim_{x\to 0} F_N(x) = \infty$ and $\lim_{x\to\infty} F_N(x) = \max\{\rho - c_N(1-\rho), 0\}$. As $N \to \infty$, $c_N \to c$, and $F_N(x) \to F(x) = F(x;\rho)$ defined in the statement of the theorem which therefore satisfies $F(x) = (1-\rho)\frac{1}{x}\frac{1}{F(x)} + \rho - c(1-\rho)$ and is decreasing with $\lim_{x\to 0} F(x) = \infty$ and $\lim_{x\to\infty} F(x) = \max\{\rho - c(1-\rho), 0\}$. This implies in particular that the function

$$G : x \mapsto \int \frac{t}{x\rho + \frac{1-\rho}{(1-\rho)c+F(x)}t}\nu(dt) \tag{5.9}$$

is decreasing with $\lim_{x\to 0} G(x) = \infty$ and $\lim_{x\to\infty} G(x) = 0$. Hence the existence and uniqueness of $\check{\gamma}(\rho)$ as defined in the theorem.

Now consider the function $H_N : x \mapsto xF_N(x)$ for $x > 0$ and $\rho < 1$. Then, for $x > 0$,

$$H_N'(x) = \frac{1}{2} \frac{A(x) + B(x)}{\sqrt{\left(\frac{\rho-(1-\rho)c_N}{2}\right)^2 x^2 + (1-\rho)x}}$$

where

$$A(x) = 2\left(\frac{\rho-(1-\rho)c_N}{2}\right)\sqrt{\left(\frac{\rho-(1-\rho)c_N}{2}\right)^2 x^2 + (1-\rho)x}$$
$$B(x) = 1-\rho+2\left(\frac{\rho-(1-\rho)c_N}{2}\right)^2 x.$$

Although $A(x)$ may be negative, it is easily verified that $B(x)^2 = A(x)^2 + (1-\rho)^2$ for all $x \geq 0$. Therefore, if $\rho < 1$, for each $w_0 > 0$, there exists $\varepsilon > 0$ such that

$$\liminf_N \sup_{w_0-\varepsilon<x<w_0+\varepsilon} H_N'(x) > 0 \tag{5.10}$$

a relation which will be useful in the core of the proof of Theorem 5.1.2.

To prove continuity of $\check{\gamma}$, the same arguments as in the proof of Theorem 5.1.1 hold. That is, take $\rho_0, \rho \in (0,1]$ and denote $\check{\gamma}_0 = \check{\gamma}(\rho_0)$ and $\check{\gamma} = \check{\gamma}(\rho)$. Then, by definition of $\check{\gamma}(\rho)$, using $F(x) = (1-\rho)\frac{1}{x}\frac{1}{F(x)} + \rho - c(1-\rho)$,

$$\int \frac{t}{\check{\gamma}_0\rho_0 + \frac{(1-\rho_0)\check{\gamma}_0 F(\check{\gamma}_0)}{1-\rho_0+\rho_0\check{\gamma}_0 F(\check{\gamma}_0)}t}\nu(dt) - \int \frac{t}{\check{\gamma}\rho + \frac{(1-\rho)\check{\gamma} F(\check{\gamma})}{1-\rho+\rho\check{\gamma} F(\check{\gamma})}t}\nu(dt) = 0.$$

Setting these to a common denominator gives, after some calculus,

$$\begin{aligned}
&[(\check{\gamma}_0 - \check{\gamma})\rho_0 + \check{\gamma}(\rho_0 - \rho)] \int \frac{t}{D(t)}\nu(dt) \\
&= \frac{(1-\rho)(1-\rho_0)(\check{\gamma}F(\check{\gamma}) - \check{\gamma}_0 F(\check{\gamma}_0)) + (\rho_0-\rho)\check{\gamma}\check{\gamma}_0 F(\check{\gamma})F(\check{\gamma}_0)}{(1-\rho+\rho\check{\gamma}F(\check{\gamma}))(1-\rho_0+\rho_0\check{\gamma}_0F(\check{\gamma}_0))} \int \frac{t^2}{D(t)}\nu(dt)
\end{aligned} \tag{5.11}$$

where

$$D(t) = \left(\check{\gamma}_0\rho_0 + \frac{(1-\rho_0)\check{\gamma}_0 F(\check{\gamma}_0)}{1-\rho_0+\rho_0\check{\gamma}_0 F(\check{\gamma}_0)}t\right)\left(\check{\gamma}\rho + \frac{(1-\rho)\check{\gamma} F(\check{\gamma})}{1-\rho+\rho\check{\gamma} F(\check{\gamma})}t\right) > 0.$$

Note now that $\check{\gamma}(\rho) \leq \rho^{-1}\limsup_N \|C_N\|$ and, on a small neighborhood of $\rho_0 \in (0,1]$, $\check{\gamma} = \check{\gamma}(\rho)$ is uniformly away from zero. Indeed, if this were not the case, on some subsequence $\rho_k \to \rho_0$ such that $\check{\gamma}(\rho_k) \to 0$, the definition of $\check{\gamma}$ would imply

$$1 = \int \frac{t}{\check{\gamma}(\rho_k)\rho_k + \frac{1-\rho}{(1-\rho_k)c+F(\check{\gamma}(\rho_k))}}\nu(dt) \to 0$$

which is a contradiction. This implies as a consequence that $F(\check{\gamma})$ is bounded on a neighborhood of ρ_0. All this implies that all terms proportional to $\rho_0 - \rho$ in (5.11) tend to zero as $\rho \to \rho_0$, so that, in the limit $\rho \to \rho_0$,

$$(\check{\gamma}_0 - \check{\gamma})\rho_0 \int \frac{t\nu(dt)}{D(t)} + \frac{(1-\rho)(1-\rho_0)(\check{\gamma}_0 F(\check{\gamma}_0) - \check{\gamma}F(\check{\gamma}))}{(1-\rho+\rho\check{\gamma}F(\check{\gamma}))(1-\rho_0+\rho_0\check{\gamma}_0F(\check{\gamma}_0))} \int \frac{t^2\nu(dt)}{D(t)} \to 0.$$

But, since $x \mapsto xF(x)$ is increasing, $\check{\gamma}_0 F(\check{\gamma}_0) - \check{\gamma}F(\check{\gamma})$ is of the same sign as $\check{\gamma}_0 - \check{\gamma}$. As $D(t)$ is uniformly bounded for ρ in a small neighborhood of ρ_0, this induces $\check{\gamma}_0 - \check{\gamma} \to 0$, which concludes the proof of continuity.

Main proof Let us now work on the matrix $\check{B}_N$. From the definition of $\check{C}_N$,

$$\check{B}_N = \frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N} \frac{1}{n}\sum_{i=1}^{n} \frac{z_i z_i^*}{\frac{1}{N} z_i^* \check{B}_N^{-1} z_i} + \rho I_N.$$

Denoting $\check{B}_{(i)} = \check{B}_N - \frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N}\frac{1}{n}\frac{z_i z_i^*}{\frac{1}{N}z_i^*\check{B}_N^{-1}z_i}$ and using again $(A + txx^*)^{-1}x = A^{-1}x/(1 + tx^*A^{-1}x)$, we have this time

$$\frac{1}{N} z_i^* \check{B}_N^{-1} z_i = \frac{\frac{1}{N} z_i^* \check{B}_{(i)}^{-1} z_i}{1 + (1-\rho)c_N \frac{\frac{1}{N} z_i^* \check{B}_{(i)}^{-1} z_i}{\frac{1}{N} z_i^* \check{B}_N^{-1} z_i} \frac{1}{\frac{1}{N}\operatorname{tr}\check{B}_N}}$$

so that

$$\frac{1}{N} z_i^* \check{B}_N^{-1} z_i = \frac{1}{N} z_i^* \check{B}_{(i)}^{-1} z_i \left(1 - c_N(1-\rho)\frac{1}{\frac{1}{N}\operatorname{tr}\check{B}_N}\right). \tag{5.12}$$

From the positivity of both quadratic forms above, this implies in particular that $\frac{1}{N}\operatorname{tr}\check{B}_N - c(1-\rho) > 0$.

Replacing the quadratic forms $\frac{1}{N} z_i^* \check{B}_N^{-1} z_i$ in the expression of $\check{B}_N$, we can now rewrite $\check{B}_N$ as

$$\check{B}_N = \frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)} \frac{1}{n}\sum_{i=1}^{n} \frac{z_i z_i^*}{\frac{1}{N} z_i^* \check{B}_{(i)}^{-1} z_i} + \rho I_N. \tag{5.13}$$

Denote now $\check{d}_i \triangleq \frac{1}{N} z_i^* \check{B}_{(i)}^{-1} z_i$ and assume, up to relabeling, that $\check{d}_1 \leq \ldots \leq \check{d}_n$ for all n. Then, with the definition of $\check{B}_{(i)}$, we have

$$\begin{aligned}
\check{d}_n &= \frac{1}{N} z_n^* \left(\frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)} \frac{1}{n}\sum_{i=1}^{n-1} \frac{z_i z_i^*}{\check{d}_i} + \rho I_N \right)^{-1} z_n \\
&\leq \frac{1}{N} z_n^* \left(\frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)} \frac{1}{n}\sum_{i=1}^{n-1} \frac{z_i z_i^*}{\check{d}_n} + \rho I_N \right)^{-1} z_n \\
&= \frac{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)}{1-\rho} \frac{1}{N} z_n^* \left(\frac{1}{n}\sum_{i=1}^{n-1} \frac{z_i z_i^*}{\check{d}_n} + \rho \frac{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)}{1-\rho} I_N \right)^{-1} z_n
\end{aligned}$$

where the inequality follows from the initial quadratic form being increasing when seen as a function of $\check{d}_i$ for each i. This can be equivalently written

$$1 \leq \frac{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)}{1-\rho}\frac{1}{N}z_n^*\left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \check{d}_n \rho \frac{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)}{1-\rho} I_N\right)^{-1} z_n. \tag{5.14}$$

At this point, it is convenient to express (5.14) as a function of F_N defined in (5.8). From (5.13), note indeed that

$$\frac{1}{N}\operatorname{tr}\check{B}_N = \frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho)}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} + \rho$$

so that, since $\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho) > 0$,

$$\frac{1}{N}\operatorname{tr}\check{B}_N - c_N(1-\rho) = F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right). \tag{5.15}$$

Since F_N is decreasing, the term on the right-hand side is decreasing in $\check{d}_i$ for each i. Hence

$$F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right) \geq F_N\left(\check{d}_n\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right).$$

This implies, returning to (5.14)

$$\begin{aligned} 1 \leq{}& \frac{1}{1-\rho}F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right) \\ &\times \frac{1}{N}z_n^*\left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \check{d}_n\frac{\rho}{1-\rho}F_N\left(\check{d}_n\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right) I_N\right)^{-1} z_n. \end{aligned} \tag{5.16}$$

With this, similar to the proof of Theorem 5.1.1, we will now show via a contradiction argument that $\sup_{\rho\in\check{\mathcal{R}}_\varepsilon}\max_{1\leq i\leq n}|\check{d}_i(\rho) - \check{\gamma}(\rho)| \xrightarrow{\text{a.s.}} 0$. Let us then assume that, on a sequence $\{\rho_n\}_{n=1}^\infty$, $\check{d}_n = \check{d}_n(\rho_n) > \check{\gamma}(\rho_n) + \ell = \check{\gamma} + \ell$ infinitely often, for some $\ell > 0$, and let us consider a subsequence on which $\rho_n \to \rho_0 \in \check{\mathcal{R}}_\varepsilon$ and $\check{d}_n(\rho_n) > \check{\gamma}(\rho_n) + \ell$. Then, from the fact that $H_N(x) = xF_N(x)$ is increasing for $x > 0$, we have

$$\begin{aligned} 1 \leq{}& \frac{1}{1-\rho}F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right) \\ &\times \frac{1}{N}z_n^*\left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \frac{(\check{\gamma}+\ell)\rho}{1-\rho}F_N\left((\check{\gamma}+\ell)\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right) I_N\right)^{-1} z_n. \end{aligned} \tag{5.17}$$

Assume first that $\rho_0 < 1$. We will deal with each factor involving F_N on the right-hand side of (5.17). We start with the right-most factor. Using $\max_{1\le i\le n}\{\frac{1}{N}\|z_i\|^2\} \xrightarrow{\text{a.s.}} 1$ since $\frac{1}{N}\operatorname{tr} C_N = 1$ for each N, $\check{\gamma}(\rho_n) \to \check{\gamma}(\rho_0)$ (by continuity of $\check{\gamma}$) and also the fact that $\lim_N \inf_{\{\check{\gamma}(\rho_0)-\eta<x<\check{\gamma}(\rho_0)+\eta\}} H_N'(x) > 0$ for some $\eta > 0$ small (from (5.10)), from (3.5), we obtain, with probability one

$$\begin{aligned}
&\lim_n \frac{1}{N} z_n^* \left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \frac{(\check{\gamma}+\ell)\rho_n}{1-\rho_n} F_N\left((\check{\gamma}+\ell)\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right) I_N \right)^{-1} z_n \\
&< \lim_n \frac{1}{N} z_n^* \left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \frac{\check{\gamma}\rho_n}{1-\rho_n} F_N\left(\check{\gamma}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right) I_N \right)^{-1} z_n \qquad (5.18)\\
&= \delta
\end{aligned}$$

where δ is the unique positive solution to

$$\delta = \int \frac{t}{\frac{\rho_0\check{\gamma}(\rho_0)F(\check{\gamma}(\rho_0))}{1-\rho_0} + \frac{t}{1+c\delta}}\nu(dt).$$

Note here the fundamental importance of having H_N' uniformly positive in a neighborhood of $\check{\gamma}(\rho_0)$ to ensure the inequality sign in (5.18) remains strict when passing to the limit over n. We will now show that $e \triangleq \frac{F(\check{\gamma}(\rho_0))}{1-\rho_0}\delta = 1$. Indeed, from the above equation,

$$e = \int \frac{t}{\rho_0\check{\gamma}(\rho_0) + \frac{(1-\rho_0)t}{F(\check{\gamma}(\rho_0))+(1-\rho_0)ce}}\nu(dt)$$

or equivalently

$$1 = \int \frac{t}{e\rho_0\check{\gamma}(\rho_0) + \frac{(1-\rho_0)te}{F(\check{\gamma}(\rho_0))+(1-\rho_0)ce}}\nu(dt). \qquad (5.19)$$

The right-hand side of (5.19) is a decreasing function of e with limits ∞ as $e \to 0$ and 0 as $e \to \infty$. As an equation of e, (5.19) therefore has a unique positive solution which happens to be 1 by definition of $\check{\gamma}(\rho_0)$ in the theorem statement. Therefore, $e = 1$.

Now consider the leading factor involving F_N in (5.17). We will show that this factor is uniformly bounded. For this, proceeding similarly as above with $\check{d}_1$ instead of $\check{d}_n$, note that (5.16), with $\rho = \rho_n$, becomes (this is obtained by reverting all inequality signs in the preceding derivations)

$$\begin{aligned}
1 &\ge \frac{1}{1-\rho_n} F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right)\\
&\quad\times \frac{1}{N} z_1^* \left(\frac{1}{n}\sum_{i=1}^{n-1} z_i z_i^* + \check{d}_1 \frac{\rho_n}{1-\rho_n} F_N\left(\check{d}_1\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right) I_N \right)^{-1} z_1. \qquad (5.20)
\end{aligned}$$

Assume $\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\to\infty$ on some subsequence (of probability one) over which $\max_i \frac{1}{N}\|z_i\|^2 \to 1$. In particular $\check{d}_1 \to 0$. Then, from the limiting values taken by F_N and H_N, the quadratic form in (5.20) has positive limit (even infinite if $c > 1$) while the first term on the right-hand side tends to infinity. This contradicts (5.20) altogether and therefore $\limsup_n \frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} < \infty$.

Since in addition $\check{d}_i \leq \rho_n^{-1}\frac{1}{N}\|z_i\|^2$ (using $\|(A+\rho_n I_N)^{-1}\| \leq \rho_n^{-1}$ for nonnegative Hermitian A) is uniformly bounded a.s. for all large n, it follows that $\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}$ is uniformly bounded and bounded away from zero. This implies that $F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right)$ is uniformly bounded, as desired.

Getting back to (5.17) with $\rho = \rho_n$, we can therefore extract a further subsequence on which the latter converges to F^∞ and $\check{d}_1$ converges to $\check{d}_1^\infty$ ($\check{d}_1^\infty$ can be zero) and we then have along this subsequence

$$1 < \frac{F^\infty}{1-\rho_0}\delta = \frac{F^\infty}{F(\check{\gamma}(\rho_0))} \tag{5.21}$$

with the equality arising from $F(\check{\gamma}(\rho_0))\delta = 1-\rho_0$.

Since F_N is increasing,

$$F_N\left(\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}\right]^{-1}\right) \leq F_N\left(\check{d}_i\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\|z_i\|^2\right]^{-1}\right)$$

so that, taking the limit over n, $F^\infty \leq F(\check{d}_1^\infty)$ (set equal to ∞ if $\check{d}_1^\infty = 0$). This further implies

$$F(\check{\gamma}(\rho_0)) < F(\check{d}_1^\infty)$$

so that, if $\check{d}_1^\infty > 0$, inverting the above inequality, gives $\check{d}_1^\infty < \check{\gamma}(\rho_0)$. Obviously, if $\check{d}_1^\infty = 0$, this is still true. Therefore $\check{d}_1(\rho_n) < \check{\gamma}(\rho_0) - \ell'$ infinitely often for some $\ell' > 0$ along the considered subsequence.

Conserving the same subsequence and reproducing the same steps for the sequence $\check{d}_1(\rho_n)$ instead of $\check{d}_n(\rho_n)$ (from (5.20), use $\check{d}_1(\rho_n) < \check{\gamma}(\rho_n) - \ell'$ infinitely often and the growth of H_N similar to before), we obtain this time

$$1 > \frac{F^\infty}{F(\check{\gamma}(\rho_0))}$$

which contradicts (5.21).

Assume now $\rho_0 = 1$. Starting from (5.14) with $\rho = \rho_n$ and the expression of F_N, we have

$$
\begin{aligned}
1 &\le \limsup_N F_N \left(\left[\frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} \right]^{-1} \right) \\
&\times \frac{1}{N} z_n^* \left((1-\rho_n) \frac{1}{n} \sum_{i=1}^{n-1} z_i z_i^* + \check{d}_n \rho_n F_N \left(\left[\frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} \right]^{-1} \right) I_N \right)^{-1} z_n \\
&\le \limsup_N F_N \left(\left[\frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} \right]^{-1} \right) \\
&\times \frac{1}{N} z_n^* \left((1-\rho_n) \frac{1}{n} \sum_{i=1}^{n-1} z_i z_i^* + (\check{\gamma} + \ell) \rho_n F_N \left(\left[\frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i} \right]^{-1} \right) I_N \right)^{-1} z_n \\
&= \frac{1}{\check{\gamma}(\rho_0) + \ell}
\end{aligned}
$$

since $\rho_n \to \rho_0 = 1$, since $\frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{N}\|z_i\|^2}{\check{d}_i}$ is uniformly away from zero (as shown previously), and since $\limsup_n \|\frac{1}{n}\sum_{i=1}^{n} z_i z_i^*\| < \infty$ (Bai and Silverstein, 1998). But then, the fact that $\check{\gamma}(\rho_0) = 1$ by definition along with the above relation leads to $1 \le 1/(1+\ell)$, again a contradiction.

Therefore, gathering the results, our very initial hypothesis that there exists a subsequence of n and ρ_n over which $\check{d}_n(\rho_n) > \gamma(\rho_n) + \ell$ infinitely often is invalid and we conclude that, instead, $\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \check{d}_n(\rho) - \check{\gamma}(\rho) \le \ell$ for all large n a.s.

The same procedure works similarly when starting over with $\check{d}_1$ and assuming with the same contradiction argument that $\check{d}_1(\rho_n') < \check{\gamma}(\rho_n') - \ell$ infinitely often on some sequence ρ_n'. Taking a subsequence over which $\rho_n' \to \rho_0'$, this will imply this time that $\check{d}_n(\rho_0') > \check{\gamma}(\rho_0') + \ell'$ for some $\ell' > 0$ for all large n a.s. which we now know is invalid.

Gathering the results, we finally obtain

$$
\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \max_{1 \le i \le n} |\check{d}_i(\rho) - \check{\gamma}(\rho)| \xrightarrow{\text{a.s.}} 0 \tag{5.22}
$$

as desired. This implies from (5.15) that

$$
\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \left| \frac{1}{N} \operatorname{tr} \check{B}_N - c(1-\rho) - F(\check{\gamma}(\rho)) \right| \xrightarrow{\text{a.s.}} 0
$$

with $\inf_{\rho \in \check{\mathcal{R}}_\varepsilon} F(\check{\gamma}(\rho)) > 0$ so that, from (5.13), Assumption 5.2–b., and (Bai and Silverstein, 1998),

$$
\sup_{\rho \in \check{\mathcal{R}}_\varepsilon} \left\| \check{B}_N - \left[\frac{1-\rho}{F(\check{\gamma}(\rho))\check{\gamma}(\rho)} \frac{1}{n} \sum_{i=1}^{n} z_i z_i^* + \rho I_N \right] \right\| \xrightarrow{\text{a.s.}} 0.
$$

Dividing the expression inside the norm by $\frac{1}{N}\operatorname{tr}\check{B}_N$ and taking the limit finally gives

$$\sup_{\rho\in\check{\mathcal{R}}_\varepsilon}\left\|\check{C}_N-\left[\frac{1-\rho}{\rho F(\check{\gamma})\check{\gamma}+(1-\rho)}\frac{1}{n}\sum_{i=1}^n z_iz_i^*+\frac{\rho\check{\gamma}F(\check{\gamma})}{\rho\check{\gamma}F(\check{\gamma})+(1-\rho)}I_N\right]\right\|\xrightarrow{\text{a.s.}}0$$

with $\check{\gamma}=\check{\gamma}(\rho)$, which is the expected result.

5.1.3.3 Proof of Corollary 5.2

We only give the proof for $\hat{C}_N(\rho)$. Similar arguments hold for $\check{C}_N(\rho)$. The joint eigenvalue convergence is an application of (Horn and Johnson, 1985, Theorem 4.3.7) on the spectral norm convergence of Theorems 5.1.1 and 5.1.2. The norm boundedness results from $\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}|\|\hat{C}_N(\rho)\|-\|\hat{S}_N(\rho)\||\xrightarrow{\text{a.s.}}0$ and from $\limsup_N\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}\|\hat{S}_N(\rho)\|<\infty$ by an application of (Bai and Silverstein, 1998). The joint convergence of moments over $\hat{\mathcal{R}}_\varepsilon$ follows first from the convergence $\hat{m}_N(z;\rho)-m_{\hat{\mu}_\rho}(z)\xrightarrow{\text{a.s.}}0$ for each z with $\Im[z]>0$ and for each $\rho\in\hat{\mathcal{R}}_\varepsilon$ where $m_N(z;\rho)=\frac{1}{N}\operatorname{tr}((\hat{S}_N(\rho)-zI_N)^{-1})$ (as a consequence of Corollary 5.1). Since this holds for each such z, the almost sure convergence is also valid uniformly on a countable set of z with $\Im[z]>0$ having a limit point away from the union $\mathcal{U}$ over $\rho\in\hat{\mathcal{R}}_\varepsilon$ of the limiting spectra of $\hat{S}_N(\rho)$, $\mathcal{U}$ being a bounded set since $\limsup_N\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}\|\hat{S}_N(\rho)\|<\infty$. But then, since

$$\frac{(1-\rho)m_N(z;\rho)}{\hat{\gamma}(\rho)(1-(1-\rho)c)}=\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n z_iz_i^*+\frac{\rho-z}{1-\rho}\hat{\gamma}(\rho)(1-(1-\rho)c)I_N\right)^{-1}\right]$$

is analytic in $\hat{z}(\rho)=\frac{\rho-z}{1-\rho}\hat{\gamma}(\rho)(1-(1-\rho)c)$ and bounded on all bounded regions away from $\mathcal{U}$, by Vitali's convergence theorem (Titchmarsh, 1939), the convergence $\hat{m}_N(z;\rho)-m_{\hat{\mu}_\rho}(z)\xrightarrow{\text{a.s.}}0$ is uniform on such bounded sets of (z,ρ). Using the Cauchy integrals $\oint z^k m_N(z;\rho)dz=\frac{1}{N}\operatorname{tr}(\hat{S}_N(\rho)^\ell)$ and $\oint z^k m_{\hat{\mu}_\rho}(z)dz=M_{\hat{\mu}_\rho,k}$ for each $k\in\mathbb{N}$ on a contour that circles around (but sufficiently away from) $\mathcal{U}$ implies $\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}|\frac{1}{N}\operatorname{tr}(\hat{S}_N(\rho)^\ell)-M_{\hat{\mu}_\rho,\ell}|\xrightarrow{\text{a.s.}}0$, from which the result unfolds.

5.1.3.4 Proof of Lemma 5.1

We start with $\hat{S}_N$. Remark first that, for $\rho\in(\max\{0,1-c^{-1}\},1]$,

$$\frac{\hat{S}_N(\rho)}{M_{\hat{\mu}_\rho,1}}=\left(1-\frac{\rho}{\frac{1}{\hat{\gamma}(\rho)}\frac{1-\rho}{1-(1-\rho)c}+\rho}\right)\frac{1}{n}\sum_{i=1}^n z_iz_i^*+\frac{\rho}{\frac{1}{\hat{\gamma}(\rho)}\frac{1-\rho}{1-(1-\rho)c}+\rho}I_N.$$

Denoting

$$\begin{aligned}\hat{f}:(\max\{0,1-c^{-1}\},1]&\to(0,1]\\ \rho&\mapsto\frac{\rho}{\frac{1}{\hat{\gamma}(\rho)}\frac{1-\rho}{1-(1-\rho)c}+\rho}=\frac{1}{\frac{1}{\rho\hat{\gamma}(\rho)}\frac{1-\rho}{1-(1-\rho)c}+1}\end{aligned}$$

we have $\frac{\hat{S}_N(\rho)}{M_{\hat{\mu}_\rho,1}} = (1-\hat{f}(\rho))\frac{1}{n}\sum_{i=1}^n z_i z_i^* + \hat{f}(\rho) I_N$ and it therefore suffices to show that $\hat{f}$ is continuously increasing and onto. The continuity of $\hat{f}$ unfolds immediately from the continuity of $\hat{\gamma}$. By the definition of $\hat{\gamma}$, the function $\rho \mapsto \rho\hat{\gamma}(\rho)$ is increasing and nonnegative (since ν is distinct from $\boldsymbol{\delta}_0$ almost everywhere) while $\rho \mapsto \frac{1-\rho}{1-(1-\rho)c}$ is decreasing and nonnegative. Therefore, $\hat{f}$ is increasing and nonnegative. It remains to show that $\hat{f}$ is onto. Clearly $\hat{f}(1) = 1$ since $\hat{\gamma}(1) = M_{\nu,1} = 1$. To handle the lower limit, let us rewrite

$$\hat{f}(\rho) = \frac{\rho\hat{\gamma}(\rho)(1-(1-\rho)c)}{1-\rho+\rho\hat{\gamma}(\rho)(1-(1-\rho)c)}$$

which we aim to show approaches zero as $\rho \downarrow \max\{0, 1-c^{-1}\}$. For this, assume $\rho_k\hat{\gamma}(\rho_k)(1-(1-\rho_k)c) \to \ell \in (0,\infty]$ for a sequence $\rho_k \downarrow \max\{0, 1-c^{-1}\}$. Then, from the defining equation of $\hat{\gamma}(\rho)$ in Theorem 5.1.1,

$$\begin{aligned}
1 &= \int \frac{(1-(1-\rho_k)c)t}{\rho_k\hat{\gamma}(\rho_k)(1-(1-\rho_k)c) + (1-\rho_k)(1-(1-\rho_k)c)t}\nu(dt) \\
&\leq \frac{(1-(1-\rho_k)c)\limsup_N \|C_N\|}{\rho_k\hat{\gamma}(\rho_k)(1-(1-\rho_k)c) + (1-\rho_k)(1-(1-\rho_k)c)\limsup_N \|C_N\|} \\
&\to \frac{\lim_k(1-(1-\rho_k)c)\limsup_N \|C_N\|}{\ell + \lim_k(1-\rho_k)(1-(1-\rho_k)c)\limsup_N \|C_N\|} \\
&< 1
\end{aligned}$$

since the limit is either zero (when $c \geq 1$) or $(1-c)\limsup_N \|C_N\|/(\ell+(1-c)\limsup_N \|C_N\|) < 1$ (when $c < 1$). But this is a contradiction. This implies that $\rho\hat{\gamma}(\rho)(1-(1-\rho)c) \to 0$ and consequently $\hat{f}(\rho) \to 0$ as $\rho \downarrow \max\{0, 1-c^{-1}\}$, which completes the proof for $\hat{S}(\rho)$.

Similarly, for $\check{S}(\rho)$, define

$$\begin{aligned}
\check{f} &: (0,1] \to (0,1] \\
&\rho \mapsto \frac{T_\rho}{1-\rho+T_\rho}
\end{aligned}$$

where we recall that $T_\rho = \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)$ and which is such that $\check{S}_N(\rho) = (1-\check{f}(\rho))\frac{1}{n}\sum_{i=1}^n z_i z_i^* + \check{f}(\rho)I_N$. We will show that $\check{f}$ is continuously increasing and onto. The continuity arises from the continuity of $\check{\gamma}$. We first show that $\check{\gamma}$ is onto. For the upper limit, $\check{f}(1) = 1$. For the lower limit, assume $T_{\rho_k} \to \ell \in (0,\infty]$ over a sequence $\rho_k \to 0$, so that in particular $T_{\rho_k}\rho_k^{-1} \to \infty$. Then, by the definition of $\check{\gamma}(\rho)$ and since $F(x;\rho) = (1-\rho)\frac{1}{xF(x;\rho)} + \rho - c(1-\rho)$,

$$1 = \int \frac{1}{\check{\gamma}(\rho_k)\rho_k t^{-1} + T_{\rho_k}\rho_k^{-1}\frac{1-\rho_k}{1-\rho_k+T_{\rho_k}}}\nu(dt) \to 0$$

by dominated convergence (recall that ν has bounded support), which is a contradiction. This implies $\check{f}(\rho) \to 0$ as $\rho \to 0$. It remains to show that $\check{f}$ is increasing. For this, we will rewrite the equation defining $\check{\gamma}(\rho)$ as a function of $\check{f}(\rho)$. Using again $F(x;\rho) = (1-\rho)\frac{1}{xF(x;\rho)} + \rho - c(1-\rho)$, we first have, for each $t \geq 0$,

$$\begin{aligned}
\check{\gamma}(\rho)\rho + \frac{1-\rho}{(1-\rho)c + F(\check{\gamma}(\rho);\rho)}t &= \check{\gamma}(\rho)\rho + \frac{1-\rho}{(1-\rho)\frac{1}{\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)} + \rho}t \\
&= \check{\gamma}(\rho)\rho + \frac{(1-\rho)\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)}{1-\rho+\rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)}t \\
&= \frac{\rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)}{F(\check{\gamma}(\rho);\rho)} + \frac{1-\rho}{\rho}\check{f}(\rho)t \\
&= \frac{1}{F(\check{\gamma}(\rho);\rho)}\frac{(1-\rho)\check{f}(\rho)}{1-\check{f}(\rho)} + \frac{1-\rho}{\rho}\check{f}(\rho)t
\end{aligned}$$

where in the last equality we used $(1-\rho)\check{f}(\rho) = (1-\check{f}(\rho))\rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)$. We now work on $F(\check{\gamma}(\rho);\rho)$. By its implicit definition,

$$\begin{aligned}
\frac{1}{F(\check{\gamma}(\rho);\rho)} &= \frac{1}{(1-\rho)\frac{1}{\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)} + \rho - c(1-\rho)} \\
&= \frac{\rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)}{\rho(1-\rho) + \rho^2\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho) - c(1-\rho)\rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)} \\
&= \frac{(1-\rho)\check{f}(\rho)}{1-\check{f}(\rho)}\frac{1}{\rho(1-\rho) + \rho\frac{(1-\rho)\check{f}(\rho)}{1-\check{f}(\rho)} - c(1-\rho)\frac{(1-\rho)\check{f}(\rho)}{1-\check{f}(\rho)}} \\
&= \frac{\check{f}(\rho)}{\rho - c(1-\rho)\check{f}(\rho)}
\end{aligned}$$

where the last equation follows from standard algebraic simplification. Note here in particular that, by positivity of $F(x;\rho)$ for $x > 0$, $\rho - c(1-\rho)\check{f}(\rho) > 0$. Plugging the two results above in the defining equation for $\check{\gamma}(\rho)$, we obtain

$$1 = \int \frac{t}{\frac{\check{f}(\rho)}{\rho - c(1-\rho)\check{f}(\rho)}\frac{(1-\rho)\check{f}(\rho)}{\rho(1-\check{f}(\rho))} + \frac{1-\rho}{\rho}\check{f}(\rho)t}\nu(dt). \tag{5.23}$$

Now assume that $\check{f}(\rho)$ is decreasing on an open neighborhood of $\rho_0 \in (0,1)$. Then $\rho \mapsto \frac{1-\rho}{\rho}\check{f}(\rho)$ and $\rho \mapsto \frac{(1-\rho)\check{f}(\rho)}{\rho(1-\check{f}(\rho))}$ are also decreasing. This follows from the fact that, on this neighborhood, $\rho \mapsto (1-\rho)/\rho = 1/\rho - 1$, $\rho \mapsto 1-\rho$, and $\rho \mapsto \check{f}(\rho)/(1-\check{f}(\rho)) = -1 + 1/(1-\check{f}(\rho))$ are all positive decreasing functions of ρ. Finally,

$$\frac{\check{f}(\rho)}{\rho - c(1-\rho)\check{f}(\rho)} = \frac{1}{\frac{\rho}{\check{f}(\rho)} + c(\rho - 1)}$$

which is also positive decreasing, since $\rho \mapsto \rho/\check{f}(\rho)$ and $\rho \mapsto c(\rho-1)$ are both increasing and of positive sum. But then, the right-hand side of (5.23) is increasing on a neighborhood of ρ_0 while being constant equal to one, which is a contradiction. Therefore, our initial assumption that $\check{f}(\rho)$ is locally decreasing around ρ_0 does not hold, and therefore $\check{f}(\rho)$ is increasing there and thus increasing on $(0,1]$. This completes the proof.

5.1.3.5 Proof of Proposition 5.1.1

We only prove the result for $\hat{C}_N$, the treatment for $\check{C}_N$ being the same. First observe that, denoting $A_N(\hat{\rho}) = \frac{\hat{C}_N(\hat{\rho})}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho})} - \frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}}$,

$$\begin{aligned}
&\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left|\hat{D}_N(\hat{\rho}) - \frac{1}{N}\operatorname{tr}\left(\left(\frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}} - C_N\right)^2\right)\right| \\
&= \sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left|\frac{1}{N}\operatorname{tr}\left(A_N(\hat{\rho})\left[\frac{\hat{C}_N(\hat{\rho})}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho})} + \frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}} - 2C_N\right]\right)\right| \\
&\le \sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left\{2\left|\frac{1}{N}\operatorname{tr}(A_N(\hat{\rho})C_N)\right| + \left|\frac{1}{N}\operatorname{tr}\left(A_N(\hat{\rho})\left[\frac{\hat{C}_N(\hat{\rho})}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho})} + \frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}}\right]\right)\right|\right\} \\
&\le \sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\|A_N(\hat{\rho})\| \sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left(3 + \frac{\frac{1}{N}\operatorname{tr}\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}}\right)
\end{aligned}$$

where we used $|\operatorname{tr}(AB)| \le \operatorname{tr} A\|B\|$ for nonnegative definite A along with $\frac{1}{N}\operatorname{tr} C_N = 1$. Now,

$$\begin{aligned}
\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\|A_N(\hat{\rho})\| \le{}& \frac{\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon} M_{\hat{\mu}_{\hat{\rho}},1}\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\|\hat{C}_N(\hat{\rho}) - \hat{S}_N(\hat{\rho})\|}{\inf_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho})M_{\hat{\mu}_{\hat{\rho}},1}} \\
&+ \frac{\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\|\hat{S}_N(\hat{\rho})\|\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left|\frac{1}{N}\operatorname{tr}\hat{C}_N(\hat{\rho}) - M_{\hat{\mu}_{\hat{\rho}},1}\right|}{\inf_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon} M_{\hat{\mu}_{\hat{\rho}},1}\frac{1}{N}\operatorname{tr}\left(\hat{C}_N(\hat{\rho})\right)}.
\end{aligned}$$

Since $M_{\hat{\mu}_{\hat{\rho}},1} = \frac{1}{\hat{\gamma}(\hat{\rho})}\frac{1-\hat{\rho}}{1-(1-\hat{\rho})c}$ is uniformly bounded across $\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon$, this finally implies from Theorem 5.1.1 and Corollary 5.2 that both right-hand side terms tend almost surely to zero in the large N, n limit (in particular since the denominators are bounded away from zero), and finally

$$\sup_{\hat{\rho}\in\hat{\mathfrak{R}}_\varepsilon}\left|\hat{D}_N(\hat{\rho}) - \frac{1}{N}\operatorname{tr}\left[\left(\frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}} - C_N\right)^2\right]\right| \xrightarrow{\text{a.s.}} 0.$$

Moreover, from Lemma 5.1, for each $\hat{\rho}\in(\max\{0, 1-c^{-1}\}, 1]$,

$$\frac{1}{N}\operatorname{tr}\left[\left(\frac{\hat{S}_N(\hat{\rho})}{M_{\hat{\mu}_{\hat{\rho}},1}} - C_N\right)^2\right] = \frac{1}{N}\operatorname{tr}\left[\left(\bar{S}_N(\rho) - C_N\right)^2\right]$$

with $\rho = \hat{\rho}(\frac{1}{\hat{\gamma}(\hat{\rho})}\frac{1-\hat{\rho}}{1-(1-\hat{\rho})c} + \hat{\rho})^{-1} \in (0,1]$ and with $\bar{S}_N = (1-\rho)\frac{1}{n}\sum_{i=1}^n z_i z_i^* + \rho I_N$. Also, using $\frac{1}{N}\operatorname{tr}\left(\frac{1}{n}\sum_{i=1}^n z_i z_i^*\right) \xrightarrow{\text{a.s.}} M_{\nu,1} = 1$, $\frac{1}{N}\operatorname{tr}\left[\left(\frac{1}{n}\sum_{i=1}^n z_i z_i^*\right)^2\right] \xrightarrow{\text{a.s.}} M_{\nu,2} + c$, and basic arithmetic derivations

$$\sup_{\rho\in[0,1]}\left|\frac{1}{N}\operatorname{tr}\left[\left(\bar{S}_N(\rho) - C_N\right)^2\right] - \bar{D}(\rho)\right| \xrightarrow{\text{a.s.}} 0$$

where

$$\bar{D}(\rho) = (M_{\nu,2} - 1)\rho^2 + c(1-\rho)^2.$$

Note importantly that, from the Cauchy–Schwarz inequality, $1 = M_{\nu,1}^2 \leq M_{\nu,2}$ and therefore $M_{\nu,2} - 1 \geq 0$ with equality if and only if $\nu = \boldsymbol{\delta}_a$ for some $a \geq 0$ almost everywhere. From the above convergence, we then have, for any $\varepsilon > 0$ small

$$\sup_{\hat{\rho} \in \hat{\mathcal{R}}_\varepsilon} \left| \hat{D}_N(\hat{\rho}) - \bar{D}(\rho) \right| \xrightarrow{\text{a.s.}} 0. \tag{5.24}$$

Now, call $\rho^\star$ the minimizer of $\bar{D}(\rho)$ over $[0,1]$. It is easily verified that $\rho^\star \in (0,1]$ is as defined in the theorem. Also denote $\hat{\rho}^\star$ the unique value such that $\rho^\star = \hat{\rho}^\star(\frac{1}{\hat{\gamma}(\hat{\rho}^\star)} \frac{1-\hat{\rho}^\star}{1-(1-\hat{\rho}^\star)c} + \hat{\rho}^\star)^{-1}$, which is well defined according to Lemma 5.1. Call also $\hat{\rho}_N^\circ$ the minimizer of $\hat{D}_N(\hat{\rho})$ over $\hat{\mathcal{R}}_\varepsilon$ and $\rho_N^\circ = \hat{\rho}_N^\circ(\frac{1}{\hat{\gamma}(\hat{\rho}_N^\circ)} \frac{1-\hat{\rho}_N^\circ}{1-(1-\hat{\rho}_N^\circ)c} + \hat{\rho}_N^\circ)^{-1}$. If ε is as given in the theorem statement, $\hat{\rho}^\star \in \hat{\mathcal{R}}_\varepsilon$ and then

$$\begin{aligned}
\bar{D}(\rho^\star) &\leq \bar{D}(\rho_N^\circ) \\
\hat{D}_N(\hat{\rho}_N^\circ) &\leq \hat{D}_N(\hat{\rho}^\star) \\
\hat{D}_N(\hat{\rho}^\star) - \bar{D}(\rho^\star) &\xrightarrow{\text{a.s.}} 0 \\
\hat{D}_N(\hat{\rho}_N^\circ) - \bar{D}(\rho_N^\circ) &\xrightarrow{\text{a.s.}} 0
\end{aligned}$$

the last two equations following from (5.24) (the joint convergence in (5.24) is fundamental since ρ_N° and $\hat{\rho}_N^\circ$ are not constant with N). These four relations together ensure that

$$\begin{aligned}
\hat{D}_N(\hat{\rho}_N^\circ) - \bar{D}(\rho^\star) &\xrightarrow{\text{a.s.}} 0 \\
\hat{D}_N(\hat{\rho}_N^\circ) - \hat{D}_N(\hat{\rho}^\star) &\xrightarrow{\text{a.s.}} 0.
\end{aligned}$$

These and the fact that $\bar{D}(\rho^\star) = D^\star$ as defined in the theorem statement conclude the proof of the first part of the theorem.

For the second part, denoting $\rho_N = \hat{\rho}_N(\frac{1}{\hat{\gamma}(\hat{\rho}_N)} \frac{1-\hat{\rho}_N}{1-(1-\hat{\rho}_N)c} + \hat{\rho}_N)^{-1}$, we have that $\bar{D}(\rho_N) - \bar{D}(\rho^\star) \xrightarrow{\text{a.s.}} 0$ by continuity of $\bar{D}$ since $\rho_N \xrightarrow{\text{a.s.}} \rho^\star$ and therefore, since $\hat{D}_N(\hat{\rho}_N) - \bar{D}(\rho_N) \xrightarrow{\text{a.s.}} 0$ by (5.24), $\hat{D}_N(\hat{\rho}_N) - \bar{D}(\rho^\star) \xrightarrow{\text{a.s.}} 0$ which is the expected result.

5.1.3.6 Proof of Proposition 5.1.2

We first show the following identities

$$\frac{1}{n} \operatorname{tr} \left[\left(\frac{1}{n} \sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N}\|x_i\|^2} \right)^2 \right] - c_N \xrightarrow{\text{a.s.}} M_{\nu,2} \tag{5.25}$$

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} \left| T_\rho - \rho \frac{1}{n} \sum_{i=1}^n \frac{x_i^* \check{C}_N(\rho)^{-1} x_i}{\|x_i\|^2} \right| \xrightarrow{\text{a.s.}} 0. \tag{5.26}$$

Equation (5.25) unfolds from $\frac{1}{n}\operatorname{tr}\left[(\frac{1}{n}\sum_{i=1}^n z_i z_i^*)^2\right] \xrightarrow{\text{a.s.}} M_{\nu,2} + cM_{\nu,1}^2 = M_{\nu,2} + c$ and from $\max_{1\leq i\leq n} |\frac{1}{N}\|z_i\|^2 - 1| \xrightarrow{\text{a.s.}} 0$. As for Equation (5.26), it is a consequence of the elements of the proof of Theorem 5.1.2. Indeed, from (5.12),

$$\rho\frac{1}{N}x_i^*\check{C}_N(\rho)^{-1}x_i = \rho\frac{1}{N}x_i^*\check{B}_{(i)}(\rho)^{-1}x_i\left(\frac{1}{N}\operatorname{tr}\check{B}_N(\rho) - c_N(1-\rho)\right)$$

where $\check{B}_{(i)}(\rho) = \check{B}_N(\rho) - \frac{1}{n}\frac{1-\rho}{\frac{1}{N}\operatorname{tr}\check{B}_N}\frac{x_i x_i^*}{\frac{1}{N}x_i^*\check{B}_N(\rho)^{-1}x_i}$, which according to (5.15) further reads

$$\rho\frac{1}{N}x_i^*\check{C}_N(\rho)^{-1}x_i = \rho\frac{1}{N}x_i^*\check{B}_{(i)}(\rho)^{-1}x_i F_N\left(\left[\frac{1}{n}\sum_{i=1}^n\frac{\|x_i\|^2}{\frac{1}{N}x_i^*\check{B}_{(i)}(\rho)^{-1}x_i}\right]^{-1};\rho\right)$$

with $F_N(x;\rho)$ the same function as F but with c_N in place of c (recall that in (5.15), $\check{d}_i = \frac{1}{N}z_i^*\check{B}_{(i)}(\rho)^{-1}z_i$). Since the τ_i normalization is irrelevant in the expression above, x_i can be replaced by z_i. Using the convergence result (5.22) and the continuity and boundedness of $x\mapsto xF_N(x)$, we then have

$$\sup_{\rho\in\check{\mathcal{R}}_\varepsilon}\max_{1\leq i\leq n}\left|\rho\frac{1}{N}z_i^*\check{C}_N(\rho)^{-1}z_i - \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)\right| \xrightarrow{\text{a.s.}} 0.$$

As a consequence,

$$\begin{aligned}
&\sup_{\rho\in\check{\mathcal{R}}_\varepsilon}\left|\rho\frac{1}{n}\sum_{i=1}^n\frac{1}{N}z_i^*\check{C}_N(\rho)^{-1}z_i - \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)\right| \\
&\leq \sup_{\rho\in\check{\mathcal{R}}_\varepsilon}\max_{1\leq i\leq n}\left|\rho\frac{1}{N}z_i^*\check{C}_N(\rho)^{-1}z_i - \rho\check{\gamma}(\rho)F(\check{\gamma}(\rho);\rho)\right| \\
&\xrightarrow{\text{a.s.}} 0.
\end{aligned}$$

This, and the fact that $\max_{1\leq i\leq n}|\frac{1}{N}\|z_i\|^2 - 1| \xrightarrow{\text{a.s.}} 0$ gives the result.

It remains to prove that $\hat{\rho}_N \xrightarrow{\text{a.s.}} \hat{\rho}^\star$ and $\check{\rho}_N \xrightarrow{\text{a.s.}} \check{\rho}^\star$. We only prove the first convergence, the second one unfolding along the same lines. First observe from Corollary 5.2 that the defining equation of $\hat{\rho}_N$ implies

$$\hat{f}(\hat{\rho}_N) = \frac{c}{M_{\nu,2}+c-1} + \ell_n$$

for some sequence $\ell_n \xrightarrow{\text{a.s.}} 0$, with $\hat{f}: x\mapsto x(\frac{1}{\check{\gamma}(x)}\frac{1-x}{1-(1-x)c}+x))^{-1}$. Since $\hat{f}$ is a one-to-one growing map from $(\max\{0,1-c^{-1}\},1]$ onto $(0,1]$ (Lemma 5.1) and $\frac{c}{M_{\nu,2}+c-1}\in(0,1)$, such a $\hat{\rho}_N$ exists (not necessarily uniquely though) for all large N almost surely. Taking such a ρ_N, by definition of $\hat{\rho}^\star$, we further have

$$\hat{f}(\hat{\rho}_N) - \hat{f}(\hat{\rho}^\star) \xrightarrow{\text{a.s.}} 0$$

which, by the continuous growth of $\hat{f}$, ensures that $\hat{\rho}_N \xrightarrow{\text{a.s.}} \hat{\rho}^\star$. The convergence $\hat{D}_N(\hat{\rho}_N) \xrightarrow{\text{a.s.}} D^\star$ is then an application of Proposition 5.1.1.

5.2 Application to portfolio optimization

In terms of applications, Proposition 5.1.2 allows for the design of covariance matrix estimators, with minimal Frobenius distance to the population covariance matrix for impulsive i.i.d. samples but in the absence of outliers, and having robustness properties in the presence of outliers. This is fundamental to those scientific fields where the covariance matrix is the object of central interest. More generally though, Theorems 5.1.1 and 5.1.2 can be used to design optimal covariance matrix estimators under other metrics than the Frobenius norm. This is in particular the case in applications to finance where a possible target consists in the minimization of the risk induced by portfolios built upon such covariance matrix estimates.

Precisely, this section aims at designing a covariance estimation technique to optimize large portfolios based on impulsive market return observations and under the assumption that the number of observed samples is of the same order as the number of assets in the portfolio. The covariance estimation shall rely on Abramovich–Pascal's estimator $\hat{C}_N(\rho)$. Since the results provided in this section mimic closely those of Section 5.1 for a portfolio-based metric instead of the Frobenius norm minimization, we omit most of proofs here or only point out the minor differences in the present setting if any.

We shall first characterize the out-of-sample performance of minimum variance portfolios based on $\hat{C}_N(\rho)$ by analyzing the convergence of the achieved realized risk as $N, n \to \infty$, with $c_N = N/n \to c \in (0, \infty)$. We subsequently provide a consistent estimator of the realized portfolio risk that is defined only in terms of the observed market returns. Minimizing this estimated risk then brings a minimum risk covariance matrix estimate. Performance comparisons versus previously proposed schemes on artificial and actual stock returns from the Hang Seng Index (HSI) will show a competitive advantage of the new estimator.

5.2.1 System model and results

Consider the successive and independent returns $x_1, \ldots, x_n$ of N financial assets to be modelled as

$$x_i = \mu + \sqrt{\tau_i} C_N^{\frac{1}{2}} w_i \tag{5.27}$$

where $\mu \in \mathbb{R}^N$ is the mean vector of the asset returns, $\tau_i > 0$ is an impulsiveness scalar, $C_N \in \mathbb{R}^{N \times N}$ is a positive definite covariance matrix of the returns, and $w_i \in \mathbb{R}^N$ is a zero mean unitarily invariant norm $\|w_i\|^2 = N$ random vector independent of τ_i. Moreover denote $z_i = C_N^{\frac{1}{2}} w_i$. This statistical modelling of impulsive financial stock returns goes in line with previous works, as in e.g., (Ruppert, 2010). Although it is not practically tenable to assume independent x_i's, this assumption leads to tractable design solutions and is a commonly used assumption (Ledoit and Wolf, 2003).

Let now $h \in \mathbb{R}^N$ denote the portfolio selection, i.e., the vector of asset holdings in units of currency normalized by the total outstanding wealth, such that $h^* 1_N = 1$ (with $1_N = [1, \ldots, 1]^* \in \mathbb{R}^N$). Then the portfolio variance (or risk) over the investment period of interest is given by

$\sigma^2(h) = h^* C_N h$. Accordingly, the global minimum variance portfolio (GMVP) selection problem we ought to solve can be formulated as

$$\min_h \sigma^2(h), \text{ such that } h^* 1_N = 1. \tag{5.28}$$

The solution to (5.28) is explicitly given by

$$h_N = \frac{C_N^{-1} 1_N}{1_N^* C_N^{-1} 1_N}$$

and the corresponding portfolio risk by

$$\sigma^2(h_N) = \frac{1}{1_N^* C_N^{-1} 1_N}. \tag{5.29}$$

Here, (5.29) represents the theoretical minimum portfolio risk bound achievable upon knowing the population covariance matrix C_N exactly. In practice, C_N being unknown, one resorts to a plug-in estimator by substituting any valid estimator for C_N in (5.29). We propose here to consider such a plug-in estimator to be the Abramovich–Pascal estimator $\hat{C}_N(\rho)$ for the centered data $\tilde{x}_i = x_i - \frac{1}{n}\sum_{j=1}^n x_j$, i.e., $\hat{C}_N(\rho)$ is defined, for each $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1]$ as the unique solution to[4]

$$\hat{C}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{\tilde{x}_i \tilde{x}_i^*}{\frac{1}{N}\tilde{x}_i^* \hat{C}_N^{-1}(\rho)\tilde{x}_i} + \rho I_N. \tag{5.30}$$

The corresponding estimated GMVP selection is thus

$$\hat{h}_N(\rho) = \frac{\hat{C}_N^{-1}(\rho) 1_N}{1_N^* \hat{C}_N^{-1}(\rho) 1_N}$$

with realized portfolio risk

$$\sigma^2(\hat{h}_N(\rho)) = \frac{1_N^* \hat{C}_N^{-1}(\rho) C_N \hat{C}_N^{-1}(\rho) 1_N}{(1_N^T \hat{C}_N^{-1}(\rho) 1_N)^2}. \tag{5.31}$$

Our goal is to select ρ to be such that (5.31) reaches a minimum. This is not obvious as the dependence over C_N requires to first and foremost determine a consistent estimator for the non-observable $\sigma^2(\hat{h}_N(\rho))$ (it can be shown that substituting $\hat{C}_N(\rho)$ for C_N in this expression yields a so-called in-sample risk which tends to underestimate the realized portfolio risk, leading to overly-optimistic investment decisions, as shown in (Rubio et al., 2012)). To this end, we first derive a deterministic equivalent for (5.31) from which the consistent estimator will be deduced.

Since the model for $\tilde{x}_i$ differs from that considered in Section 5.1, we first need some additional technical assumptions which we gather along with classical hypotheses below.

[4] Since the x_i are linearly independent with probability one, $\hat{C}_N(\rho)$ remains well defined.

Assumption 5.3. *1. As $N, n \to \infty$, $N/n = c_N \to c \in (0, \infty)$;*

2. The τ_i's are i.i.d., and $E[\tau_1], E[\frac{1}{\tau_1}] < \infty$;[5]

3. Denoting $\lambda_1 \leq \ldots \leq \lambda_N$ the ordered eigenvalues of C_N, $\nu_N \triangleq \frac{1}{N}\sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i}$ satisfies $\nu_N \to \nu$ weakly with $\nu \neq \boldsymbol{\delta}_0$ almost everywhere; moreover, $\limsup_N \lambda_N < \infty$.

We will also write $k(\rho) = \frac{1-\rho}{1-(1-\rho)c}$ for short and define α as the unique positive solution to

$$\alpha = \frac{1}{n}\operatorname{tr}\left[C_N\left(\frac{k(\rho)}{(\gamma+\alpha k(\rho))}C_N + \rho I_N\right)^{-1}\right]$$

with γ the unique positive solution to

$$1 = \int \frac{t}{\gamma\rho + (1-\rho)t}\nu(dt).$$

We also define

$$\beta = \frac{1}{n}\operatorname{tr}\left[C_N^2\left(\frac{k(\rho)}{(\gamma+\alpha k(\rho))}C_N + \rho I_N\right)^{-2}\right].$$

The following theorem is our main technical result.

Theorem 5.2.1. *Let Assumption 5.3 hold. For $\varepsilon \in (0, \min\{1, c^{-1}\})$, define $\hat{\mathcal{R}}_\varepsilon = [\varepsilon + \max\{0, 1-c^{-1}\}, 1]$. Then,*

$$\sup_{\rho\in\hat{\mathcal{R}}_\varepsilon}\left|\sigma^2(\hat{h}_N(\rho)) - \bar{\sigma}^2(\hat{h}_N(\rho))\right| \xrightarrow{\text{a.s.}} 0$$

where

$$\bar{\sigma}^2(\hat{h}_N(\rho)) \triangleq \frac{1}{1-\frac{\beta k(\rho)^2}{(\gamma+\alpha k(\rho))^2}}\frac{1_N^*\left(\frac{k(\rho)}{(\gamma+\alpha k(\rho))}C_N+\rho I_N\right)^{-1}C_N\left(\frac{k(\rho)}{(\gamma+\alpha k(\rho))}C_N+\rho I_N\right)^{-1}1_N}{\left(1_N^*\left(\frac{k(\rho)}{(\gamma+\alpha k(\rho))}C_N+\rho I_N\right)^{-1}1_N\right)^2}$$

Proof. The proof of Theorem 5.2.1 draws on the one hand from the asymptotic properties of $\hat{C}_N(\rho)$ for the centered variables $\tilde{x}_i$ and on the other hand from the work (Rubio et al., 2012) where the same result is derive but for a sample covariance matrix based shrinkage model. The main difficulty here lies precisely in the former aspect which brings some critical differences versus Section 5.1. In particular, Theorem 5.1.1 is not expected to hold any longer for lack of control of the $\tilde{x}_i$ terms. Nonetheless, the difference between bilinear forms or linear statistics of $\hat{C}_N(\rho)$ and $\hat{S}_N(\rho)$ still remain, which are sufficient for the purpose of the present result. We do not provide any further detail on this technical aspect. As for the latter aspect, it presents little difficulty as it only requires to adapt (Rubio et al., 2012) to $\hat{S}_N(\rho)$ instead of the plain Ledoit–Wolf shrinkage estimator. □

[5] Note that, unlike Section 5.1 where the τ_i's had no incidence, they do play a role here due to the structure of $\tilde{x}_i$, which can be shown asymptotically negligible under these assumptions.

Along the same lines as in (Rubio et al., 2012), the deterministic equivalent provided in Theorem 5.2.1 will now be broken down into individual pieces which can be each estimated consistently from the x_i's. This estimator will then further allow for an optimal tuning of the shrinkage parameter ρ for GMVP performance. Let us then provide the various estimators necessary to approximate consistently $\bar{\sigma}^2(\hat{h}_N(\rho))$.

Denoting $\kappa = \int t\nu(dt)$, we start with the following estimators of γ/κ and α/κ, where we use the subscript "sc" for "scaled". The proof of these results follows easily from the elements of Section 5.1 and is omitted.

Lemma 5.2. *As $N, n \to \infty$ with $c_N \to c \in (0, \infty)$,*

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} |\hat{\gamma}_{\rm sc} - \gamma/\kappa| \xrightarrow{\rm a.s.} 0$$
$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} |\hat{\alpha}_{\rm sc} - \alpha/\kappa| \xrightarrow{\rm a.s.} 0,$$

where

$$\hat{k}(\rho) = \frac{1-\rho}{1-(1-\rho)c_N}$$
$$\hat{\gamma}_{\rm sc} = \frac{1}{1-(1-\rho)c_N} \frac{1}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \frac{\tilde{x}_i^* \hat{C}_N^{-1}(\rho) \tilde{x}_i}{\|\tilde{x}_i\|^2}$$
$$\hat{\alpha}_{\rm sc} = \frac{\hat{\gamma}_{\rm sc} \frac{1}{N} \operatorname{tr}\left[I_N - \rho \hat{C}_N^{-1}(\rho)\right]}{\hat{k}(\rho) \left(\frac{n}{N} - \frac{1}{N} \operatorname{tr}\left[I_N - \rho \hat{C}_N^{-1}(\rho)\right]\right)}$$

and $\mathcal{B} \triangleq \{t : \|\tilde{x}_t\|^2 > \xi\}$ with $\xi > 0$ sufficiently small.

From Lemma 5.2 and similar derivations as in (Rubio et al., 2012), we then obtain the following estimator.

Theorem 5.2.2. *As $N, n \to \infty$ with $c_N \to c \in (0, \infty)$,*

$$\sup_{\rho \in \hat{\mathcal{R}}_\varepsilon} \left| \hat{\sigma}^2_{\rm sc}(\hat{h}_N(\rho)) - \frac{1}{\kappa} \sigma^2(\hat{h}_N(\rho)) \right| \xrightarrow{\rm a.s.} 0 \tag{5.32}$$

where

$$\hat{\sigma}^2_{\rm sc}(\hat{h}_N(\rho)) = \frac{(\hat{\gamma}_{\rm sc} + \hat{\alpha}_{\rm sc}\hat{k}(\rho))^2}{\hat{k}(\rho)\hat{\gamma}_{\rm sc}} \frac{1_N^* \hat{C}_N^{-1}(\rho) \left(\hat{C}_N(\rho) - \rho I_N\right) \hat{C}_N^{-1}(\rho) 1_N}{(1_N^* \hat{C}_N^{-1}(\rho) 1_N)^2}.$$

Note now that, since κ is independent of ρ, if ρ minimizes $\sigma^2(\hat{h}_N(\rho))$ then it also minimizes $\sigma^2(\hat{h}_N(\rho))/\kappa$. Thus it follows, with the same arguments as in Section 4.1 that

$$|\sigma^2(\hat{h}_N(\hat{\rho}^\star)) - \sigma^2(\hat{h}_N(\rho^\star))| \xrightarrow{\rm a.s.} 0$$

where $\hat{\rho}^\star$ is the minimizer of $\hat{\sigma}^2_{\rm sc}(\hat{h}_N(\rho))$ and $\rho^\star$ that of $\sigma^2(\hat{h}_N(\rho))$. The problem of obtaining the best asset allocation, as measured by the minimum realized portfolio risk, thus asymptotically reduces to minimizing $\hat{\sigma}^2_{\rm sc}(\hat{h}_N(\rho))$ with regard to ρ, which may be performed by a numerical search.

In summary, given n past return observations of N considered market assets, we propose the following optimized portfolio

$$\hat{h}_N^\star = \frac{\hat{C}_N(\hat{\rho}^\star)^{-1}1_N}{1_N^*\hat{C}_N(\hat{\rho}^\star)^{-1}1_N}.$$

5.2.2 Simulation results

We provide here simulation results both for synthetic and real market data to compare the performances of $\hat{h}_N^\star$ versus competing methods. The latter are composed of: (i) the original Ledoit–Wolf estimator consisting in replacing C_N by $(1-\rho)\frac{1}{n}\sum_{i=1}^n \tilde{x}_i\tilde{x}_i^* + \rho I_N$ in the expression of h_N with ρ taken to minimize the expected Frobenius norm error with C_N, (ii) the Rubio estimator taken from (Rubio et al., 2012) consisting in the same estimator but for ρ taken to minimize the portfolio risk, and (iii) the estimator consisting in replacing C_N by $\hat{C}_N(\rho)$ in the expression of h_N with ρ minimizing the expected Frobenius norm of $\hat{C}_N(\rho) - C_N$, which we refer to as the Abramovich–Pascal estimate.

The synthetic data are i.i.d. multivariate Student-T, i.e., with $\tau_i = d/\chi^2_d$ in distribution with $d = 3$, $N = 200$. We assume the population covariance matrix C_N to be based on a one-factor return structure, see e.g., (DeMiguel et al., 2009), $C_N = bb^*\sigma^2 + \Sigma$, where $\sigma = 0.16$, $b \in \mathbb{R}^N$ with uniform random entries in $[0.5, 1.5]$, and Σ diagonal with uniform independent entries supported in $[0.1, 0.3]$. The results are provided in Figure 5.6 which illustrates the performance gain achieved by our proposed estimator.

For real world data, we consider the stocks conforming the HSI. Precisely, we use the dividend-adjusted daily closing prices downloaded from the Yahoo Finance database to obtain the continuously compounded (logarithmic) returns for the 45 constituents of the HSI over $L = 736$ working days from January 3rd, 2011 to December 31st, 2013 (excluding the weekends and public holidays). As conventionally done in the financial literature, the out-of-sample evaluation is defined in terms of a rolling window method. At a particular day t, we use the previous n days (i.e., $t - n$ to $t - 1$) as the training window for covariance estimation and construct and compare the performance of the portfolio selection $\hat{h}_N^\star$ against the aforementioned competing approaches. We then evaluate the various portfolio returns achieved in the following 20 days. Next the window is shifted 20 days forward and the portfolio returns for another 20 days are computed. This procedure is repeated until exhausting the data. The realized risk is computed conventionally as the annualized sample standard deviation of the corresponding GMVP returns. In our tests, different training window lengths are considered. Figure 5.7 provides the results. Again, it is observed that the proposed $\hat{h}_N^\star$ achieves the smallest realized risk, seemingly uniformly so over all window sizes. As opposed to synthetic data though, observe that for long windows, the performance degrades presumably due to a loss of stationarity in the long run.

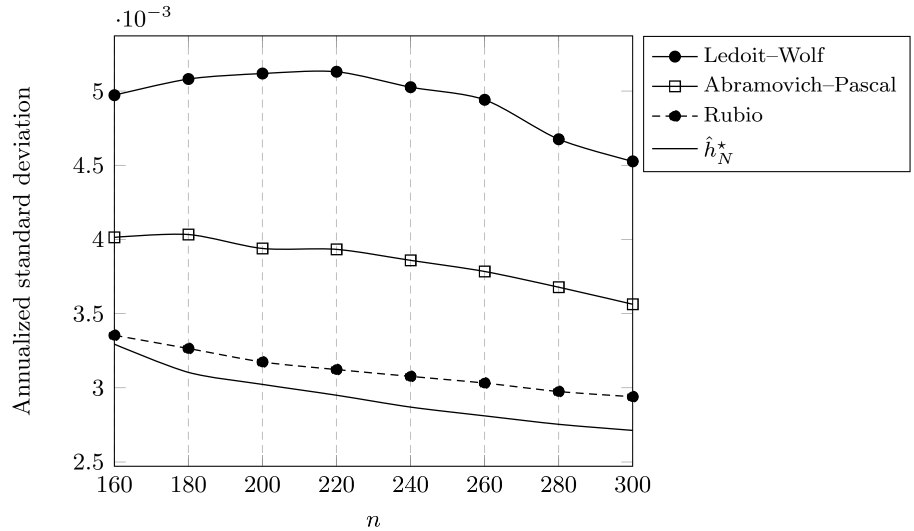


Figure 5.6: Average realized portfolio risk of different covariance estimators in the GMVP framework using synthetic data.

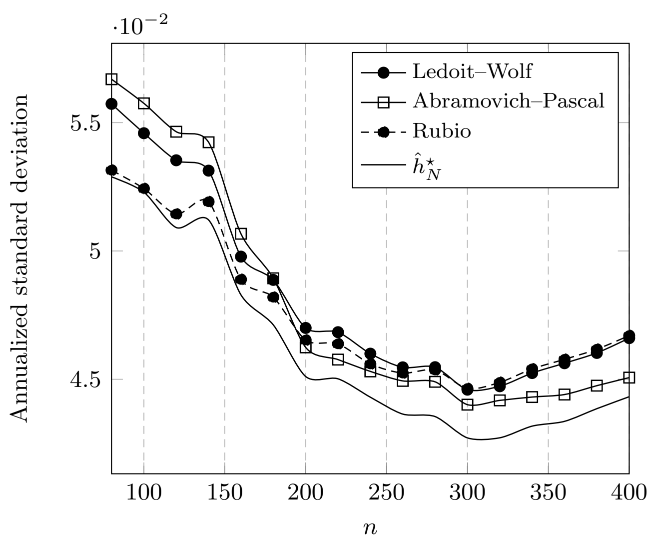


Figure 5.7: Realized portfolio risks achieved out-of-sample over 736 days of HSI real market data (from Jan. 3rd, 2011 to Dec. 31st, 2013) by a GMVP implemented using different covariance estimators.

Chapter 6

Second-order statistics

We have seen that robust estimators of scatter $\hat{C}_N$, be they Maronna's or robust shrinkage estimators, can be straightforwardly substituted by tractable random matrices, that we generically denote $\hat{S}_N$, to derive new consistent robust estimators of functionals of the population scatter or covariance matrix. This enfolds from the sufficiently strong convergence in spectral norm $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ along with identities relating $\hat{S}_N$ to the sought for functional.

Nonetheless, if the replacement of $\hat{C}_N$ by $\hat{S}_N$ helps in deriving consistent estimates, the convergence $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ is in general not sufficient to assess the performance of the estimator for large but finite N, n. Indeed, when second order results such as central limit theorems need be established, say at rate $N^{-\frac{1}{2}}$, to proceed similarly to the replacement of $\hat{C}_N$ by $\hat{S}_N$ in the analysis, one would ideally demand that $\|\hat{C}_N - \hat{S}_N\| = o(N^{-\frac{1}{2}})$; but such a result, we believe, unfortunately does not hold. This constitutes a severe limitation in the exploitation of robust estimators as their performance as well as optimal fine-tuning often rely on second order performance. Concretely, for the robust GMUSIC algorithm derived in Section 4.2, one may naturally ask which choice of the u function is optimal to minimize the variance of (consistent) power and angle estimates. This question remains unanswered to this point for lack of better theoretical results.

The main purpose of this chapter is twofold. From a technical aspect, taking the robust shrinkage estimator $\hat{C}_N(\rho)$ studied in Chapter 5 as an example, we first show that, although the convergence $\|\hat{C}_N(\rho) - \hat{S}_N(\rho)\| \xrightarrow{\text{a.s.}} 0$ (from Theorem 5.1.1) may not be extensible to a rate $O(N^{1-\varepsilon})$, one has the bilinear form convergence $N^{1-\varepsilon} a^*(\hat{C}_N^k(\rho) - \hat{S}_N^k(\rho))b \xrightarrow{\text{a.s.}} 0$ for each $\varepsilon > 0$, each $a, b \in \mathbb{C}^N$ of unit norm, and each $k \in \mathbb{Z}$. This result implies that, if $\sqrt{N} a^* \hat{S}_N^k(\rho) b$ satisfies a central limit theorem, then so does $\sqrt{N} a^* \hat{C}_N^k(\rho) b$ with the same limiting variance. This result is of fundamental importance to any statistical application based on such quadratic forms. Our second contribution is to exploit this result for the specific problem of signal detection in impulsive noise environments via the generalized likelihood-ratio test, particularly suited for radar signals detection under elliptical noise (Conte et al., 1995; Pascal et al., 2013). In this context, we determine the shrinkage parameter ρ which minimizes the probability of false detections and provide an empirical consistent estimate for this parameter, thus improving significantly over traditional sample covariance matrix-based estimators.

6.1 CLT for quadratic forms

We start with the theoretical part of the work. We first recall the notations and assumptions considered in this chapter.

Let $N, n \in \mathbb{N}$, $c_N \triangleq N/n$, and $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1]$. Let also $x_1, \ldots, x_n \in \mathbb{C}^N$ be n independent random vectors defined by the following assumptions.

Assumption 6.1 (Data vectors)**.** *For $i \in \{1, \ldots, n\}$, $x_i = \sqrt{\tau_i} A_N w_i = \sqrt{\tau_i} z_i$, where*

- *$w_i \in \mathbb{C}^N$ is Gaussian with zero mean and covariance I_N, independent across i;*
- *$A_N A_N^* \triangleq C_N \in \mathbb{C}^{N \times N}$ is such that $\nu_N \triangleq \frac{1}{N} \sum_{i=1}^N \boldsymbol{\delta}_{\lambda_i(C_N)} \to \nu$ weakly, $\limsup_N \|C_N\| < \infty$, and $\frac{1}{N} \operatorname{tr} C_N = 1$;*
- *$\tau_i > 0$ are random or deterministic scalars.*

Under Assumption 6.1, letting $\tau_i = \tilde{\tau}_i / \|w_i\|$ for some $\tilde{\tau}_i$ independent of w_i, x_i belongs to the class of elliptically distributed random vectors. Note that the normalization $\frac{1}{N} \operatorname{tr} C_N = 1$ is not a restricting constraint since the scalars τ_i may absorb any other normalization.

In this section, we shall consider the Abramovich–Pascal robust shrinkage estimator of scatter $\hat{C}_N(\rho)$, that we recall is defined as the unique solution to

$$\hat{C}_N(\rho) = (1-\rho)\frac{1}{n}\sum_{i=1}^n \frac{x_i x_i^*}{\frac{1}{N} x_i \hat{C}_N^{-1}(\rho) x_i} + \rho I_N.$$

Remember from the previous chapter that, for any $\kappa > 0$ small, defining $\mathcal{R}_\kappa \triangleq [\kappa + \max\{0, 1 - c^{-1}\}, 1]$, as $N, n \to \infty$ with $c_N = N/n \to c \in (0, \infty)$,

$$\sup_{\rho \in \mathcal{R}_\kappa} \left\| \hat{C}_N(\rho) - \hat{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N(\rho) = \frac{1}{\gamma_N(\rho)} \frac{1-\rho}{1-(1-\rho)c_N} \frac{1}{n} \sum_{i=1}^n z_i z_i^* + \rho I_N$$

with $\gamma_N(\rho)$ the unique solution to

$$1 = \int \frac{t}{\gamma_N(\rho)\rho + (1-\rho)t} \nu_N(dt).$$

A careful analysis of the proof of Theorem 5.1.1 (which is performed in Section 6.1.1) reveals that the above convergence can be refined as

$$\sup_{\rho \in \mathcal{R}_\kappa} N^{\frac{1}{2}-\varepsilon} \left\| \hat{C}_N(\rho) - \hat{S}_N(\rho) \right\| \xrightarrow{\text{a.s.}} 0 \tag{6.1}$$

for each $\varepsilon > 0$. This suggests that (well-behaved) functionals of $\hat{C}_N(\rho)$ fluctuating at a slower speed than $N^{-\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$ follow the same statistics as the same functionals with $\hat{S}_N(\rho)$ in place of $\hat{C}_N(\rho)$. However, this result is quite weak as most limiting theorems (starting with the classical central limit theorems for independent scalar variables) deal with fluctuations of order $N^{-\frac{1}{2}}$ and sometimes in random matrix theory of order N^{-1}. In our opinion, the convergence speed (6.1) cannot be improved to a rate $N^{-\frac{1}{2}}$. Nonetheless, thanks to an averaging effect documented in Section 6.1.1, the fluctuation of special forms of functionals of $\hat{C}_N(\rho)$ can be proved to be much slower. Although among these functionals we could have considered linear functionals of the eigenvalue distribution of $\hat{C}_N(\rho)$, our present concern (driven by more obvious applications) is rather on bilinear forms of the type $a^*\hat{C}_N^k(\rho)b$ for some $a, b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$, $k \in \mathbb{Z}$.

Our main result is the following.

Theorem 6.1.1 (Fluctuation of bilinear forms)**.** *Let $a, b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$. Then, as $N, n \to \infty$ with $c_N \to c \in (0, \infty)$, for any $\varepsilon > 0$ and every $k \in \mathbb{Z}$,*

$$\sup_{\rho \in \mathcal{R}_\kappa} N^{1-\varepsilon} \left| a^*\hat{C}_N^k(\rho)b - a^*\hat{S}_N^k(\rho)b \right| \xrightarrow{\text{a.s.}} 0.$$

Some comments and remarks are in order. First, we recall that central limit theorems involving bilinear forms of the type $a^*\hat{S}_N^k(\rho)b$ are classical objects in random matrix theory (see e.g. (Kammoun et al., 2009; Mestre, 2008a) for $k = -1$), particularly common in signal processing and wireless communications. These central limit theorems in general show fluctuations at speed $N^{-\frac{1}{2}}$. This indicates, taking $\varepsilon < \frac{1}{2}$ in Theorem 6.1.1 and using the fact that almost sure convergence implies weak convergence, that $a^*\hat{C}_N^k(\rho)b$ exhibits the same fluctuations as $a^*\hat{S}_N^k(\rho)b$, the latter being classical and tractable while the former is quite intricate at the onset, due to the implicit nature of $\hat{C}_N(\rho)$.

Of practical interest to many applications in signal processing is the case where $k = -1$. In Section 6.2, we present a classical generalized maximum likelihood signal detection in impulsive noise, for which we shall characterize the shrinkage parameter ρ that meets minimum false alarm rates. Meanwhile, the next section provides the proof of Theorem 6.1.1.

6.1.1 Proof

In this section, we prove Theorem 6.1.1, which is in particular based on the important Lemma 6.2, the proof of which is deferred to Section 6.1.1.2.

Before delving into the core of the proofs, let us introduce a few notations that shall be used throughout the section. Specifically, for each $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1]$, we define

$$\alpha(\rho) = \frac{1-\rho}{1-(1-\rho)c_N}$$

$$d_i(\rho) = \frac{1}{N} z_i^* \hat{C}_{(i)}^{-1}(\rho) z_i = \frac{1}{N} z_i^* \left(\alpha(\rho) \frac{1}{n} \sum_{j \neq i} \frac{z_j z_j^*}{d_j(\rho)} + \rho I_N \right)^{-1} z_i$$

$$\tilde{d}_i(\rho) = \frac{1}{N} z_i^* \hat{S}_{(i)}^{-1}(\rho) z_i = \frac{1}{N} z_i^* \left(\alpha(\rho) \frac{1}{n} \sum_{j \neq i}^{n} \frac{z_j z_j^*}{\gamma_N(\rho)} + \rho I_N \right)^{-1} z_i$$

with $\hat{C}_{(i)}(\rho) = \hat{C}_N(\rho) - (1-\rho)\frac{1}{n}\frac{z_i z_i^*}{\frac{1}{N} z_i^* \hat{C}_N^{-1}(\rho) z_i}$. Recall from Section 5.1 that $d_1(\rho), \ldots, d_n(\rho)$ are well defined as the unique solution of their n defining equations. We shall also discard the parameter ρ for readability whenever not needed.

6.1.1.1 Main proof

As shall become clear, the proof unfolds similarly for each $k \in \mathbb{Z} \setminus \{0\}$ and we can therefore restrict ourselves to a single value for k. As Theorem 6.2.1 in Section 6.2 relies on $k = -1$, for consistency, we take $k = -1$ from now on. Thus, our objective is to prove that, for $a, b \in \mathbb{C}^N$ with $\|a\| = \|b\| = 1$, and for any $\varepsilon > 0$,

$$\sup_{\rho \in \mathcal{R}_\kappa} N^{1-\varepsilon} \left| a^* \hat{C}_N^{-1}(\rho) b - a^* \hat{S}_N^{-1}(\rho) b \right| \xrightarrow{\text{a.s.}} 0.$$

For this, forgetting for some time the index ρ, first write

$$a^* \hat{C}_N^{-1} b - a^* \hat{S}_N^{-1} b = a^* \hat{C}_N^{-1} \left(\frac{\alpha}{n} \sum_{i=1}^{n} \left[\frac{1}{\gamma_N} - \frac{1}{d_i} \right] z_i z_i^* \right) \hat{S}_N^{-1} b \tag{6.2}$$

$$= \frac{\alpha}{n} \sum_{i=1}^{n} a^* \hat{C}_N^{-1} z_i \frac{d_i - \gamma_N}{\gamma_N d_i} z_i^* \hat{S}_N^{-1} b. \tag{6.3}$$

In Chapter 5, where we showed that $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ (that is the spectral norm of the inner parenthesis in (6.2) vanishes), the core of the proof was to show that $\max_{1 \leq i \leq n} |d_i - \gamma_N| \xrightarrow{\text{a.s.}} 0$ which, along with the convergence of γ_N away from zero and the almost sure boundedness of $\|\frac{1}{n}\sum_{i=1}^n z_i z_i^*\|$ for all large N (from e.g. (Bai and Silverstein, 1998)), gives the result. A thorough inspection of the proof in Theorem 5.1.1 reveals that $\max_{1 \leq i \leq n} |d_i - \gamma_N| \xrightarrow{\text{a.s.}} 0$ may be improved into $\max_{1 \leq i \leq n} N^{\frac{1}{2}-\varepsilon} |d_i - \gamma_N| \xrightarrow{\text{a.s.}} 0$ for any $\varepsilon > 0$ but that this speed cannot be further improved beyond $N^{\frac{1}{2}}$. The latter statement is rather intuitive since γ_N is essentially a sharp deterministic approximation for $\frac{1}{N} \operatorname{tr} \hat{C}_N^{-1}$ while d_i is a quadratic form on $\hat{C}_{(i)}^{-1}$; classical random matrix results involving fluctuations of such quadratic forms, see e.g. (Kammoun et al., 2009), indeed show that these fluctuations are of order $N^{-\frac{1}{2}}$. As a consequence, $\max_{1 \leq i \leq n} N^{1-\varepsilon} |d_i - \gamma_N|$ and thus $N^{1-\varepsilon} \|\hat{C}_N - \hat{S}_N\|$ are not expected to vanish for small ε.

This being said, when it comes to bilinear forms, for which we shall naturally have $N^{\frac{1}{2}-\varepsilon}|a^*\hat{C}_N^{-1}b - a^*\hat{S}_N^{-1}b| \xrightarrow{\text{a.s.}} 0$, seeing the difference in absolute values as the n-term average (6.3), one may expect that the fluctuations of $d_i - \gamma_N$ are sufficiently loosely dependent across i to further increase the speed of convergence from $N^{\frac{1}{2}-\varepsilon}$ to $N^{1-\varepsilon}$ (which is the best one could expect from a law of large numbers aspect if the $d_i - \gamma_N$ were truly independent). It turns out that this intuition is correct.

Nonetheless, to proceed with the proof, it shall be quite involved to work directly with (6.3) which involves the rather intractable terms d_i (as the random solutions to an implicit equation). As in Chapter 5, our approach will consist in first approximating d_i by a much more tractable quantity. Letting γ_N be this approximation is however not good enough this time since $\gamma_N - d_i$ is a non-obvious quantity of amplitude $O(N^{-\frac{1}{2}})$ which, due to intractability, we shall not be able to average across i into a $O(N^{-1})$ quantity. Thus, we need a refined approximation of d_i which we shall take to be $\tilde{d}_i$ defined above. Intuitively, since $\tilde{d}_i$ is also a quadratic form closely related to d_i, we expect $d_i - \tilde{d}_i$ to be of order $O(N^{-1})$, which we shall indeed observe. With this approximation in place, d_i can be replaced by $\tilde{d}_i$ in (6.3), which now becomes a more tractable random variable (as it involves no implicit equation) that fluctuates around γ_N at the expected $O(N^{-1})$ speed.

Let us then introduce the variable $\tilde{d}_i$ in (6.2) to obtain

$$\begin{aligned}
a^*\hat{C}_N^{-1}b - a^*\hat{S}_N^{-1}b &= a^*\hat{C}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\gamma_N} - \frac{1}{\tilde{d}_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b \\
&\quad + a^*\hat{C}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\tilde{d}_i} - \frac{1}{d_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b \\
&\triangleq \xi_1 + \xi_2.
\end{aligned}$$

We will now show that $\xi_1 = \xi_1(\rho)$ and $\xi_2 = \xi_2(\rho)$ vanish at the appropriate speed and uniformly so on $\mathcal{R}_\kappa$.

Let us first progress in the derivation of $\xi_1(\rho)$ from which we wish to discard the explicit dependence on $\hat{C}_N$. We have

$$\begin{aligned}
\xi_1 &= a^*\hat{C}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\gamma_N} - \frac{1}{\tilde{d}_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b \\
&= a^*\hat{S}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\gamma_N} - \frac{1}{\tilde{d}_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b + a^*(\hat{C}_N^{-1} - \hat{S}_N^{-1})\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\gamma_N} - \frac{1}{\tilde{d}_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b \\
&= a^*\hat{S}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\frac{\tilde{d}_i - \gamma_N}{\gamma_N^2}z_iz_i^*\right)\hat{S}_N^{-1}b - a^*\hat{S}_N^{-1}\left(\frac{\alpha}{n}\sum_{i=1}^n\frac{(\tilde{d}_i - \gamma_N)^2}{\gamma_N^2\tilde{d}_i}z_iz_i^*\right)\hat{S}_N^{-1}b \\
&\quad + a^*(\hat{C}_N^{-1} - \hat{S}_N^{-1})\left(\frac{\alpha}{n}\sum_{i=1}^n\left[\frac{1}{\gamma_N} - \frac{1}{\tilde{d}_i}\right]z_iz_i^*\right)\hat{S}_N^{-1}b \\
&\triangleq \xi_{11} + \xi_{12} + \xi_{13}.
\end{aligned}$$

The terms ξ_{12} and ξ_{13} exhibit products of two terms that are expected to be of order $O(N^{-\frac{1}{2}})$ and which are thus easily handled. As for ξ_{11}, it no longer depends on $\hat{C}_N$ and is therefore a standard random variable which, although involved, is technically tractable via standard random matrix methods. In order to show that $N^{1-\varepsilon}\max\{|\xi_{12}|,|\xi_{13}|\} \xrightarrow{\text{a.s.}} 0$ uniformly in ρ, we use the following lemma.

Lemma 6.1. *For any $\varepsilon > 0$,*

$$\max_{1\leq i\leq n}\sup_{\rho\in\mathcal{R}_\kappa} N^{\frac{1}{2}-\varepsilon}|\tilde{d}_i(\rho)-\gamma_N(\rho)| \xrightarrow{\text{a.s.}} 0$$

$$\max_{1\leq i\leq n}\sup_{\rho\in\mathcal{R}_\kappa} N^{\frac{1}{2}-\varepsilon}|d_i(\rho)-\gamma_N(\rho)| \xrightarrow{\text{a.s.}} 0.$$

Note that, while the first result is a standard, easily established, random matrix result, the second result is the aforementioned refinement of the core result in the proof of Theorem 5.1.1.

Proof of Lemma 6.1. We start by proving the first identity. From (5.5) in Section 5.1 (taking $w = -\gamma_N\rho\alpha^{-1}$), we have, for each $p \geq 2$ and for each $1 \leq k \leq n$,

$$\mathrm{E}\left[\left|\tilde{d}_k(\rho)-\gamma_N(\rho)\right|^p\right] = O(N^{-\frac{p}{2}})$$

where the bound does not depend on $\rho > \max\{0, 1-1/c\}+\kappa$. Let now $\max\{0,1-1/c\}+\kappa = \rho_0 < \ldots < \rho_{\sqrt{n}} = 1$ be a regular sampling of $\mathcal{R}_\kappa$ in $\sqrt{n}$ intervals. We then have, from Markov inequality and the union bound on $n(\sqrt{n}+1)$ events

$$P\left(\max_{1\leq k\leq n, 0\leq i\leq\sqrt{n}}\left|N^{\frac{1}{2}-\varepsilon}(\tilde{d}_k(\rho_i)-\gamma_N(\rho_i))\right|\right) \leq KN^{-p\varepsilon+\frac{3}{2}}$$

for some $K > 0$ only dependent on p. From the Borel Cantelli lemma, we then have $\max_{k,i}|N^{\frac{1}{2}-\varepsilon}(\tilde{d}_k(\rho_i)-\gamma_N(\rho_i))| \xrightarrow{\text{a.s.}} 0$ as long as $-p\varepsilon+3/2 < -1$, which is obtained for $p > 5/(2\varepsilon)$. Using $|\gamma_N(\rho)-\gamma_N(\rho')| \leq K|\rho-\rho'|$ for some constant K and each $\rho,\rho'\in\mathcal{R}_\kappa$ (see the proof of Theorem 5.1.1) and similarly $\max_{1\leq k\leq n}|\tilde{d}_k(\rho)-\tilde{d}_k(\rho')| \leq K|\rho-\rho'|$ for all large n a.s. (obtained by explicitly writing the difference and using the fact that $\|z_k\|^2/N$ is asymptotically bounded almost surely), we get

$$\begin{aligned}\max_{1\leq k\leq n}\sup_{\rho\in\mathcal{R}_\kappa} N^{\frac{1}{2}-\varepsilon}|\tilde{d}_k(\rho)-\gamma_N(\rho)| &\leq \max_{k,i} N^{\frac{1}{2}-\varepsilon}|\tilde{d}_k(\rho_i)-\gamma_N(\rho_i)| + KN^{-\varepsilon}\\ &\xrightarrow{\text{a.s.}} 0.\end{aligned}$$

The second result relies on revisiting the proof of Theorem 5.1.1 incorporating the convergence speed on $\tilde{d}_k - \gamma_N$. For convenience and compatibility with similar derivations that appear later in the proof, we slightly modify the original proof of Theorem 5.1.1 presented in Section 5.1. We first define $f_i(\rho) = d_i(\rho)/\gamma_N(\rho)$ and relabel the $d_i(\rho)$ in such a way that $f_1(\rho) \leq \ldots \leq f_n(\rho)$

(the ordering may then depend on ρ). Then, we have by definition of $d_n(\rho) = \gamma_N(\rho) f_n(\rho)$

$$\gamma_N(\rho) f_n(\rho) = \frac{1}{N} z_n^* \left(\alpha(\rho) \frac{1}{n} \sum_{i<n} \frac{z_i z_i^*}{\gamma_N(\rho) f_i(\rho)} + \rho I_N \right)^{-1} z_n$$
$$\leq \frac{1}{N} z_n^* \left(\alpha(\rho) \frac{1}{f_n(\rho)} \frac{1}{n} \sum_{i<n} \frac{z_i z_i^*}{\gamma_N(\rho)} + \rho I_N \right)^{-1} z_n$$

where we used $f_n(\rho) \geq f_i(\rho)$ for each i. The above is now equivalent to

$$\gamma_N(\rho) \leq \frac{1}{N} z_n^* \left(\alpha(\rho) \frac{1}{n} \sum_{i<n} \frac{z_i z_i^*}{\gamma_N(\rho)} + f_n(\rho) \rho I_N \right)^{-1} z_n.$$

We now make the assumption that there exists $\eta > 0$ and a sequence $\{\rho^{(n)}\} \in \mathcal{R}_\kappa$ such that $f_n(\rho^{(n)}) > 1 + N^{\eta - \frac{1}{2}}$ infinitely often, which is equivalent to saying $d_n(\rho^{(n)}) > \gamma_N(\rho^{(n)})(1 + N^{\eta - \frac{1}{2}})$ infinitely often (i.o.). Then, from these assumptions and the above first convergence result

$$\gamma_N(\rho^{(n)}) \leq \frac{1}{N} z_n^* \left(\alpha(\rho^{(n)}) \frac{1}{n} \sum_{i<n} \frac{z_i z_i^*}{\gamma_N(\rho^{(n)})} + \rho^{(n)}(1 + N^{\eta - \frac{1}{2}}) I_N \right)^{-1} z_n$$
$$= \tilde{d}_n(\rho^{(n)}) - N^{\eta - \frac{1}{2}} \frac{1}{N} z_n^* \left(\frac{1}{n} \sum_{i<n} \frac{\alpha(\rho^{(n)}) z_i z_i^*}{\rho^{(n)} \gamma_N(\rho^{(n)})} + (1 + N^{\eta - \frac{1}{2}}) I_N \right)^{-1}$$
$$\times \left(\frac{1}{n} \sum_{i<n} \frac{\alpha(\rho^{(n)}) z_i z_i^*}{\gamma_N(\rho^{(n)})} + \rho^{(n)} I_N \right)^{-1} z_n. \tag{6.4}$$

Now, by the first result of the lemma, letting $0 < \varepsilon < \eta$, we have

$$\left| \tilde{d}_n(\rho^{(n)}) - \gamma_N(\rho^{(n)}) \right| \leq \max_{\rho \in \mathcal{R}_\kappa} \left| \tilde{d}_n(\rho) - \gamma_N(\rho) \right| \leq N^{\varepsilon - \frac{1}{2}}$$

for all large n a.s., so that, for these large n, $\tilde{d}_n(\rho^{(n)}) \leq \gamma_N(\rho^{(n)}) + N^{\varepsilon - \frac{1}{2}}$. Applying this inequality to the first right-end side term of (6.4) and using the almost sure boundedness of the rightmost right-end side term entails

$$0 \leq N^{\varepsilon - \frac{1}{2}} - K N^{\eta - \frac{1}{2}}$$

for some $K > 0$ for all large n a.s. But, $N^{\varepsilon/2 - 1/2} - K N^{\eta/2 - 1/2} < 0$ for all large N, which contradicts the inequality. Thus, our initial assumption is wrong and therefore, for each $\eta > 0$, we have for all large n a.s., $d_n(\rho) < \gamma_N(\rho) + N^{\eta - \frac{1}{2}}$ uniformly on $\rho \in \mathcal{R}_\kappa$. The same calculus can be performed for $d_1(\rho)$ by assuming that $f_1(\rho'^{(n)}) < 1 - N^{\eta - \frac{1}{2}}$ i.o. over some sequence $\rho'^{(n)}$; by reverting all inequalities in the derivation above, we similarly conclude by contradiction that $d_1(\rho) > \gamma_N(\rho) - N^{\eta - \frac{1}{2}}$ for all large n, uniformly so in $\mathcal{R}_\kappa$. Together, both results finally lead, for each $\varepsilon > 0$, to

$$\max_{1 \leq k \leq n} \sup_{\rho \in \mathcal{R}_\kappa} \left| N^{\frac{1}{2} - \varepsilon} \left(d_k(\rho) - \gamma_N(\rho) \right) \right| \xrightarrow{\text{a.s.}} 0$$

obtained by fixing ε, taking η such that $0 < \eta < \varepsilon$, and using $\max_k \sup_\rho |d_k(\rho) - \gamma_N(\rho)| < N^{\eta - \frac{1}{2}}$ for all large n a.s. $\square$

Thanks to Lemma 6.1, expressing $\hat{C}_N^{-1}(\rho) - \hat{S}_N^{-1}(\rho)$ as a function of $d_i(\rho) - \gamma_N(\rho)$ and using the (almost sure) boundedness of the various terms involved, we finally get $N^{1-\varepsilon}\xi_{12} \xrightarrow{\text{a.s.}} 0$ and $N^{1-\varepsilon}\xi_{13} \xrightarrow{\text{a.s.}} 0$ uniformly on ρ.

It then remains to handle the more delicate term ξ_{11}, which can be further expressed as

$$\begin{aligned}\xi_{11} &= \frac{\alpha}{\gamma_N^2} a^* \hat{S}_N^{-1} \left(\frac{1}{n} \sum_{i=1}^n (\tilde{d}_i - \gamma_N) z_i z_i^* \right) \hat{S}_N^{-1} b \\ &= \frac{\alpha}{\gamma_N^2} \frac{1}{n} \sum_{i=1}^n a^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} b \left(\tilde{d}_i - \gamma_N \right).\end{aligned}$$

For that, we will resort to the following lemma, whose proof is postponed to Section 6.1.1.2.

Lemma 6.2. *Let c and d be random or deterministic vectors, independent of $z_1, \cdots, z_n$, such that* $\max\left(\mathrm{E}[\|c\|^k], \mathrm{E}[\|d\|^k]\right) \leq K$ *for some $K > 0$ and all integer k. Then, for each integer p,*

$$\mathrm{E}\left[\left| \frac{1}{n} \sum_{i=1}^n c^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} d \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \gamma_N(\rho) \right) \right|^{2p} \right] = O\left(N^{-2p}\right)$$

By the Markov inequality and the union bound, similar to the proof of Lemma 6.1, we get from Lemma 6.2 (with $a = c$ and $d = b$) that, for each $\eta > 0$ and for each integer $p \geq 1$,

$$P\left(\sup_{\rho \in \{\rho_0 < \ldots < \rho_{\sqrt{n}}\}} N^{1-\varepsilon} |\xi_{11}| > \eta \right) \leq K N^{-p\varepsilon + \frac{1}{2}}$$

with K only function of η and $\rho_0 < \ldots < \rho_{\sqrt{n}}$ a regular sampling of $\mathcal{R}_\kappa$. Taking $p > 3/(2\varepsilon)$, we finally get from the Borel Cantelli lemma that

$$N^{1-\varepsilon}\xi_{11} \xrightarrow{\text{a.s.}} 0$$

uniformly on $\{\rho_0, \ldots, \rho_{\sqrt{n}}\}$ and finally, using Lipschitz arguments as in the proof of Lemma 6.1, uniformly on $\mathcal{R}_\kappa$. Putting all results together, we finally have

$$\sup_{\rho \in \mathcal{R}_\kappa} N^{1-\varepsilon} |\xi_1(\rho)| \xrightarrow{\text{a.s.}} 0$$

which concludes the first part of the proof.

We now continue with $\xi_2(\rho)$. In order to prove $N^{1-\varepsilon}\xi_2(\rho) \xrightarrow{\text{a.s.}} 0$ uniformly on $\rho \in \mathcal{R}_\kappa$, it is sufficient (thanks to the boundedness of the various terms involved) to prove that

$$\max_{1 \leq i \leq n} \sup_{\rho \in \mathcal{R}_\kappa} \left| N^{1-\varepsilon} \left(\tilde{d}_i(\rho) - d_i(\rho) \right) \right| \xrightarrow{\text{a.s.}} 0.$$

To obtain this result, we first need the following fundamental proposition.

Proposition 6.1.1. *For any $\varepsilon > 0$,*

$$\max_{1\le k\le n}\sup_{\rho\in\mathcal{R}_\kappa}\left|N^{1-\varepsilon}\left(\tilde{d}_k(\rho)-\frac{1}{N}z_k^*\left(\alpha(\rho)\frac{1}{n}\sum_{i\neq k}\frac{z_iz_i^*}{\tilde{d}_i(\rho)}+\rho I_N\right)^{-1}z_k\right)\right|\xrightarrow{\text{a.s.}}0.$$

Proof. By expanding the definition of $\tilde{d}_k$, first observe that

$$\begin{aligned}&\tilde{d}_k-\frac{1}{N}z_k^*\left(\alpha\frac{1}{n}\sum_{i\neq k}\frac{z_iz_i^*}{\tilde{d}_i}+\rho I_N\right)^{-1}z_k\\&=\alpha\frac{1}{n}\sum_{i\neq k}\frac{1}{N}z_k^*\hat{S}_{(k)}^{-1}z_iz_i^*\frac{\gamma_N-\tilde{d}_i}{\gamma_N\tilde{d}_i}\left(\alpha\frac{1}{n}\sum_{j\neq k}\frac{z_jz_j^*}{\tilde{d}_j}+\rho I_N\right)^{-1}z_k.\end{aligned}$$

Similar to the derivation of ξ_1, we now proceed to approximating $\tilde{d}_i$ in the central denominator and each $\tilde{d}_j$ in the rightmost inverse matrix by the non-random γ_N. We obtain (from Lemma 6.1)

$$\begin{aligned}&\tilde{d}_k-\frac{1}{N}z_k^*\left(\alpha\frac{1}{n}\sum_{i\neq k}\frac{z_iz_i^*}{\tilde{d}_i}+\rho I_N\right)^{-1}z_k\\&=\frac{\alpha}{\gamma_N^2}\frac{1}{n}\sum_{i\neq k}\frac{1}{N}z_k^*\hat{S}_{(k)}^{-1}z_iz_i^*(\gamma_N-\tilde{d}_i)\hat{S}_{(k)}^{-1}z_k+o(N^{\varepsilon-1})\end{aligned}$$

almost surely, for $\varepsilon>0$ and uniformly so on ρ. The objective is then to show that the first right-hand side term is $o(N^{\varepsilon-1})$ almost surely and that this holds uniformly on k and ρ. This is achieved by applying Lemma 6.2 with $c=d=z_k$. Indeed, Lemma 6.2 ensures that, for each integer p,[1]

$$\mathrm{E}\left[\left|\frac{1}{n}\sum_{i\neq k}\frac{1}{N}z_k^*S_{(k)}^{-1}(\rho)z_iz_i^*S_{(k)}^{-1}(\rho)z_k\left(\frac{1}{N}z_i^*S_{(i,k)}^{-1}(\rho)z_i-\gamma_N(\rho)\right)\right|^p\right]=O(N^{-p})$$

From this lemma, applying Markov's inequality, we have for each k,

$$P\left(N^{1-\varepsilon}\left|\frac{1}{n}\sum_{i\neq k}\frac{1}{N}z_k^*\hat{S}_{(k)}^{-1}z_iz_i^*\hat{S}_{(k)}^{-1}z_k\left(\frac{1}{N}z_i^*\hat{S}_{(i,k)}^{-1}z_i-\gamma_N\right)\right|>\eta\right)\le KN^{-p\varepsilon}$$

for some $K>0$ only dependent on $\eta>0$. Applying the union bound on the $n(n+1)$ events for $k=1,\ldots,n$ and for $\rho\in\{\rho_0,\ldots,\rho_n\}$, regular n-discretization of $\mathcal{R}_\kappa$, we then have

$$\begin{aligned}&P\left(\max_{k,j}N^{1-\varepsilon}\left|\frac{1}{n}\sum_{i\neq k}\frac{1}{N}z_k^*\hat{S}_{(k)}^{-1}z_iz_i^*\hat{S}_{(k)}^{-1}z_k\left(\frac{1}{N}z_i^*\hat{S}_{(i,k)}^{-1}z_i-\gamma_N(\rho_j)\right)\right|>\eta\right)\\&\le KN^{-p\varepsilon+2}.\end{aligned}$$

[1] Note that Lemma 6.2 can strictly be applied here for $n-1$ instead of n; but since $1/n-1/(n-1)=O(n^{-2})$, this does not affect the result.

Taking $p > 3/\varepsilon$, by the Borel Cantelli lemma the above convergence holds almost surely, we finally get

$$\max_{k,j}\left| N^{1-\varepsilon}\left(\tilde{d}_k(\rho_j) - \frac{1}{N}z_k^*\left(\alpha(\rho_j)\frac{1}{n}\sum_{i\neq k}\frac{z_iz_i^*}{\tilde{d}_i(\rho_j)} + \rho_j I_N\right)^{-1} z_k\right)\right| \xrightarrow{\text{a.s.}} 0.$$

Using the ρ-Lipschitz property (which holds almost surely so for all large n a.s.) on both terms in the above difference concludes the proof of the proposition. □

The crux of the proof for the convergence of ξ_2 starts now. In a similar manner as in the proof of Lemma 6.1, we define $\tilde{f}_i(\rho) = d_i(\rho)/\tilde{d}_i(\rho)$ and reorder the indexes in such a way that $\tilde{f}_1(\rho) \leq \ldots \leq \tilde{f}_n(\rho)$ (this ordering depending on ρ). Then, by definition of $d_n(\rho) = \tilde{f}_i(\rho)\tilde{d}_i(\rho)$,

$$\begin{aligned}\tilde{d}_n(\rho)\tilde{f}_n(\rho) &= \frac{1}{N}z_n^*\left(\alpha(\rho)\frac{1}{n}\sum_{i<n}\frac{z_iz_i^*}{\tilde{d}_i(\rho)\tilde{f}_i(\rho)} + \rho I_n\right)^{-1} z_n \\ &\leq \frac{1}{N}z_n^*\left(\alpha(\rho)\frac{1}{\tilde{f}_n(\rho)}\frac{1}{n}\sum_{i<n}\frac{z_iz_i^*}{\tilde{d}_i(\rho)} + \rho I_n\right)^{-1} z_n\end{aligned}$$

where we used $\tilde{f}_n(\rho) \geq \tilde{f}_i(\rho)$ for each i. This inequality is equivalent to

$$\tilde{d}_n(\rho) \leq \frac{1}{N}z_n^*\left(\alpha(\rho)\frac{1}{n}\sum_{i<n}\frac{z_iz_i^*}{\tilde{d}_i(\rho)} + \tilde{f}_n(\rho)\rho I_n\right)^{-1} z_n.$$

Assume now that, over some sequence $\{\rho^{(n)}\} \in \mathcal{R}_\kappa$, $\tilde{f}_n(\rho^{(n)}) > 1 + N^{\eta-1}$ infinitely often for some $\eta > 0$ (or equivalently, $d_n(\rho^{(n)}) > \tilde{d}_n(\rho^{(n)}) + N^{\eta-1}$ i.o.). Then we would have

$$\begin{aligned}\tilde{d}_n(\rho^{(n)}) &\leq \frac{1}{N}z_n^*\left(\alpha(\rho^{(n)})\frac{1}{n}\sum_{i<n}\frac{z_iz_i^*}{\tilde{d}_i(\rho^{(n)})} + \rho^{(n)}(1+N^{\eta-1})I_N\right)^{-1} z_n \\ &= \tilde{d}_n(\rho^{(n)}) - N^{\eta-1}\frac{1}{N}z_n^*\left(\frac{1}{n}\sum_{i<n}\frac{\alpha(\rho^{(n)})z_iz_i^*}{\rho^{(n)}\tilde{d}_i(\rho^{(n)})} + (1+N^{\eta-1})I_N\right)^{-1} \\ &\times\left(\frac{1}{n}\sum_{i<n}\frac{\alpha(\rho^{(n)})z_iz_i^*}{\tilde{d}_i(\rho^{(n)})} + \rho I_N\right)^{-1} z_n.\end{aligned}$$

But, by Proposition 6.1.1, letting $0 < \varepsilon < \eta$, we have, for all large n a.s.,

$$\frac{1}{N}z_n^*\left(\alpha(\rho^{(n)})\frac{1}{n}\sum_{i<n}\frac{z_iz_i^*}{\tilde{d}_i(\rho^{(n)})} + \rho^{(n)}I_n\right)^{-1} z_n \leq \tilde{d}_n(\rho^{(n)}) + N^{\varepsilon-1}$$

which, along with the uniform boundedness of the $\tilde{d}_i$ away from zero, leads to

$$\tilde{d}_n(\rho^{(n)}) \leq \tilde{d}_n(\rho^{(n)}) + N^{\varepsilon-1} - KN^{\eta-1}$$

for some $K > 0$. But, as $N^{\varepsilon-1} - KN^{\eta-1} < 0$ for all large N, we obtain a contradiction. Hence, for each $\eta > 0$, we have for all large n a.s., $d_n(\rho) < \tilde{d}_n(\rho) + N^{\eta-1}$ uniformly on $\rho \in \mathcal{R}_\kappa$. Proceeding similarly with $d_1(\rho)$, and exploiting $\limsup_n \sup_\rho \max_i |\tilde{d}_i(\rho)| = O(1)$ a.s., we finally have, for each $0 < \varepsilon < \frac{1}{2}$, that

$$\max_{1\leq k\leq n} \sup_{\rho\in\mathcal{R}_\kappa} \left| N^{1-\varepsilon} \left(d_k(\rho) - \tilde{d}_k(\rho) \right) \right| \xrightarrow{\text{a.s.}} 0$$

(for this, take an η such that $0 < \eta < \varepsilon$ and use $\max_k \sup_\rho |d_k(\rho) - \tilde{d}_k(\rho)| < N^{\eta-1}$ for all large n a.s.).

Getting back to ξ_2, we now have

$$N^{1-\varepsilon}|\xi_2(\rho)| = N^{1-\varepsilon} \left| a^*\hat{C}_N^{-1}(\rho) \left(\frac{\alpha(\rho)}{n} \sum_{i=1}^n \frac{d_i(\rho) - \tilde{d}_i(\rho)}{d_i(\rho)\tilde{d}_i(\rho)} z_i z_i^* \right) \hat{S}_N^{-1}(\rho) b \right|.$$

But, from the above result,

$$N^{1-\varepsilon} \left\| \frac{\alpha(\rho)}{n} \sum_{i=1}^n \frac{d_i(\rho) - \tilde{d}_i(\rho)}{d_i(\rho)\tilde{d}_i(\rho)} z_i z_i^* \right\| \leq N^{1-\varepsilon} \max_{1\leq k\leq n} \left| \frac{d_k(\rho) - \tilde{d}_k(\rho)}{d_k(\rho)\tilde{d}_k(\rho)} \right| \left\| \frac{\alpha(\rho)}{n} \sum_{i=1}^n z_i z_i^* \right\| \xrightarrow{\text{a.s.}} 0$$

uniformly so on $\rho \in \mathcal{R}_\kappa$ which, along with the boundedness of $\|\hat{C}_N^{-1}\|$, $\|\hat{S}_N^{-1}\|$, $\|a\|$, and $\|b\|$, finally gives $N^{1-\varepsilon}\xi_2 \xrightarrow{\text{a.s.}} 0$ uniformly on $\rho \in \mathcal{R}_\kappa$ as desired.

We have then proved that for each $\varepsilon > 0$,

$$\sup_{\rho\in\mathcal{R}_\kappa} \left| N^{1-\varepsilon} \left(a^*\hat{C}_N^{-1}(\rho)b - a^*\hat{S}_N^{-1}(\rho)b \right) \right| \xrightarrow{\text{a.s.}} 0$$

which proves Theorem 6.1.1 for $k = -1$. The generalization to arbitrary k is rather immediate. Writing recursively $\hat{C}_N^k - \hat{S}_N^k = \hat{C}_N^{k-1}(\hat{C}_N - \hat{S}_N) + (\hat{C}_N^{k-1} - \hat{S}_N^{k-1})\hat{S}_N$, for positive k or $\hat{C}_N^k - \hat{S}_N^k = \hat{C}_N^k(\hat{S}_N - \hat{C}_N)\hat{S}_N^{-1} + (\hat{C}_N^{k-1} - \hat{S}_N^{k-1})\hat{S}_N^{-1}$, (6.2) becomes a finite sum of terms that can be treated almost exactly as in the proof. This concludes the proof of Theorem 6.1.1.

6.1.1.2 Proof of Lemma 6.2

This section is devoted to the proof of the key Lemma 6.2. The proof relies on an appropriate decomposition of the quantity under study as a sum of martingale differences. Before delving into the core of the proofs, we introduce some notations along with some of the key-lemmas that will be extensively used in this section.

In this section, E_j will denote the conditional expectation with respect to the $\sigma-$ field $\mathcal{F}_j$ generated by the vectors $(z_\ell, 1 \leq \ell \leq j)$. By convention, $\mathrm{E}_0 = \mathrm{E}$.

Preliminaries. We start the proof by some preliminary results.

Lemma 6.3. *Let $z_1, \cdots, z_n$ be as in Assumption 6.1. Let $c \in \mathbb{C}^{N\times 1}$ be independent of $z_1, \cdots, z_n$ and such that $\mathrm{E}\|c\|^k$ is bounded uniformly in N for all order k. Then, for any integer p, there exists K_p such that*

$$\mathrm{E}\left[\left|z_i^* \hat{S}_N^{-1} c\right|^p\right] \leq \mathrm{E}\left[\left|z_i^* \hat{S}_{(i)}^{-1} c\right|^p\right] \leq K_p.$$

Proof. The first inequality can be obtained from the following decomposition:

$$\hat{S}_N^{-1} z_i = \frac{\hat{S}_{(i)}^{-1} z_i}{1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} z_i^* \hat{S}_{(i)}^{-1} z_i}$$

while the second follows by noticing that $\mathrm{E}\,|z_i^* c|^p \leq \mathrm{E}\,(c^* C_N c)^{\frac{p}{2}}$. □

Using the same kind of calculations, we can also control the order of magnitude of some interesting quantities.

Lemma 6.4. *The following statements hold true:*

1. *Denote by $\Delta_{i,j}$ the quantity:*

$$\Delta_{i,j} = \frac{1}{n} z_j^* \hat{S}_{(i,j)}^{-1} z_j - \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}.$$

   Then, for any $p \geq 2$.

$$\mathrm{E}\,|\Delta_{i,j}|^p = O(n^{-\frac{p}{2}}).$$

2. *Let i and j be two distinct integers from $\{1, \cdots, n\}$. Then,*

$$\mathrm{E}\left|z_i^* \hat{S}_{(i,j)}^{-1} z_j\right|^p = O(n^{\frac{p}{2}}).$$

3. *Let $z_i \in \mathbb{C}^{N\times 1}$ be as in Assumption 6.1 and A be a $N \times N$ random matrix independent of z_i and having a bounded spectral norm. Then,*

$$\mathrm{E}\,|z_i^* A z_i|^p = O(n^p).$$

4. *Let $j \in \{1, \cdots, n\}$ and i and k two distinct integers different from j. Then:*

$$\mathrm{E}\left|z_i^* \hat{S}_{(i,j)}^{-1} \hat{S}_{(j,k)}^{-1} z_k\right|^p = O(n^{\frac{p}{2}}).$$

Proof. Item 1) and 3) are standard results that are a by-product of (Bai and Silverstein, 2009, Lemma B.26), while Item 2) can be easily obtained from Lemma 6.3. As for item 4), it follows by first decomposing $\hat{S}_{(i,j)}^{-1}$ and $\hat{S}_{(j,k)}^{-1}$ as:

$$\hat{S}_{(i,j)}^{-1} = \hat{S}_{(i,j,k)}^{-1} - \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{\hat{S}_{(i,j,k)}^{-1} z_k z_k^* \hat{S}_{(i,j,k)}^{-1}}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_k^* \hat{S}_{(i,j,k)}^{-1} z_k}$$
$$\hat{S}_{(j,k)}^{-1} = \hat{S}_{(i,j,k)}^{-1} - \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{\hat{S}_{(i,j,k)}^{-1} z_i z_i^* \hat{S}_{(i,j,k)}^{-1}}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_i^* \hat{S}_{(i,j,k)}^{-1} z_i}$$

The above relations serve to better control the dependencies of $\hat{S}_{(i,j)}^{-1}$ and $\hat{S}_{(j,k)}^{-1}$ on z_k and z_i. Plugging the above decompositions on $z_i^* \hat{S}_{(i,j)}^{-1} \hat{S}_{(j,k)}^{-1} z_k$, we obtain

$$\begin{aligned}
z_i^* \hat{S}_{(i,j)}^{-1} \hat{S}_{(j,k)}^{-1} z_k &= z_i^* \hat{S}_{(i,j,k)}^{-2} z_k - \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{z_i^* \hat{S}_{(i,j,k)}^{-1} z_k z_k^* \hat{S}_{(i,j,k)}^{-2} z_k}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_k^* \hat{S}_{(i,j,k)}^{-1} z_k} \\
&- \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{z_i^* \hat{S}_{(i,j,k)}^{-2} z_i z_i^* \hat{S}_{(i,j,k)}^{-1} z_k}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_i^* \hat{S}_{(i,j,k)}^{-1} z_i} \\
&+ \frac{1}{n^2}\left(\frac{\alpha(\rho)}{\gamma_N(\rho)}\right)^2 \frac{z_i^* \hat{S}_{(i,j,k)}^{-1} z_k z_k^* \hat{S}_{(i,j,k)}^{-2} z_i z_i^* \hat{S}_{(i,j,k)}^{-1} z_k}{\left(1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_k^* \hat{S}_{(i,j,k)}^{-1} z_k\right)\left(1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} z_i^* \hat{S}_{(i,j,k)}^{-1} z_i\right)}.
\end{aligned}$$

The control of these four terms follows from a direct application of item 2) and 3) along with possibly the use of the generalized Hölder inequality in Lemma A.6. □

Core of the proof. With these preliminaries results at hand, we are now in position to get into the core of the proof. Let β_N be given by

$$\beta_N = \frac{1}{n}\sum_{i=1}^n c^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} d \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \gamma_N(\rho)\right).$$

Decompose β_N as

$$\begin{aligned}
\beta_N &= \frac{1}{n}\sum_{i=1}^n c^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} d \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N}\operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right) \\
&+ \frac{1}{n}\sum_{i=1}^n c^* \hat{S}_N^{-1} z_i z_i^* \hat{S}_N^{-1} d \left(\frac{1}{N}\operatorname{tr} C_N \hat{S}_{(i)}^{-1} - \gamma_N(\rho)\right) \\
&\triangleq \beta_{N,1} + \beta_{N,2}.
\end{aligned}$$

The control of $\beta_{N,2}$ follows from a direct application of Lemma A.5 and Lemma A.6, that is

$$
\begin{aligned}
\mathrm{E}\left[|\beta_{N,2}|^{2p}\right] &\leq \frac{n^{2p-1}}{n^{2p}} \sum_{i=1}^{n} \mathrm{E}\left|c^* \hat{S}_N^{-1} z_i\right|^{2p} \left|z_i^* \hat{S}_N^{-1} d\right|^{2p} \left|\frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} - \gamma_N(\rho)\right|^{2p} \\
&\leq \frac{n^{2p-1}}{n^{2p}} \sum_{i=1}^{n} \left(\mathrm{E}\left|c^* \hat{S}_N^{-1} z_i\right|^{6p}\right)^{\frac{1}{3}} \left(\mathrm{E}\left|z_i^* \hat{S}_N^{-1} d\right|^{6p}\right)^{\frac{1}{3}} \left(\mathrm{E}\left|\frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} - \gamma_N(\rho)\right|^{6p}\right)^{\frac{1}{3}}
\end{aligned}
$$

By standard results from random matrix theory (e.g. (Najim and Yao, 2013, Prop. 7.1)), we know that

$$
\mathrm{E}\left|\frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} - \gamma_N(\rho)\right|^{6p} = O(n^{-6p})
$$

Hence, by Lemma 6.3, we finally get:

$$
\mathrm{E}\left|\beta_{N,2}\right|^{2p} = O(n^{-2p}).
$$

While the control of $\beta_{N,2}$ requires only the manipulation of conventional moment bounds due to the rapid convergence of $\frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} - \gamma_N(\rho)$, the analysis of $\beta_{N,1}$ is more intricate since

$$
\mathrm{E}\left|\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right|^{p} = O(n^{-\frac{p}{2}})
$$

a convergence rate which seems insufficient at the onset. The averaging occurring in $\beta_{N,2}$ shall play the role of improving this rate. To control $\beta_{N,1}$, one needs to resort to advanced tools based on Burkhölder inequalities. First, decompose $\beta_{N,1}$ as

$$
\beta_{N,1} = \overset{o}{\beta}_{N,1} + \mathrm{E}\left[\beta_{N,1}\right].
$$

As in Lemma 6.4, define $\Delta_i \triangleq \frac{1}{n} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}$. Using the relation

$$
\hat{S}_N^{-1} z_i = \frac{\hat{S}_{(i)}^{-1} z_i}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_i^* \hat{S}_{(i)}^{-1} z_i}
$$

we get

$$
\begin{aligned}
\mathrm{E}\left[\beta_{N,1}\right] &= \mathrm{E}\left[\frac{1}{N} \sum_{i=1}^{n} \frac{c^* \hat{S}_{(i)}^{-1} z_i z_i^* \hat{S}_{(i)}^{-1} d}{\left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_i^* \hat{S}_{(i)}^{-1} z_i\right)^2} \Delta_i\right] \\
&= \mathrm{E}\left[\frac{1}{N} \sum_{i=1}^{n} \frac{c^* \hat{S}_{(i)}^{-1} z_i z_i^* \hat{S}_{(i)}^{-1} d}{\left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right)^2} \Delta_i\right] \\
&\quad - \frac{\alpha(\rho)}{\gamma_N(\rho)} \mathrm{E}\left[\frac{1}{N} \sum_{i=1}^{n} \frac{c^* \hat{S}_{(i)}^{-1} z_i z_i^* \hat{S}_{(i)}^{-1} d \Delta_i^2 \left(2 + \left(\frac{\alpha(\rho)}{\gamma_N(\rho)}\right) \left(\frac{1}{n} z_i^* \hat{S}_{(i)}^{-1} z_i + \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right)\right)}{\left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right)^2 \left(1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} z_i^* \hat{S}_{(i)}^{-1} z_i\right)^2}\right] \\
&\triangleq \beta_{N,1,1} + \beta_{N,1,2}
\end{aligned}
$$

Since $\mathrm{E}\left[w^* A w \left(w^* B w - \operatorname{tr} B\right)\right] = \mathrm{E} \operatorname{tr} AB$ when w is standard complex Gaussian vector and A, B random matrices independent of w, we have

$$\mathrm{E}\left[\beta_{N,1,1}\right] = \frac{1}{Nn}\mathrm{E}\left[\operatorname{tr} \frac{C_N \hat{S}_{(i)}^{-1} C_N \hat{S}_{(i)}^{-1} dc^* \hat{S}_{(i)}^{-1}}{\left(1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right)^2}\right] = O(n^{-1}).$$

As for $\beta_{N,1,2}$, we have for some $K > 0$, again by Lemma 6.4

$$\begin{aligned}
|\beta_{N,1,2}| \leq & \frac{K}{n} \sum_{i=1}^{n} \left(\mathrm{E}\left|c^* \hat{S}_{(i)}^{-1} z_i\right|^4\right)^{\frac{1}{4}} \left(\mathrm{E}\left|z_i^* \hat{S}_{(i)}^{-1} d\right|^4\right)^{\frac{1}{4}} \left(\mathrm{E}\left|\Delta_i\right|^8\right)^{\frac{1}{4}} \\
& \times \left(\mathrm{E}\left|2 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \left(\frac{1}{n} z_i^* \hat{S}_{(i)}^{-1} z_i + \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i)}^{-1}\right)\right|^4\right)^{\frac{1}{4}} = O(\frac{1}{n}).
\end{aligned}$$

We therefore have

$$\left|\mathrm{E}\left[\beta_{N,1}\right]\right|^{2p} = O(n^{-2p}).$$

Let's turn to the control of $\overset{o}{\beta}_{N,1}$. For that, we decompose $\overset{o}{\beta}_{N,1}$ as a sum of martingale differences as

$$\overset{o}{\beta}_{N,1} = \sum_{j=1}^{n} \left(\mathrm{E}_j - \mathrm{E}_{j-1}\right) \beta_{N,1}$$

The control of $\mathrm{E}\left[\left|\overset{o}{\beta}_{N,1}\right|^p\right]$ requires the convergence rate of two kinds of martingale differences:

- Sum of martingale differences with a quadratic form representation of the form

$$\sum_{j=1}^{n} \left(\mathrm{E}_j - \mathrm{E}_{j-1}\right) z_j^* A_j z_j.$$

  For these terms, from Lemma A.8, it will be sufficient to show that $\max_j \mathrm{E}\|A_j\|_{\mathrm{Fro}}^{2p} = O(n^{-3p})$ in order to obtain the required convergence rate.

- Sum of martingale differences with more than one occurrence of z_j and z_j^*. In this case, this sum is given by:

$$\sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \sum_{i=1, i \neq j}^{n} \varepsilon_i$$

  where ε_j are small random quantities depending on $z_1, \cdots, z_n$. According to Lemma A.7, we have

$$\left|\sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \sum_{i=1, i \neq j}^{n} \varepsilon_i\right|^{2p} = O(n^{-2p})$$

provided that

$$\mathrm{E}\left|\sum_{i=1, i\neq j} \varepsilon_i\right|^{2p} = O(n^{-3p}).$$

The control of the above sum will rely on successively using Lemma A.5 to get

$$\mathrm{E}\left|\sum_{i=1, i\neq j} \varepsilon_i\right|^{2p} \leq n^{2p-1} \sum_{i=1}^{n} \mathrm{E}\left|\varepsilon_i\right|^{2p}$$

and controlling $\max_i \mathrm{E}\left|\varepsilon_i\right|^{2p}$.

With this explanation at hand, we will now get into the core of the proofs. We first have

$$\begin{aligned}
\overset{o}{\beta}_{N,1} &= \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{1}{N} \sum_{i=1}^{n} c^* \hat{S}_N^{-1} z_i z_i^* d \Delta_i \\
&= \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) c^* \hat{S}_N^{-1} z_j z_j^* d \Delta_j \\
&+ \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{1}{N} \sum_{i=1, i\neq j}^{n} c^* \hat{S}_N^{-1} z_i z_i^* d \Delta_i \\
&\triangleq \sum_{j=1}^{n} W_{j,1} + \sum_{j=1}^{n} W_{j,2}.
\end{aligned}$$

In order to prove that $\mathrm{E}\left|\sum_{j=1}^{n} W_{j,1}\right| = O(n^{-2p})$, it is sufficient to show

$$\mathrm{E}\left|W_{j,1}\right| = O(n^{-3p})$$

a statement which holds true since, by Lemma A.6

$$\begin{aligned}
\mathrm{E}\left|W_{j,1}\right|^{2p} &\leq \frac{K}{n^{2p}} \mathrm{E}\left|c^* \hat{S}_N^{-1} z_j\right|^{2p} \left|z_j^* \hat{S}_N^{-1} d\right|^{2p} \Delta_j^{2p} \\
&\leq \frac{K}{n^{2p}} \left(\mathrm{E}\left|c^* \hat{S}_N^{-1} z_j\right|^{6p}\right)^{\frac{1}{3}} \left(\mathrm{E}\left|z_j^* \hat{S}_N^{-1} d\right|^{6p}\right)^{\frac{1}{3}} \left(\mathrm{E}\Delta_j^{6p}\right)^{\frac{1}{3}} \\
&= O(n^{-3p}).
\end{aligned}$$

We now consider the more involved term $\sum_{j=1}^{n} W_{j,2}$. Using the relation

$$\hat{S}_N^{-1} = \hat{S}_{(j)}^{-1} - \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} \frac{\hat{S}_{(j)}^{-1} z_j z_j^* \hat{S}_{(j)}^{-1}}{1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} z_j^* \hat{S}_{(j)}^{-1} z_j}$$

to let the independent $\hat{S}_{(j)}^{-1}$ and z_j variables appear, we write

$$\begin{aligned}
&\sum_{j=1}^{n} W_{j,2} = \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{1}{n} \sum_{i=1, i\neq j}^{n} c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} \right) \\
&- \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n^2} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_j z_j^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(j)}^{-1} z_j} \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} \right) \\
&- \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n^2} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} z_j z_j^* \hat{S}_{(j)}^{-1} d}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(j)}^{-1} z_j} \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} \right) \\
&+ \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \left(\frac{\alpha(\rho)}{\gamma_N(\rho)} \right)^2 \frac{1}{n^3} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_j z_j^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} z_j z_j^* \hat{S}_{(j)}^{-1} d}{\left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(j)}^{-1} z_j\right)^2} \left(\frac{1}{N} z_i^* \hat{S}_{(i)}^{-1} z_i - \frac{1}{N} \operatorname{tr} C_N \hat{S}_{(i)}^{-1} \right) \\
&\triangleq \chi_1 + \chi_2 + \chi_3 + \chi_4.
\end{aligned}$$

Next, we will sequentially control $\chi_i, i = 1, \cdots, 4$.

Control of χ_1. Using the relation

$$\hat{S}_{(i)}^{-1} = \hat{S}_{(i,j)}^{-1} - \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} \frac{\hat{S}_{(i,j)}^{-1} z_j z_j^* \hat{S}_{(i,j)}^{-1}}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(i,j)}^{-1} z_j}$$

the quantity χ_1 can be decomposed as

$$\begin{aligned}
\chi_1 &= \sum_{j=1}^{n} -(\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n^2 N} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \left| z_i^* \hat{S}_{(i,j)}^{-1} z_j \right|^2}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(i,j)}^{-1} z_j} \\
&+ \sum_{j=1}^{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{1}{n^2 N} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d z_j^* \hat{S}_{(i,j)}^{-1} C_N \hat{S}_{(i,j)}^{-1} z_j}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(i,j)}^{-1} z_j} \\
&\triangleq \chi_{1,1} + \chi_{1,2}.
\end{aligned}$$

where we used the fact that for r_j random quantity independent of z_j, $(\mathrm{E}_j - \mathrm{E}_{j-1})(r_j) = 0$. We will begin by controlling $\chi_{1,1}$. To handle the quadratic forms in the denominator, we further develop $\chi_{1,1}$ as

$$\begin{aligned}
\chi_{1,1} &= -\sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n^2 N} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \left| z_i^* \hat{S}_{(i,j)}^{-1} z_j \right|^2}{1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}} \\
&+ \sum_{j=1}^{n} (\mathrm{E}_j - \mathrm{E}_{j-1}) \left(\frac{\alpha(\rho)}{\gamma_N(\rho)} \right)^2 \frac{1}{n^2 N} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} \left| z_i^* \hat{S}_{(i,j)}^{-1} z_j \right|^2 \Delta_{i,j}}{\left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right) \left(1 + \frac{1}{n} \frac{\alpha(\rho)}{\gamma_N(\rho)} z_j^* \hat{S}_{(i,j)}^{-1} z_j\right)} \\
&= \sum_{j=1}^{n} X_{j,1} + \sum_{j=1}^{n} X_{j,2}.
\end{aligned}$$

To control $\sum_{j=1}^n X_{j,1}$, we resort to Lemma A.8. Indeed, $X_{j,1}$ can be written as $X_{j,1} = -\frac{\alpha(\rho)}{\gamma_N(\rho)}(\mathrm{E}_j - \mathrm{E}_{j-1})z_j^* A_j z_j$ where A_j is given by

$$A_j = \frac{1}{n^2 N} \sum_{i=1, i\neq j}^{n} \frac{c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d}{1 + \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}} \hat{S}_{(i,j)}^{-1} z_i z_i^* \hat{S}_{(i,j)}^{-1}.$$

According to Lemma A.8, it is sufficient to prove that $\mathrm{E}\,\|A_j\|_{\mathrm{Fro}}^{2p} = O(n^{-3p})$. Expanding $\mathrm{E}\,\|A_j\|_{\mathrm{Fro}}^{2p}$, we indeed get

$$\begin{aligned}
\mathrm{E}\,\|A_j\|_{\mathrm{Fro}}^{2p} &\leq \frac{K}{n^{6p}} \mathrm{E} \left| \sum_{i\neq j} \sum_{k \neq j} \frac{\left| z_k^* \hat{S}_{(j,k)}^{-1} \hat{S}_{(i,j)}^{-1} z_i \right|^2 c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d d^* \hat{S}_{(j)}^{-1} z_k z_k^* \hat{S}_{(j)}^{-1} c}{\left(1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right)\left(1 + \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(j,k)}^{-1}\right)} \right|^p \\
&\leq \frac{K}{n^{6p}} \mathrm{E} \left| \sum_{i \neq j} \left| z_i^* \hat{S}_{(i,j)}^{-2} z_i \right|^2 \left| c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \right|^2 \right|^p \\
&+ \frac{K}{n^{6p}} \mathrm{E} \left| \sum_{i\neq j} \sum_{\substack{k \neq j \\ k \neq i}} \frac{\left| z_k^* \hat{S}_{(j,k)}^{-1} \hat{S}_{(i,j)}^{-1} z_i \right|^2 c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d d^* \hat{S}_{(j)}^{-1} z_k z_k^* \hat{S}_{(j)}^{-1} c}{\left(1 + \frac{\alpha(\rho)}{\gamma_N(\rho)} \frac{1}{n} \operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right)\left(1 + \frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)} \operatorname{tr} C_N \hat{S}_{(j,k)}^{-1}\right)} \right|^p \\
&\leq \frac{K n^{p-1}}{n^{6p}} \mathrm{E} \left| z_i^* \hat{S}_{(i,j)}^{-2} z_i \right|^{2p} \left| c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \right|^{2p} \\
&+ \frac{K n^{2(p-1)}}{n^{6p}} \sum_{i \neq j} \sum_{\substack{k \neq j \\ k \neq i}} \mathrm{E} \left| z_k^* \hat{S}_{(j,k)}^{-1} \hat{S}_{(i,j)}^{-1} z_i \right|^{2p} \left| c^* \hat{S}_{(j)}^{-1} z_i z_i^* \hat{S}_{(j)}^{-1} d \right|^p \left| d^* \hat{S}_{(j,k)}^{-1} z_k z_k^* \hat{S}_{(j,k)}^{-1} c \right|^p
\end{aligned}$$

which is further bounded as

$$\begin{aligned}
\mathrm{E}\,\|A_j\|_{\mathrm{Fro}}^{2p} &\leq \frac{K n^{p-1}}{n^{6p}} \sum_{i \neq j} \left(\mathrm{E} \left| z_i^* \hat{S}_{(i,j)}^{-2} z_i \right|^{6p} \right)^{\frac{1}{3}} \left(\left| c^* \hat{S}_{(j)}^{-1} z_i \right|^{6p} \right)^{\frac{1}{3}} \left(\left| z_i^* \hat{S}_{(j)}^{-1} d \right|^{6p} \right)^{\frac{1}{3}} \\
&+ \frac{K n^{2(p-1)}}{n^{6p}} \sum_{i \neq j} \sum_{\substack{k \neq j \\ k \neq i}} \left(\mathrm{E} \left| z_k^* \hat{S}_{(j,k)}^{-1} \hat{S}_{(i,j)}^{-1} z_i \right|^{10p} \right)^{\frac{1}{5}} \left(\mathrm{E} \left| c^* \hat{S}_{(i,j)}^{-1} z_i \right|^{5p} \right)^{\frac{1}{5}} \\
&\times \left(\mathrm{E} \left| z_i^* \hat{S}_{(j)}^{-1} d \right|^{5p} \right)^{\frac{1}{5}} \left(\mathrm{E} \left| d^* \hat{S}_{(j,k)}^{-1} z_k \right|^{5p} \right)^{\frac{1}{5}} \left(\mathrm{E} \left| z_k^* \hat{S}_{(j,k)}^{-1} c \right|^{5p} \right)^{\frac{1}{5}} \\
&= O(n^{-3p}).
\end{aligned}$$

As for $X_{j,1}$, we can show that $\mathrm{E}\,|X_{j,1}|^{2p} = O(n^{-3p})$. Indeed, we have

$$\begin{aligned}
\mathrm{E}\,|X_{j,2}|^{2p} &\leq \frac{K n^{2p-1}}{n^{6p}} \sum_{i \neq j} \left(\mathrm{E} \left| c^* \hat{S}_{(j)}^{-1} z_i \right|^{8p} \right)^{\frac{1}{4}} \left(\mathrm{E} \left| z_i^* \hat{S}_{(j)}^{-1} d \right|^{8p} \right)^{\frac{1}{4}} \left(\mathrm{E} \left| z_i^* \hat{S}_{(j)}^{-1} z_j \right|^{16p} \right)^{\frac{1}{4}} \left(\mathrm{E}\,|\Delta_{i,j}|^{8p} \right)^{\frac{1}{4}} \\
&= O(n^{-3p}).
\end{aligned}$$

The Burkhölder inequality shows that this rate of convergence of the moment of $X_{j,1}$ and $X_{j,2}$ is sufficient to finally ensure that $\mathrm{E}\left|\chi_{1,1}\right|^{2p} = O(n^{-2p})$.

We study next $\chi_{1,2}$. First, decompose $\chi_{1,2}$ as

$$\begin{aligned}
\chi_{1,2} &= \sum_{j=1}^{n}(\mathrm{E}_j - \mathrm{E}_{j-1})\frac{1}{n^2 N}\sum_{i\neq j}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{c^*\hat{S}_{(j)}^{-1}z_i z_i^*\hat{S}_{(j)}^{-1}dz_j^*\hat{S}_{(i,j)}^{-1}C_N\hat{S}_{(i,j)}^{-1}z_j}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\operatorname{tr}C_N\hat{S}_{(i,j)}^{-1}} \\
&\quad - \sum_{j=1}^{n}(\mathrm{E}_j - \mathrm{E}_{j-1})\frac{1}{n^2 N}\sum_{i\neq j}\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{c^*\hat{S}_{(j)}^{-1}z_i z_i^*\hat{S}_{(j)}^{-1}d\Delta_{i,j}z_j^*\hat{S}_{(i,j)}^{-1}C_N\hat{S}_{(i,j)}^{-1}z_j}{\left(1+\frac{\alpha(\rho)}{\gamma_N(\rho)}\frac{1}{n}z_j^*\hat{S}_{(i,j)}^{-1}z_j\right)\left(1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\operatorname{tr}C_N\hat{S}_{(i,j)}^{-1}\right)} \\
&\triangleq \sum_{j=1}^{n}Y_{j,1} + \sum_{j=1}^{n}Y_{j,2}.
\end{aligned}$$

The quantities $\sum_{j=1}^{n}Y_{j,1}$ and $\sum_{j=1}^{n}Y_{j,2}$ are differences of martingales whose controls follow the same procedure as above. While $\sum_{j=1}^{n}Y_{j,1}$ can be controlled using Lemma A.8, the convergence of $\sum_{j=1}^{n}Y_{j,2}$ is faster due to the term $\Delta_{i,j}$. Details are thus omitted.

Control of χ_2. The control of χ_2 cannot be exactly dealt with using the same procedure. As for χ_1, one works out χ_2 by substituting $\frac{1}{n}z_j^*\hat{S}_{(j)}^{-1}z_j$ by its approximate $\frac{1}{n}\operatorname{tr}C_N\hat{S}_{(j)}^{-1}$ and using the decomposition of $\hat{S}_{(i)}^{-1}$ as a function of $\hat{S}_{(i,j)}^{-1}$ to get

$$\chi_2 = -\frac{\alpha(\rho)}{\gamma_N(\rho)}\sum_{j=1}^{n}(\mathrm{E}_j - \mathrm{E}_{j-1})\frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}\frac{c^*\hat{S}_{(j)}^{-1}z_j z_j^*\hat{S}_{(j)}^{-1}z_i z_i^*\hat{S}_{(j)}^{-1}d\left(\frac{1}{N}z_i^*\hat{S}_{(i,j)}^{-1}z_i - \frac{1}{N}\operatorname{tr}C_N\hat{S}_{(i,j)}^{-1}\right)}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\operatorname{tr}C_N\hat{S}_{(j)}^{-1}} + \varepsilon$$

where we easily obtain that $\mathrm{E}[|\varepsilon|^{2p}] = O(n^{-2p})$. We omit the details of this step, since the calculations are the same as those used for the control of χ_1. The control of the Frobenius norm of the underlying matrices using the same techniques as above does not yield the required convergence rate. We will thus pursue a different approach. Precisely, we write χ_2 as

$$\chi_2 = -\frac{\alpha(\rho)}{\gamma_N(\rho)}\sum_{j=1}^{n}(\mathrm{E}_j - \mathrm{E}_{j-1})T_j + \varepsilon$$

with

$$T_j = \frac{1}{n^2}\frac{c^*\hat{S}_{(j)}^{-1}z_j z_j^*\hat{S}_{(j)}^{-1}Z_j D_j Z_j^*\hat{S}_{(j)}^{-1}d}{1+\frac{1}{n}\frac{\alpha(\rho)}{\gamma_N(\rho)}\operatorname{tr}\hat{S}_{(j)}^{-1}}$$

where $Z_j = [z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_n]$ and D_j is a diagonal matrix with diagonal elements: $[D_j]_{i,i} = \frac{n}{N}\Delta_{j,i}$. Hence, by Lemma A.7

$$\begin{aligned}
\mathrm{E}\left|T_j\right|^{2p} &\leq \frac{1}{n^{4p}}\mathrm{E}\left|c^*\hat{S}_{(j)}^{-1}z_j\right|^{2p}\left|z_j^*\hat{S}_{(j)}^{-1}Z_j D_j Z_j^*\hat{S}_{(j)}^{-1}d\right|^{2p} \\
&\leq \frac{1}{n^{4p}}\left(\mathrm{E}\left|c^*\hat{S}_{(j)}^{-1}z_j\right|^{4p}\right)^{\frac{1}{2}}\left(\mathrm{E}\left|z_j^*\hat{S}_{(j)}^{-1}Z_j D_j Z_j^*d\right|^{4p}\right)^{\frac{1}{2}}
\end{aligned}$$

Since D_j is independent of z_j, applying the inequality $\mathrm{E}\left|z_j^* u\right|^p \leq \mathrm{E}\left(u^* C_N u\right)^{\frac{p}{2}}$, we finally get

$$\begin{aligned}
\mathrm{E}\left|T_j\right|^{2p} &\leq \frac{K}{n^{4p}}\left(\mathrm{E}\left|d^*\hat{S}_{(j)}^{-1} Z_j D_j Z_j^* \hat{S}_{(j)}^{-1} C_N \hat{S}_{(j)}^{-1} Z_j D_j Z_j^* \hat{S}_{(j)}^{-1} d\right|^{2p}\right)^{\frac{1}{2}} \\
&= \frac{K}{n^{3p}}\left(\mathrm{E}\left|d^*\hat{S}_{(j)}^{-1} Z_j D_j \frac{Z_j^* \hat{S}_{(j)}^{-1} C_N \hat{S}_{(j)}^{-1} Z_j}{n} D_j Z_j^* \hat{S}_{(j)}^{-1} d\right|^{2p}\right)^{\frac{1}{2}} \\
&\overset{(a)}{\leq} \frac{K}{n^{3p}}\left(\mathrm{E}\left\|D_j Z_j^* \hat{S}_{(j)}^{-1} d\right\|^{4p}\right)^{\frac{1}{2}}
\end{aligned}$$

where (a) follows since $\left\|\frac{Z_j^* \hat{S}_{(j)}^{-1} C_N \hat{S}_{(j)}^{-1} Z_j}{n}\right\|$ is bounded. In order to prove that $\mathrm{E}[|T_j|^{2p}] = O(n^{-3p})$, it suffices to check that $\mathrm{E}[\left\|D_j Z_j^* \hat{S}_{(j)}^{-1} d\right\|^{4p}]$ is uniformly bounded in N. Expanding this quantity, we indeed get

$$\begin{aligned}
\mathrm{E}\left\|D_j Z_j^* \hat{S}_{(j)}^{-1} d\right\|^{4p} &= \mathrm{E}\left|\sum_{\substack{i=1 \\ i\neq j}}^{n}\left(\frac{1}{N} z_i^* \hat{S}_{(i,j)}^{-1} z_i - \frac{1}{N}\operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right)^2 \left|z_i^* \hat{S}_{(j)}^{-1} d\right|^2\right|^{2p} \\
&\leq n^{2p-1}\sum_{i=1}^{n} \mathrm{E}\left(\frac{1}{N} z_i^* \hat{S}_{(i,j)}^{-1} z_i - \frac{1}{N}\operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right)^{4p} \left|z_i^* \hat{S}_{(j)}^{-1} d\right|^{4p} \\
&\leq n^{2p-1}\sum_{i=1}^{n}\left(\mathrm{E}\left(\frac{1}{N} z_i^* \hat{S}_{(i,j)}^{-1} z_i - \frac{1}{N}\operatorname{tr} C_N \hat{S}_{(i,j)}^{-1}\right)^{8p}\right)^{\frac{1}{2}}\left(\mathrm{E}\left|z_i^* \hat{S}_{(j)}^{-1} d\right|^{8p}\right)^{\frac{1}{2}} \\
&= O(1).
\end{aligned}$$

The control of χ_3 is similar to that of χ_2, while that of χ_4 follows immediately by using sequentially Lemma A.5 along with the generalized Hölder inequality in Lemma A.6. This completes the proof.

6.2 Application to FAR minimization

In this section, we consider the hypothesis testing scenario by which an N-sensor array receives a vector $y \in \mathbb{C}^N$ according to the following hypotheses

$$y = \begin{cases} x & , \mathcal{H}_0 \\ \alpha p + x & , \mathcal{H}_1 \end{cases}$$

in which $\alpha > 0$ is some unknown scaling factor constant while $p \in \mathbb{C}^N$ is deterministic and known at the sensor array (which often corresponds to a steering vector arising from a specific known angle), and x is an impulsive noise distributed as x_1 according to Assumption 6.1. For convenience, we shall take $\|p\| = 1$.

Under $\mathcal{H}_0$ (the null hypothesis), a noisy observation from an impulsive source is observed while under $\mathcal{H}_1$ both information and noise are collected at the array. The objective is to decide on $\mathcal{H}_1$ versus $\mathcal{H}_0$ upon the observation y and prior pure-noise observations $x_1, \ldots, x_n$ distributed according to Assumption 6.1. When $\tau_1, \ldots, \tau_n$ and C_N are unknown, the corresponding generalized likelihood ratio test, derived in (Conte et al., 1995), reads

$$T_N(\rho) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \Gamma$$

for some detection threshold Γ where

$$T_N(\rho) \triangleq \frac{|y^* \hat{C}_N^{-1}(\rho) p|}{\sqrt{y^* \hat{C}_N^{-1}(\rho) y} \sqrt{p^* \hat{C}_N^{-1}(\rho) p}}.$$

More precisely, (Conte et al., 1995) derived the detector $T_N(0)$ only valid when $n \geq N$. The relaxed detector $T_N(\rho)$ allows for a better conditioning of the estimator, in particular for $n \simeq N$. In (Pascal et al., 2013), $T_N(\rho)$ is used explicitly in a space-time adaptive processing setting but only simulation results were provided. Alternative metrics for similar array processing problems involve the signal-to-noise ratio loss minimization rather than likelihood ratio tests; in (Abramovich and Besson, 2012; Besson and Abramovich, 2013), the authors exploit the estimators $\hat{C}_N(\rho)$ but restrict themselves to the less tractable finite dimensional analysis.

Our objective is to characterize the false alarm performance of the detector. That is, provided $\mathcal{H}_0$ is the actual scenario (i.e. $y = x$), we shall evaluate $P(T_N(\rho) > \Gamma)$. Since it shall appear that, under $\mathcal{H}_0$, $T_N(\rho) \xrightarrow{\text{a.s.}} 0$ for every fixed $\Gamma > 0$ and every ρ, by dominated convergence $P(T_N(\rho) > \Gamma) \to 0$ which does not say much about the actual test performance for large but finite N, n. To avoid such empty statements, we shall then consider the non-trivial case where $\Gamma = N^{-\frac{1}{2}}\gamma$ for some fixed $\gamma > 0$. In this case our objective is to characterize the false alarm probability

$$P\left(T_N(\rho) > \frac{\gamma}{\sqrt{N}}\right).$$

Before providing this result, we need some further reminders from Chapter 5. First define

$$\underline{\hat{S}}_N(\rho) \triangleq (1-\rho)\frac{1}{n}\sum_{i=1}^n z_i z_i^* + \rho I_N.$$

Then, from Lemma 5.1, for each $\rho \in (\max\{0, 1-c^{-1}\}, 1]$,

$$\frac{\hat{S}_N(\rho)}{\rho + \frac{1}{\gamma_N(\rho)} \frac{1-\rho}{1-(1-\rho)c}} = \underline{\hat{S}}_N(\underline{\rho})$$

where

$$\underline{\rho} \triangleq \frac{\rho}{\rho + \frac{1}{\gamma_N(\rho)} \frac{1-\rho}{1-(1-\rho)c}}.$$

Moreover, the mapping $\rho \mapsto \underline{\rho}$ is continuously increasing from $(\max\{0, 1-c^{-1}\}, 1]$ onto $(0,1]$.

From classical random matrix considerations (see Chapter 3 or (Silverstein and Bai, 1995)), letting $Z = [z_1, \ldots, z_n] \in \mathbb{C}^{N\times n}$, the empirical spectral distribution[2] of $(1-\rho)\frac{1}{n}Z^*Z$ almost surely admits a weak limit μ. The Stieltjes transform $m(z) \triangleq \int (t-z)^{-1}\mu(dt)$ of μ at $z \in \mathbb{C}\setminus \operatorname{supp}(\mu)$ is the unique complex solution with positive (resp. negative) imaginary part if $\Im[z] > 0$ (resp. $\Im[z] < 0$) and unique real positive solution if $\Im[z] = 0$ and $\Re[z] < 0$ to

$$m(z) = \left(-z + c\int \frac{(1-\rho)t}{1+(1-\rho)tm(z)}\nu(dt)\right)^{-1}.$$

We denote $m'(z)$ the derivative of $m(z)$ with respect to z (recall that the Stieltjes transform of a positively supported measure is analytic, hence continuously differentiable, away from the support of the measure).

With these definitions in place and with the help of Theorem 6.1.1, we are now ready to introduce the main result of this section.

Theorem 6.2.1 (Asymptotic detector performance)**.** *Under hypothesis $\mathcal{H}_0$, as $N, n \to \infty$ with $c_N \to c \in (0,\infty)$,*

$$\sup_{\rho\in\mathcal{R}_\kappa}\left| P\left(T_N(\rho) > \frac{\gamma}{\sqrt{N}}\right) - \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho)}\right)\right| \to 0$$

where $\rho \mapsto \underline{\rho}$ is the aforementioned mapping and

$$\sigma_N^2(\rho) \triangleq \frac{1}{2}\frac{p^*C_NQ_N^2(\underline{\rho})p}{p^*Q_N(\underline{\rho})p \cdot \frac{1}{N}\operatorname{tr} C_NQ_N(\underline{\rho}) \cdot \left(1 - c(1-\underline{\rho})^2 m(-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2Q_N^2(\underline{\rho})\right)}$$

with $Q_N(\underline{\rho}) \triangleq (I_N + (1-\underline{\rho})m(-\underline{\rho})C_N)^{-1}$.

Otherwise stated, $\sqrt{N}T_N(\rho)$ is uniformly well approximated by a Rayleigh distributed random variable $R_N(\rho)$ with parameter $\sigma_N(\rho)$. Simulation results are provided in Figure 6.1 and Figure 6.2 which corroborate the results of Theorem 6.2.1 for $N = 20$ and $N = 100$, respectively (for a single value of ρ though). Comparatively, it is observed, as one would expect, that larger values for N induce improved approximations in the tails of the approximating distribution.

The result of Theorem 6.2.1 provides an analytical characterization of the performance of the GLRT for each ρ which suggests in particular the existence of values for ρ which minimize the false alarm probability for given γ. Note in passing that, independently of γ, minimizing the false alarm rate is asymptotically equivalent to minimizing $\sigma_N^2(\rho)$ over ρ. However, the expression of $\sigma_N^2(\rho)$ depends on the covariance matrix C_N which is unknown to the array and therefore does not allow for an immediate online choice of an appropriate ρ. To tackle this problem, the following proposition provides a consistent estimate for $\sigma_N^2(\rho)$ based on $\hat{C}_N(\rho)$ and p.

[2]That is the normalized counting measure of the eigenvalues.

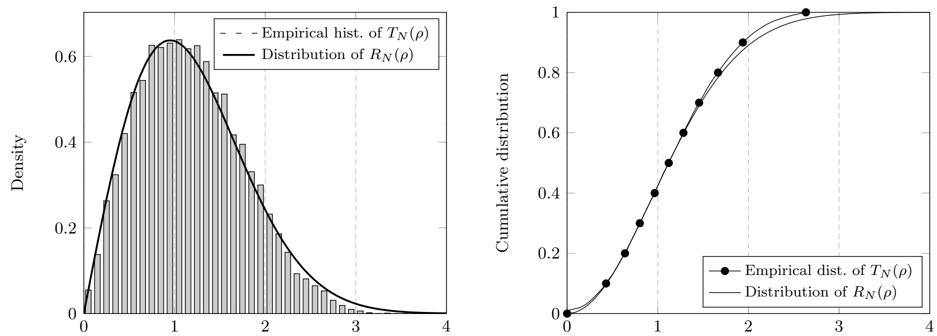


Figure 6.1: Histogram and distribution function of the $\sqrt{N}T_N(\rho)$ versus $R_N(\rho)$, $N = 20$, $p = N^{-\frac{1}{2}}[1, \ldots, 1]^\mathsf{T}$, $[C_N]_{ij} = 0.7^{|i-j|}$, $c_N = 1/2$, $\rho = 0.2$.

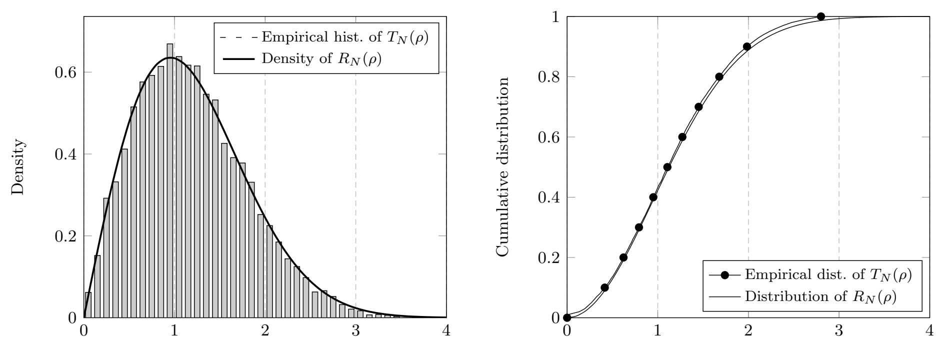


Figure 6.2: Histogram and distribution function of the $\sqrt{N}T_N(\rho)$ versus $R_N(\rho)$, $N = 100$, $p = N^{-\frac{1}{2}}[1, \ldots, 1]^\mathsf{T}$, $[C_N]_{ij} = 0.7^{|i-j|}$, $c_N = 1/2$, $\rho = 0.2$.

Proposition 6.2.1 (Empirical performance estimation)**.** *For $\rho \in (\max\{0, 1 - c_N^{-1}\}, 1)$, define*

$$\hat{\sigma}_N^2(\rho) \triangleq \frac{1}{2} \frac{1 - \rho \frac{p^* \hat{C}_N^{-2}(\rho) p}{p^* \hat{C}_N^{-1}(\rho) p}}{(1 - c_N + c_N \rho)(1 - \rho)}.$$

Also let $\hat{\sigma}_N^2(1) \triangleq \lim_{\rho \uparrow 1} \hat{\sigma}_N^2(\rho) < \infty$ a.s. Then we have

$$\sup_{\rho \in \mathcal{R}_\kappa} \left| \sigma_N^2(\rho) - \hat{\sigma}_N^2(\rho) \right| \xrightarrow{\text{a.s.}} 0.$$

Since both the estimation of $\sigma_N^2(\rho)$ in Proposition 6.2.1 and the convergence in Theorem 6.2.1 are uniform over $\rho \in \mathcal{R}_\kappa$, we have the following result.

Corollary 6.1 (Empirical performance optimum)**.** *Let $\hat{\sigma}_N^2(\rho)$ be defined as in Proposition 6.2.1 and define $\hat{\rho}_N^*$ as any value satisfying*

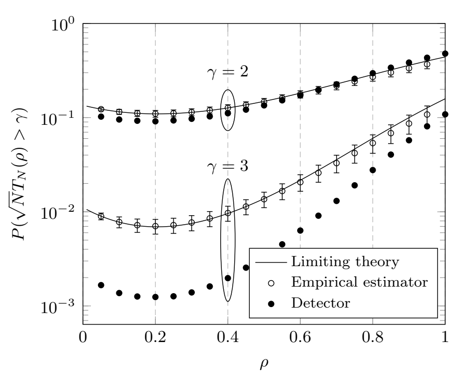
$$\hat{\rho}_N^* \in \operatorname{argmin}_{\rho \in \mathcal{R}_\kappa} \left\{ \hat{\sigma}_N^2(\rho) \right\}$$

(this set being in general a singleton). Then, for every $\gamma > 0$,

$$P\left(\sqrt{N}T_N(\hat{\rho}_N^*) > \gamma\right) - \inf_{\rho \in \mathcal{R}_\kappa} \left\{ P\left(\sqrt{N}T_N(\rho) > \gamma\right)\right\} \to 0.$$

This last result states that, for N, n sufficiently large, it is increasingly close-to-optimal to use the detector $T_N(\hat{\rho}_N^*)$ in order to reach minimal false alarm probability. A practical graphical confirmation of this fact is provided in Figure 6.3 where, in the same scenario as in Figures 6.1–6.2, the false alarm rates for various values of γ are depicted. In this figure, the black dots correspond to the actual values taken by $P(\sqrt{N}T_N(\rho) > \gamma)$ empirically obtained out of 10^6 Monte Carlo simulations. The plain curves are the approximating values $\exp(-\gamma^2/(2\sigma_N(\rho)^2))$. Finally, the white dots with error bars correspond to the mean and standard deviations of $\exp(-\gamma^2/(2\hat{\sigma}_N(\rho)^2))$ for each ρ, respectively. It is first interesting to note that the estimates $\hat{\sigma}_N(\rho)$ are quite accurate, especially so for N large, with standard deviations sufficiently small to provide good estimates, already for small N, of the false alarm minimizing ρ. However, similar to Figures 6.1–6.2, we observe a particularly weak approximation in the (small) $N = 20$ setting for large values of γ, corresponding to tail events, while for $N = 100$, these values are better recovered. This behavior is obviously explained by the fact that $\gamma = 3$ is not small compared to $\sqrt{N}$ when $N = 20$.

Nonetheless, from an error rate viewpoint, it is observed that errors of order 10^{-2} are rather well approximated for $N = 100$. In Figure 6.4, we consider this observation in depth by displaying $P(T_N(\hat{\rho}_N^*) > \Gamma)$ and its approximation $\min_\rho \exp(-N\Gamma^2/(2\sigma_N^2(\rho)))$ for $N = 20$ and $N = 100$, for various values of Γ. This figures shows that even errors of order 10^{-4} are well approximated for large N, while only errors of order 10^{-2} can be evaluated for small N.[3]



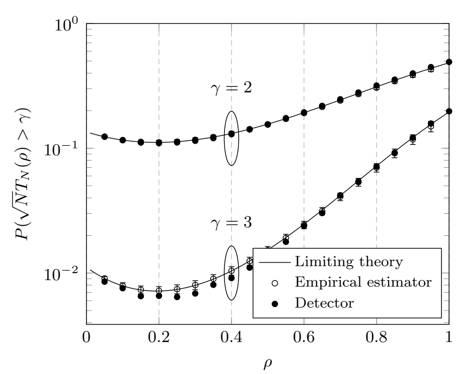


Figure 6.3: False alarm rate $P(\sqrt{N}T_N(\rho) > \gamma)$, $N = 20$ (left), $N = 100$ (right), $p = N^{-\frac{1}{2}}[1, \ldots, 1]^\mathsf{T}$, $[C_N]_{ij} = 0.7^{|i-j|}$, $c_N = 1/2$.

[3] Note that a comparison against alternative algorithms that would use no shrinkage (i.e., by setting $\rho = 0$) or that would not implement a robust estimate is not provided here, being of little relevance. Indeed, a proper selection of c_N to a large value or C_N with condition number close to one would provide an arbitrarily large gain of shrinkage-based methods, while an arbitrarily heavy-tailed choice of the τ_i distribution would provide a huge performance gain for robust methods. It is therefore not possible to compare such methods on fair grounds.

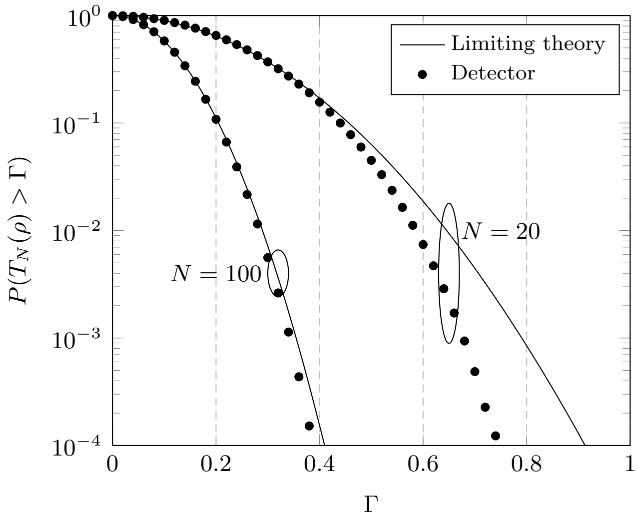


Figure 6.4: False alarm rate $P(T_N(\hat{\rho}_N^*) > \Gamma)$ for $N = 20$ and $N = 100$, $p = N^{-\frac{1}{2}}[1, \dots, 1]^\mathsf{T}$, $[C_N]_{ij} = 0.7^{|i-j|}$, $c_N = 1/2$.

6.2.1 Proofs

6.2.1.1 Fluctuations of the GLRT detector

This section is devoted to the proof of Theorem 6.2.1, which shall fundamentally rely on Theorem 6.1.1. The proof will be established in two steps. First, we shall prove the convergence for each $\rho \in \mathcal{R}_\kappa$, which we then generalize to the uniform statement of the theorem.

Let us then fix $\rho \in \mathcal{R}_\kappa$ for the moment. In anticipation of the eventual replacement of $\hat{C}_N(\rho)$ by $\underline{\hat{S}}_N(\underline{\rho})$, we start by studying the fluctuations of the bilinear forms involved in $T_N(\rho)$ but with $\hat{C}_N(\rho)$ replaced by $\underline{\hat{S}}_N(\underline{\rho})$ (note that $T_N(\rho)$ remains constant when scaling $\hat{C}_N(\rho)$ by any constant, so that replacing $\hat{C}_N(\rho)$ by $\underline{\hat{S}}_N(\underline{\rho})$ instead of by $\underline{\hat{S}}_N(\underline{\rho}) \cdot \frac{1}{N} \operatorname{tr} \hat{S}_N(\rho)$ as one would expect comes with no effect).

Our first goal is to show that the vector $\sqrt{N}(\Re[y^* \underline{\hat{S}}_N^{-1}(\underline{\rho}) p], \Im[y^* \underline{\hat{S}}_N^{-1}(\underline{\rho}) p])$ is asymptotically well approximated by a zero mean Gaussian vector with given covariance matrix. To this end, let us denote $A = [y \ p] \in \mathbb{C}^{N \times 2}$ and $Q_N = Q_N(\underline{\rho}) = (I_N + (1-\underline{\rho}) m(-\underline{\rho}) C_N)^{-1}$. Then, from (Chapon et al., 2014, Lemma 5.3) (adapted to our current notations and normalizations), for any Hermitian $B \in \mathbb{C}^{2 \times 2}$ and for any $u \in \mathbb{R}$,

$$\begin{aligned} &\mathrm{E}\left[\exp\left(\imath \sqrt{N} u \operatorname{tr} B A^* \left[\underline{\hat{S}}_N(\underline{\rho})^{-1} - \frac{1}{\underline{\rho}} Q_N(\underline{\rho})\right] A\right) \;\Big|\; y\right] \\ &= \exp\left(-\frac{1}{2} u^2 \Delta_N^2(B; y; p)\right) + O(N^{-\frac{1}{2}}) \end{aligned} \tag{6.5}$$

where we denote by $\mathrm{E}[\cdot|y]$ the conditional expectation with respect to the random vector y and where

$$\Delta_N^2(B;y;p) \triangleq \frac{cm(-\underline{\rho})^2(1-\underline{\rho})^2 \operatorname{tr}\left(ABA^* C_N Q_N^2(\underline{\rho})\right)^2}{\underline{\rho}^2\left(1-cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2 Q_N^2(\underline{\rho})\right)}.$$

Also, we have from classical central limit results on Gaussian random variables

$$\mathrm{E}\left[\exp\left(\imath\sqrt{N}u \operatorname{tr} B\left[A^* Q_N(\underline{\rho})A - \Gamma_N\right]\right)\right] = \exp\left(-\frac{1}{2}u^2\Delta_N'^2(B;p)\right) + O(N^{-\frac{1}{2}})$$

where

$$\Gamma_N \triangleq \frac{1}{\underline{\rho}}\begin{bmatrix} \frac{1}{N}\operatorname{tr} C_N Q_N(\underline{\rho}) & 0 \\ 0 & p^* Q_N(\underline{\rho})p \end{bmatrix}$$

$$\Delta_N'^2(B;p) \triangleq \frac{B_{11}^2}{\underline{\rho}^2}\frac{1}{N}\operatorname{tr} C_N^2 Q_N^2(\underline{\rho}) + \frac{2|B_{12}|^2}{\underline{\rho}^2} p^* C_N Q_N^2(\underline{\rho})p.$$

Besides, the $O(N^{-\frac{1}{2}})$ terms in the right-hand side of (6.5) remains $O(N^{-\frac{1}{2}})$ under expectation over y (for this, see the proof of (Chapon et al., 2014, Lemma 5.3)).

Altogether, we then have

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(\imath\sqrt{N}u \operatorname{tr} B\left[A^* \hat{\underline{S}}_N^{-1}(\underline{\rho})A - \Gamma_N\right]\right)\right] \\
&= \mathrm{E}\left[\exp\left(-\frac{1}{2}u^2\Delta_N^2(B;y;p)\right)\right]\exp\left(-\frac{1}{2}u^2\Delta_N'^2(B;p)\right) + O(N^{-\frac{1}{2}}).
\end{aligned}$$

Note now that

$$A^* C_N Q_N^2(\underline{\rho})A - \Upsilon_N \xrightarrow{\text{a.s.}} 0$$

where

$$\Upsilon_N \triangleq \begin{bmatrix} \frac{1}{N}\operatorname{tr} C_N^2 Q_N^2(\underline{\rho}) & 0 \\ 0 & p^* C_N Q_N^2(\underline{\rho})p \end{bmatrix}$$

so that, by dominated convergence, we obtain

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(\imath\sqrt{N}u \operatorname{tr} B\left[A^* \hat{\underline{S}}_N^{-1}(\underline{\rho})A - \Gamma_N\right]\right)\right] \\
&= \exp\left(-\frac{1}{2}u^2\left[\Delta_N^2(B;p) + \Delta_N'^2(B;p)\right]\right) + o(1)
\end{aligned}$$

where we defined

$$\Delta_N^2(B;p) \triangleq \frac{cm(-\underline{\rho})^2(1-\underline{\rho})^2 \operatorname{tr}\left(B\Upsilon_N\right)^2}{\underline{\rho}^2\left(1-cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2 Q_N^2(\underline{\rho})\right)}.$$

By a generalized Lévy's continuity theorem argument (see e.g. (Hachem et al., 2008a, Proposition 6)) and the Cramer-Wold device, we conclude that

$$\sqrt{N}\left(y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})y, \Re[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p], \Im[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p]\right) - Z_N = o_P(1)$$

where Z_N is a Gaussian random vector with mean and covariance matrix prescribed by the above approximation of $\sqrt{N}\operatorname{tr} BA^*\underline{\hat{S}}_N^{-1}A$ for each Hermitian B. In particular, taking $B_1 \in \left\{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{\imath}{2} \\ -\frac{\imath}{2} & 0 \end{bmatrix}\right\}$ to retrieve the asymptotic variances of $\sqrt{N}\Re[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p]$ and $\sqrt{N}\Im[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p]$, respectively, gives

$$\Delta_N^2(B_1;p) = \frac{1}{2\underline{\rho}^2}p^*C_NQ_N^2(\underline{\rho})p\frac{cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2Q_N^2(\underline{\rho})}{1-cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2Q_N^2(\underline{\rho})}$$

$$\Delta_N'^2(B_1;p) = \frac{1}{2\underline{\rho}^2}p^*C_NQ_N^2(\underline{\rho})p$$

and thus $\sqrt{N}(\Re[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p], \Im[y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p])$ is asymptotically equivalent to a Gaussian vector with zero mean and covariance matrix

$$(\Delta_N^2(B_1;p) + \Delta_N'^2(B_1;p))I_2 = \frac{1}{2\underline{\rho}^2}\frac{p^*C_NQ_N^2(\underline{\rho})p}{1-cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2Q_N^2(\underline{\rho})}I_2.$$

We are now in position to apply Theorem 6.1.1. Reminding that $\hat{S}_N^{-1}(\rho)(\rho+\frac{1}{\gamma_N(\rho)}\frac{1-\rho}{1-(1-\rho)c}) = \underline{\hat{S}}_N^{-1}(\underline{\rho})$, we have by Theorem 6.1.1 for $k=-1$

$$\sqrt{N}A^*\left[\hat{C}_N^{-1}(\rho) - \frac{\underline{\hat{S}}_N(\underline{\rho})^{-1}}{\rho+\frac{1}{\gamma_N(\rho)}\frac{1-\rho}{1-(1-\rho)c}}\right]A \xrightarrow{\text{a.s.}} 0.$$

Since almost sure convergence implies weak convergence, $\sqrt{N}A^*\hat{C}_N^{-1}(\rho)A$ has the same asymptotic fluctuations as $\sqrt{N}A^*\underline{\hat{S}}_N^{-1}(\underline{\rho})A/(\frac{1}{N}\operatorname{tr}\hat{S}_N(\rho))$. Also, as $T_N(\rho)$ remains identical when scaling $\hat{C}_N^{-1}(\rho)$ by $\frac{1}{N}\operatorname{tr}\hat{S}_N(\rho)$, only the fluctuations of $\sqrt{N}A^*\underline{\hat{S}}_N^{-1}(\underline{\rho})A$ are of interest, which were previously derived. We then finally conclude by the delta method (or more directly by Slutsky's lemma) that

$$\sqrt{\frac{N}{y^*\hat{C}_N^{-1}(\rho)yp^*\hat{C}_N^{-1}(\rho)p}}\begin{bmatrix}\Re\left[y^*\hat{C}_N^{-1}(\rho)p\right] \\ \Im\left[y^*\hat{C}_N^{-1}(\rho)p\right]\end{bmatrix} - \sigma_N(\rho)Z' = o_P(1)$$

for some $Z' \sim \mathcal{N}(0, I_2)$ and

$$\sigma_N^2(\rho) \triangleq \frac{1}{2}\frac{p^*C_NQ_N^2(\underline{\rho})p}{p^*Q_N(\underline{\rho})p\cdot\frac{1}{N}\operatorname{tr} C_NQ_N(\underline{\rho})\cdot\left(1-cm(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr} C_N^2Q_N^2(\underline{\rho})\right)}.$$

It unfolds that, for $\gamma > 0$,

$$P\left(T_N(\rho) > \frac{\gamma}{\sqrt{N}}\right) - \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho)}\right) \to 0 \tag{6.6}$$

as desired.

The second step of the proof is to generalize (6.6) to uniform convergence across $\rho \in \mathcal{R}_\kappa$. To this end, somewhat similar to above, we shall transfer the distribution $P(\sqrt{N}T_N(\rho) > \gamma)$ to $P(\sqrt{N}\underline{T}_N(\rho) > \gamma)$ by exploiting the uniform convergence of Theorem 6.1.1, where we defined

$$\underline{T}_N(\rho) \triangleq \frac{\left|y^* \underline{\hat{S}}_N^{-1}(\underline{\rho})p\right|}{\sqrt{y^* \underline{\hat{S}}_N^{-1}(\underline{\rho})y}\sqrt{p^* \underline{\hat{S}}_N^{-1}(\underline{\rho})p}}$$

and exploit a ρ-Lipschitz property of $\sqrt{N}\underline{T}_N(\rho)$ to reduce the uniform convergence over $\mathcal{R}_\kappa$ to a uniform convergence over finitely many values of ρ.

The ρ-Lipschitz property we shall need is as follows: for each $\varepsilon > 0$

$$\lim_{\delta\to 0}\lim_{N\to\infty} P\left(\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa\\ |\rho-\rho'|<\delta}} \sqrt{N}\left|T_N(\rho) - T_N(\rho')\right| > \varepsilon\right) = 0. \tag{6.7}$$

Let us prove this result. By Theorem 6.1.1, since almost sure convergence implies convergence in distribution, we have

$$P\left(\sup_{\rho\in\mathcal{R}_\kappa} \sqrt{N}\left|T_N(\rho) - \underline{T}_N(\rho)\right| > \varepsilon\right) \to 0.$$

Applying this result to (6.7) induces that it is sufficient to prove (6.7) for $\underline{T}_N(\rho)$ in place of $T_N(\rho)$. Let $\eta > 0$ small and $\mathcal{A}_N^\eta \triangleq \{\exists\underline{\rho} \in \mathcal{R}_\kappa, y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})yp^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p < \eta\}$. Developing the difference $\underline{T}_N(\rho) - \underline{T}_N(\rho')$ and isolating the denominator according to its belonging to $\mathcal{A}_N^\eta$ or not, we may write

$$\begin{aligned}
&P\left(\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa\\ |\rho-\rho'|<\delta}} \sqrt{N}\left|\underline{T}_N(\rho) - \underline{T}_N(\rho')\right| > \varepsilon\right)\\
&\leq P\left(\mathcal{A}_N^\eta\right) + P\left(\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa\\ |\rho-\rho'|<\delta}} \sqrt{N}V_N(\rho,\rho') > \varepsilon\eta\right)
\end{aligned}$$

where

$$\begin{aligned}
V_N(\rho,\rho') \triangleq{}& \left|y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p\right| \sqrt{y^*\underline{\hat{S}}_N^{-1}(\underline{\rho}')y}\sqrt{p^*\underline{\hat{S}}_N^{-1}(\underline{\rho}')p}\\
&- \left|y^*\underline{\hat{S}}_N^{-1}(\underline{\rho}')p\right| \sqrt{y^*\underline{\hat{S}}_N^{-1}(\underline{\rho})y}\sqrt{p^*\underline{\hat{S}}_N^{-1}(\underline{\rho})p}.
\end{aligned}$$

From classical random matrix results, $P(\mathcal{A}_N^\eta) \to 0$ for a sufficiently small choice of η. To prove that $\lim_\delta \limsup_n P(\sup_{|\rho-\rho'|<\delta} \sqrt{N} V_N(\rho,\rho') > \varepsilon\eta) = 0$, it is then sufficient to show that

$$\lim_{\delta\to 0} \limsup_n P\left(\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa \\ |\rho-\rho'|<\delta}} \sqrt{N} |y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} p - y^* \underline{\hat{S}}_N(\underline{\rho}')^{-1} p| > \varepsilon' \right) = 0 \tag{6.8}$$

for any $\varepsilon' > 0$ and similarly for $y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} y - y^* \underline{\hat{S}}_N(\underline{\rho}')^{-1} y$ and $p^* \underline{\hat{S}}_N(\underline{\rho})^{-1} p - p^* \underline{\hat{S}}_N(\underline{\rho}')^{-1} p$. Let us prove (6.8), the other two results following essentially the same line of arguments. For this, by (Kallenberg, 2002, Corollary 16.9) (see also (Billingsley, 1968, Theorem 12.3)), it is sufficient to prove, say

$$\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa \\ \rho\neq\rho'}} \sup_n \frac{\mathrm{E}\left[\sqrt{N} \left| y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} p - y^* \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \right|^2 \right]}{|\rho-\rho'|^2} < \infty.$$

But then, remarking that

$$\begin{aligned} &\sqrt{N} y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} p - y^* \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \\ &= (\underline{\rho}' - \underline{\rho}) \sqrt{N} y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} \left(I_N - \frac{1}{n} \sum_{i=1}^n z_i z_i^* \right) \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \end{aligned}$$

this reduces to showing that

$$\sup_{\rho,\rho'\in\mathcal{R}_\kappa} \sup_n \mathrm{E}\left[N \left| y^* \underline{\hat{S}}_N(\underline{\rho})^{-1} \left(I_N - \frac{1}{n} \sum_{i=1}^n z_i z_i^* \right) \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \right|^2 \right] < \infty.$$

Conditioning first on $z_1, \ldots, z_n$, this further reduces to showing

$$\sup_{\rho,\rho'\in\mathcal{R}_\kappa} \sup_n \mathrm{E}\left[\left\| \underline{\hat{S}}_N(\underline{\rho})^{-1} \left(I_N - \frac{1}{n} \sum_{i=1}^n z_i z_i^* \right) \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \right\|^2 \right] < \infty.$$

But this is yet another standard random matrix result, obtained e.g., by noticing that

$$\left\| \underline{\hat{S}}_N(\underline{\rho})^{-1} \left(I_N - \frac{1}{n} \sum_{i=1}^n z_i z_i^* \right) \underline{\hat{S}}_N(\underline{\rho}')^{-1} p \right\|^2 \leq \frac{1}{\kappa^4} \left\| I_N - \frac{1}{n} \sum_{i=1}^n z_i z_i^* \right\|^2$$

which remains of uniformly finite expectation (left norm is vector Euclidean norm, right norm is matrix spectral norm). This completes the proof of (6.7).

Getting back to our original problem, let us now take $\varepsilon > 0$ arbitrary, $\rho_1 < \ldots < \rho_K$ be a regular sampling of $\mathcal{R}_\kappa$, and $\delta = 1/K$. Then by (6.6), K being fixed, for all $n > n_0(\varepsilon)$,

$$\max_{1\leq k\leq K} \left| P\left(T_N(\rho_i) > \frac{\gamma}{\sqrt{N}} \right) - \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho_i)} \right) \right| < \varepsilon. \tag{6.9}$$

Also, from (6.7), for small enough δ,

$$\begin{aligned}
\max_{1\leq k\leq K} P&\left(\sup_{\substack{\rho\in\mathcal{R}_\kappa\\|\rho-\rho_k|<\delta}}\sqrt{N}|T_N(\rho)-T_N(\rho_k)|>\gamma\zeta\right)\\
&\leq P\left(\sup_{\substack{\rho,\rho'\in\mathcal{R}_\kappa\\|\rho-\rho'|<\delta}}\sqrt{N}|T_N(\rho)-T_N(\rho')|>\gamma\zeta\right)\\
&<\varepsilon
\end{aligned}$$

for all large $n>n_0'(\varepsilon,\zeta)>n_0(\varepsilon)$ where $\zeta>0$ is also taken arbitrarily small. Thus we have, for each $\rho\in\mathcal{R}_\kappa$ and for $n>n_0'(\varepsilon,\zeta)$

$$\begin{aligned}
P\left(T_N(\rho)>\frac{\gamma}{\sqrt{N}}\right)&\leq P\left(T_N(\rho_i)>\frac{\gamma(1-\zeta)}{\sqrt{N}}\right)+P\left(\sqrt{N}|T_N(\rho)-T_N(\rho_i)|>\gamma\zeta\right)\\
&\leq P\left(T_N(\rho_i)>\frac{\gamma(1-\zeta)}{\sqrt{N}}\right)+\varepsilon
\end{aligned}$$

for $i\leq K$ the unique index such that $|\rho-\rho_i|<\delta$ and where the inequality holds uniformly on $\rho\in\mathcal{R}_\kappa$. Similarly, reversing the roles of ρ and ρ_i,

$$P\left(T_N(\rho)>\frac{\gamma}{\sqrt{N}}\right)\geq P\left(T_N(\rho_i)>\frac{\gamma(1+\zeta)}{\sqrt{N}}\right)-\varepsilon.$$

As a consequence, by (6.9), for $n>n_0'(\varepsilon,\zeta)$, uniformly on $\rho\in\mathcal{R}_\kappa$,

$$\begin{aligned}
P\left(T_N(\rho)>\frac{\gamma}{\sqrt{N}}\right)&\leq\exp\left(-\frac{\gamma^2(1-\zeta)^2}{2\sigma_N^2(\rho_i)}\right)+2\varepsilon\\
P\left(T_N(\rho)>\frac{\gamma}{\sqrt{N}}\right)&\geq\exp\left(-\frac{\gamma^2(1+\zeta)^2}{2\sigma_N^2(\rho_i)}\right)-2\varepsilon
\end{aligned}$$

which, by continuity of the exponential and of $\rho\mapsto\sigma_N(\rho)$,[4] letting ζ and δ small enough (up to growing $n_0'(\varepsilon,\zeta)$), leads to

$$\sup_{\rho\in\mathcal{R}_\kappa}\left|P\left(\sqrt{N}T_N(\rho)>\gamma\right)-\exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho)}\right)\right|\leq 3\varepsilon$$

for all $n>n_0'(\varepsilon,\zeta)$, which completes the proof.

6.2.1.2 Around empirical estimates

This section is dedicated to the proof of Proposition 6.2.1 and Corollary 6.1.

[4] Note that it is unnecessary to ensure $\liminf_N\sigma_N(\rho)>0$ as the exponential would tend to zero anyhow in this scenario.

We start by showing that $\hat{\sigma}_N^2(1)$ is well defined. It is easy to observe that the ratio defining $\hat{\sigma}_N^2(\rho)$ converges to an undetermined form (zero over zero) as $\rho \uparrow 1$. Applying l'Hospital's rule to the ratio, using the differentiation $\frac{d}{d\underline{\rho}}\underline{\hat{S}}_N^{-1}(\underline{\rho}) = -\underline{\hat{S}}_N^{-2}(\underline{\rho})(I_N - \frac{1}{n}\sum_i z_i z_i^*)$ and the limit $\underline{\hat{S}}_N^{-1}(\underline{\rho}) \to I_N$ as $\rho \uparrow 1$, we end up with

$$\hat{\sigma}_N^2(\rho) \to \frac{1}{2}\frac{p^*\left(\frac{1}{n}\sum_{i=1}^n z_i z_i^*\right)p}{\frac{1}{N}\operatorname{tr}\left(\frac{1}{n}\sum_{i=1}^n z_i z_i^*\right)}.$$

Letting $\varepsilon, \kappa > 0$ small, since $p^*\frac{1}{n}\sum_i z_i z_i^* p - p^* C_N p \xrightarrow{\text{a.s.}} 0$, $\frac{1}{N}\operatorname{tr}\frac{1}{n}\sum_i z_i z_i^* \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$, we immediately have, by continuity of both $\sigma_N^2(\rho)$ and $\hat{\sigma}_N^2(\rho)$,

$$\sup_{\rho \in (1-\kappa, 1]} \left|\hat{\sigma}_N^2(\rho) - \sigma_N^2(\rho)\right| \le \varepsilon$$

for all large n almost surely. From now on, it then suffices to prove Proposition 6.2.1 on the complementary set $\mathcal{R}'_\kappa \triangleq [\kappa + \min\{0, 1 - c^{-1}\}, 1 - \kappa]$. For this, we first recall the following results borrowed from Chapter 5 with slightly updated notations. First, we have

$$\sup_{\rho \in \mathcal{R}_\kappa} \left\| \frac{\hat{C}_N(\rho)}{\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)} - \underline{\hat{S}}_N(\underline{\rho}) \right\| \xrightarrow{\text{a.s.}} 0.$$

Also, for $z \in \mathbb{C} \setminus \mathbb{R}_+$, defining

$$\underline{\underline{\hat{S}}}_N(z) \triangleq (1 - \underline{\rho})\frac{1}{n}\sum_{i=1}^n z_i z_i^* - z I_N$$

(so in particular $\underline{\underline{\hat{S}}}_N(-\underline{\rho}) = \underline{\hat{S}}_N(\underline{\rho})$, for all $\underline{\rho} \in \mathcal{R}_\kappa$), we have, with $\mathcal{C}$ a compact set of $\mathbb{C} \setminus \mathbb{R}_+$ and any integer k,

$$\sup_{\bar{z} \in \mathcal{C}} \left| \frac{d^k}{dz^k}\left\{ \frac{1}{N}\operatorname{tr}\underline{\underline{\hat{S}}}_N^{-1}(z) - \frac{1}{N}\operatorname{tr}\left(-z\left[I_N + (1-\underline{\rho})m_N(z)C_N\right]\right)^{-1} \right\}_{z=\bar{z}} \right| \xrightarrow{\text{a.s.}} 0$$

$$\sup_{\bar{z} \in \mathcal{C}} \left| \frac{d^k}{dz^k}\left\{ p^*\underline{\underline{\hat{S}}}_N^{-1}(z)p - p^*\left(-z\left[I_N + (1-\underline{\rho})m_N(z)C_N\right]\right)^{-1} p \right\}_{z=\bar{z}} \right| \xrightarrow{\text{a.s.}} 0$$

where $m_N(z)$ is defined as the unique solution with positive (resp. negative) imaginary part if $\Im[z] > 0$ (resp. $\Im[z] < 0$) or unique positive solution if $z < 0$ of

$$m_N(z) = \left(-z + c\int \frac{(1-\underline{\rho})t}{1 + (1-\underline{\rho})t m_N(z)}\nu_N(dt)\right)^{-1}.$$

This expression of $m_N(z)$ can be rewritten in the more practical form

$$\begin{aligned} m_N(z) &= -\frac{1-c}{z} + c\int \frac{\nu_N(dt)}{-z - z(1-\underline{\rho})t m_N(z)} \\ &= -\frac{1-c}{z} + c\frac{1}{N}\operatorname{tr}\left(-z\left[I_N + (1-\underline{\rho})m_N(z)C_N\right]\right)^{-1} \end{aligned}$$

so that, from the above relations

$$\sup_{\rho\in\mathcal{R}'_\kappa}\left|m_N(-\underline{\rho})-\left(\frac{1-c_N}{\underline{\rho}}+c_N\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}(\rho)\cdot\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)\right)\right|\xrightarrow{\text{a.s.}}0$$
$$\sup_{\rho\in\mathcal{R}'_\kappa}\left|\int\frac{t\nu_N(dt)}{1+(1-\underline{\rho})m_N(-\underline{\rho})t}-\frac{1-\underline{\rho}\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}(\rho)\cdot\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)}{(1-\underline{\rho})m_N(-\underline{\rho})}\right|\xrightarrow{\text{a.s.}}0.$$

Differentiating along z the first expression of $m_N(z)$, we also recall that

$$m_N'(z)=\frac{m_N^2(z)}{1-c\int\frac{m_N(z)^2(1-\underline{\rho}^2)t^2\nu_N(dt)}{(1-(1-\underline{\rho})tm_N(-\underline{\rho}))^2}}.$$

Now, remark that

$$p^*\underline{\hat{S}}_N(-\underline{\rho})^{-2}p=\frac{d}{dz}\left[p^*\underline{\hat{S}}_N(z)^{-1}p\right]_{z=-\underline{\rho}}$$

which (by analyticity) is uniformly well approximated by

$$\begin{aligned}&\frac{d}{dz}\left[p^*\left(-z\left[I_N+(1-\underline{\rho})m_N(z)C_N\right]\right)^{-1}p\right]_{z=-\underline{\rho}}\\
&=\frac{1}{\underline{\rho}^2}p^*Q_N(\underline{\rho})p-\frac{1}{\underline{\rho}}(1-\underline{\rho})m_N'(-\underline{\rho})p^*C_NQ_N^2(\underline{\rho})p\\
&=\frac{1}{\underline{\rho}^2}p^*Q_N(\underline{\rho})p-\frac{1}{\underline{\rho}}(1-\underline{\rho})\frac{m_N^2(-\underline{\rho})p^*C_NQ_N^2(\underline{\rho})p}{1-cm_N(-\underline{\rho})^2(1-\underline{\rho}^2)\frac{1}{N}\operatorname{tr}Q_N^2(\underline{\rho})}.\end{aligned}$$

(recall that $Q_N(\underline{\rho})=\left(I_N+(1-\underline{\rho})m_N(-\underline{\rho})C_N\right)^{-1}$). We then conclude

$$\sup_{\rho\in\mathcal{R}'_\kappa}\left|\frac{p^*C_NQ_N^2(\underline{\rho})p}{1-cm_N(-\underline{\rho})^2(1-\underline{\rho})^2\frac{1}{N}\operatorname{tr}Q_N^2(\underline{\rho})}\right.$$
$$\left.-\frac{p^*\hat{C}_N^{-1}(\rho)p\cdot\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)-\underline{\rho}p^*\hat{C}_N^{-2}(\rho)p\cdot\left(\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)\right)^2}{(1-\underline{\rho})m_N(-\underline{\rho})^2}\right|\xrightarrow{\text{a.s.}}0.$$

Putting all results together and remarking that $\frac{1}{N}\operatorname{tr}\hat{C}_N^{-1}(\rho)=1$ for all ρ and that $\frac{1}{N}\operatorname{tr}\hat{C}_N(\rho)\to\rho\underline{\rho}^{-1}$, we obtain the expected result.

It now remains to prove Corollary 6.1. This is easily performed thanks to Theorem 6.2.1 and Proposition 6.2.1. From these, we indeed have the three relations

$$P\left(\sqrt{N}T_N(\hat{\rho}_N^*)>\gamma\right)-\exp\left(-\frac{\gamma^2}{2\sigma_N^2(\hat{\rho}_N^*)}\right)\xrightarrow{\text{a.s.}}0$$
$$P\left(\sqrt{N}T_N(\rho_N^*)>\gamma\right)-\exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho_N^*)}\right)\to0$$
$$\exp\left(-\frac{\gamma^2}{2\sigma_N^2(\hat{\rho}_N^*)}\right)-\exp\left(-\frac{\gamma^2}{2\sigma_N^{*2}}\right)\xrightarrow{\text{a.s.}}0$$

where we denoted ρ_N^* any element in the argmin over ρ of $P(\sqrt{N}T_N(\rho) > \gamma)$ and σ_N^{*2} the minimum of $\sigma_N(\rho)$ (i.e. the minimizer for $\exp(-\frac{\gamma^2}{2\sigma_N(\rho)})$). Note that the first two relations rely fundamentally on the uniform convergence $\sup_{\rho\in\mathcal{R}_\kappa} |P\left(\sqrt{N}T_N(\rho) > \gamma\right) - \exp\left(-\gamma^2/(2\sigma_N^2(\rho))\right)| \xrightarrow{\text{a.s.}} 0$. By definition of ρ_N^* and σ_N^{*2}, we also have

$$\begin{aligned}
\exp\left(-\frac{\gamma^2}{2\sigma_N^{*2}}\right) &\leq \min\left\{\exp\left(-\frac{\gamma^2}{2\sigma_N^2(\hat{\rho}_N^*)}\right), \exp\left(-\frac{\gamma^2}{2\sigma_N^2(\rho_N^*)}\right)\right\} \\
P\left(\sqrt{N}T_N(\rho_N^*) > \gamma\right) &\leq P\left(\sqrt{N}T(\hat{\rho}_N^*) > \gamma\right).
\end{aligned}$$

Putting things together then gives

$$P\left(\sqrt{N}T(\hat{\rho}_N^*) > \gamma\right) - P\left(\sqrt{N}T_N(\rho_N^*) > \gamma\right) \xrightarrow{\text{a.s.}} 0$$

which is the expected result.

Chapter 7

Outlier rejection

In Chapters 4 to 6, our interest lay on the large dimensional analysis of robust M-estimators for elliptically distributed inputs. As discussed in the introductory Chapter 2, this is mainly motivated by two reasons. For one, elliptical distributions enclose a minimum degree of parametrization to adequately model various types of impulsive data behaviors. Secondly, among other classes of such easily parametrizable distributions, elliptical vectors derive naturally from (normalized) Gaussian vectors and are, as such, more amenable to probabilistic considerations. However, the elliptical distribution fails to encompass many impulsiveness scenarios of important practical use. This is mostly due to the data needing roughly to arise from a uniquely defined linear combination (through the matrix often denoted A or $C_N^{\frac{1}{2}}$) of independent Gaussian entries times a one-dimensional impulsion-providing variable (often denoted τ_i), which are too harsh constraints.

To meet the original intention of Huber to design M-estimators for their robustness to arbitrarily outlying data vectors (model inconsistent data, missing data, noise bursts, etc.), one should instead be capable of understanding the large dimensional behavior of $\hat{C}_N$ (and $\hat{C}_N(\rho)$, $\check{C}_N(\rho)$) built upon a proportion of model-fitting vectors and a complementary proportion of deterministic but unknown outliers. This is the objective of the present chapter. What we shall precisely consider here is a scenario where Maronna's M-estimators of scatter are constructed from a given amount (in practice a majority) of i.i.d. zero-mean and covariance C_N Gaussian vectors and a complementary (usually small) amount of deterministic vectors $a_1, a_1, \ldots$. We will show, by a large random matrix approach similar to previous chapters, that the quadratic form $\frac{1}{N} a_i^* C_N^{-1} a_i$ is at the core of the outlier rejection performance of $\hat{C}_N$. By a simple analysis of the theoretical results, we shall understand in particular why the function $u_{\mathrm{H}}(x)$ proposed originally by Huber is (in some sense to be defined later) optimal among the class of Maronna's $u(x)$ functions to ensure proper outlier rejection.

We start by introducing our assumptions and main results before providing some elements of proof.

7.1 Main results

Let $\varepsilon_n \in \{k/n \mid k = 0, \ldots, n\}$ and $X \in \mathbb{C}^{N \times n}$ be the concatenation matrix of the n successive observations

$$X = \left[x_1, \ldots, x_{(1-\varepsilon_n)n}, a_1, \ldots, a_{\varepsilon_n n}\right]$$

where $x_1, \ldots, x_{(1-\varepsilon_n)n} \in \mathbb{C}^N$ are random vectors with $x_i = C_N^{\frac{1}{2}} w_i$, $C_N \in \mathbb{C}^{N \times N}$ deterministic positive definite and $w_1, \ldots, w_{(1-\varepsilon_n)n}$ i.i.d. random with i.i.d. zero mean and unit variance entries, whereas $a_1, \ldots, a_{\varepsilon_n n} \in \mathbb{C}^N$ are deterministic and such that

$$0 < \min_i \liminf_n \frac{\|a_i\|}{\sqrt{N}} \leq \max_i \limsup_n \frac{\|a_i\|}{\sqrt{N}} < \infty.$$

Under this model, the x_i's are model-fitting vectors while the a_i's represent unknown outliers. Note that we consider a particular data ordering (model-fitting data first and outliers last) which shall however be inconsequential in the remainder.

As usual, we further denote $c_N \triangleq N/n$ and shall make the following assumptions.

Assumption 7.1. *For each N, $C_N \succ 0$ and $\limsup_N \|C_N\| < \infty$.*

Assumption 7.2. *As $N, n \to \infty$, $c_N \to c$ and $\varepsilon_n \to \varepsilon \in [0, 1)$ with $0 < c < 1 - \varepsilon$.*

From Assumption 7.2, Maronna's M-estimator $\hat{C}_N$ for the column vectors of X is well-defined as the almost surely unique solution to the equation in Z

$$Z = \frac{1}{n} \sum_{i=1}^{(1-\varepsilon_n)n} u\left(\frac{1}{N} x_i^* Z^{-1} x_i\right) x_i x_i^* + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} u\left(\frac{1}{N} a_i^* Z^{-1} a_i\right) a_i a_i^* \tag{7.1}$$

where u is as usual defined on $[0, \infty)$, nonnegative, continuous and non-increasing, and such that $\phi(x) = xu(x)$ is increasing and bounded with $\lim_{x \to \infty} \phi(x) \triangleq \phi_\infty$, and $1 < \phi_\infty < c^{-1}$.

As in the previous chapters, our main objective is to find a large N, n random matrix equivalent for $\hat{C}_N$ which is more tractable and prone to analysis. This is the object of the following result, established in (Morales-Jimenez et al., 2015).

Theorem 7.1.1. *Let Assumptions 7.1–7.2 hold and let $\hat{C}_N$ be the almost sure unique solution to (7.1). Then, as $N, n \to \infty$,*

$$\left\| \hat{C}_N - \hat{S}_N \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^{(1-\varepsilon_n)n} v(\gamma_n) x_i x_i^* + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} v(\alpha_{i,n}) a_i a_i^*$$

with γ_n and $\alpha_{1,n}, \ldots, \alpha_{\varepsilon_n n,n}$ the unique positive solutions to the system of $\varepsilon_n n + 1$ equations $(i = 1, \ldots, \varepsilon_n n)$

$$\gamma_n = \frac{1}{N} \operatorname{tr} C_N \left(\frac{(1-\varepsilon)v_c(\gamma_n)}{1+cv_c(\gamma_n)\gamma_n} C_N + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} v\left(\alpha_{i,n}\right) a_i a_i^* \right)^{-1}$$

$$\alpha_{i,n} = \frac{1}{N} a_i^* \left(\frac{(1-\varepsilon)v_c(\gamma_n)}{1+cv_c(\gamma_n)\gamma_n} C_N + \frac{1}{n} \sum_{j\neq i}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^* \right)^{-1} a_i$$

and $v_c(x) = u\left(g^{-1}(x)\right)$, $g(x) = x/(1 - c\phi(x))$.

Remark that the approximation matrix $\hat{S}_N$ consists of two terms: a normalized sample covariance matrix and a weighted sum of the outlier outer products. These weights allow for an automated balancing between model-fitting data and outliers. To get some insight on the properties of $\hat{C}_N$ induced by these weights, let us consider the single-outlier case where $\varepsilon_n = 1/n \to 0$. We easily obtain by a rank-one perturbation argument that $\gamma_n \to \gamma$, where γ is the solution to $\gamma = (1 + cv_c(\gamma)\gamma)/v_c(\gamma)$. It can be seen, using the definition of v, that $\gamma = \phi^{-1}(1)/(1-c)$ and that, as a consequence, $v_c(\gamma) = 1/\phi^{-1}(1)$ (which corresponds to the result obtained in Chapter 4 in the absence of outliers). As for $\alpha_{1,n}$, it satisfies

$$\alpha_{1,n} = \left(\frac{\phi^{-1}(1)}{1-c} + o(1) \right) \frac{1}{N} a_1^* C_N^{-1} a_1.$$

As such, so long that $\liminf_n \frac{1}{N} a_1^* C_N^{-1} a_1 > 1$, $v_c(\alpha_{1,n}) \leq v_c(\gamma)$ for all large n and thus the impact of the outlier a_1 will be all the more attenuated that $\frac{1}{N} a_1^* C_N^{-1} a_1$ is large. However, if $\limsup_n \frac{1}{N} a_1^* C_N^{-1} a_1 < 1$, then $v_c(\alpha_{1,n}) \geq v_c(\gamma)$ for all large n and the impact of a_1 may be worsen. As such, we can readily make the following two important observations:

- to avoid enhancing the effect of outliers, $v_c(x)$ should be set to a constant for all $x \leq \frac{\phi^{-1}(1)}{1-c}$, or equivalently $u(x)$ should be constant for $x \leq \phi^{-1}(1)$. A particular example of such a choice is $u(x) = \min\{1, (1+t)/(t+x)\}$ for some $t > 0$, which is (almost) the original Huber function u_{H} for the estimator (2.1), where $t = 0$.[1]
- if $\lambda_1(C_N)$ and $\lambda_N(C_N)$ remain close to one, the norm of a_1 dictates most of the relative outlier impact. For C_N departing from the identity, a good rejection to outliers is expected if a_1 is not aligned to the dominant eigenvectors of C_N. On the opposite, if a_1 is to be aligned to the dominant eigenmodes of C_N, the outlier rejection would be compromised.

Other considerations are easily made. In particular, if $a_1 = \ldots = a_{\varepsilon_n n}$, it is easily seen that, as $\varepsilon_n n$ grows, the outlier-rejection gain brought by the possibly large quadratic form $\frac{1}{N} a_1^* C_N^{-1} a_1$ is quickly overrun so that, if $\varepsilon > 0$ and $\limsup_N \frac{1}{N} a_1^* C_N^{-1} a_1 < \infty$, the $\alpha_{i,n}$'s converge jointly to zero and the outliers will no longer be rejected. This scenario is all the more problematic that for ε not too small, $v_c(\gamma_n)$ may become much smaller than $v_c(\alpha_{i,n})$, which would drive the robust estimator to a starkly opposite behavior than expected.

[1] Recall that taking $t = 0$ does not ensure the uniqueness of the solution to (7.1).

Specifying scenarios more involved than $a_1 = \ldots = a_{\varepsilon_n n}$ leads to more fixed point equations and thus to a less tractable understanding. It is however sensible to assume that the outliers are themselves random and likely to differ from one observation to the next. A natural assumption that maintains a high level of simplicity is to take the outliers to be i.i.d. Gaussian distributed with zero mean and covariance matrix D_N different from C_N. This gives the following corollary to Theorem 7.1.1.

Corollary 7.1. *Let Assumptions 7.1–7.2 hold and let $a_1, \ldots, a_{\varepsilon_n n}$ be random with $a_i = D_N^{\frac{1}{2}} \tilde{w}_i$, where $D_N \in \mathbb{C}^{N \times N}$ is Hermitian positive definite with $\limsup_N \|D_N\| < \infty$ and $\hat{w}_1, \ldots, \hat{w}_{\varepsilon_n n}$ are independent with i.i.d. zero mean and unit variance entries. Then, as $N, n \to \infty$,*

$$\left\| \hat{C}_N - \hat{S}_N^{\mathrm{rnd}} \right\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N^{\mathrm{rnd}} \triangleq \frac{1}{n} \sum_{i=1}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) x_i x_i^* + \frac{1}{n} \sum_{i=1}^{\varepsilon_n n} v\left(\alpha_n\right) a_i a_i^*$$

with γ_n and α_n the unique positive solutions to

$$\begin{aligned} \gamma_n &= \frac{1}{N} \operatorname{tr} C_N \left(\frac{(1-\varepsilon) v_c(\gamma_n)}{1 + c v_c(\gamma_n) \gamma_n} C_N + \frac{\varepsilon v_c(\alpha_n)}{1 + c v_c(\alpha_n) \alpha_n} D_N \right)^{-1} \\ \alpha_n &= \frac{1}{N} \operatorname{tr} D_N \left(\frac{(1-\varepsilon) v_c(\gamma_n)}{1 + c v_c(\gamma_n) \gamma_n} C_N + \frac{\varepsilon v_c(\alpha_n)}{1 + c v_c(\alpha_n) \alpha_n} D_N \right)^{-1}. \end{aligned}$$

In this scenario, $\hat{C}_N$ is equivalent to a weighted sum of two sample covariance matrices for the model-fitting against the outlying data. Again, it is interesting to study the regime where $\varepsilon = 0$. Under this assumption, we have $\gamma_n = \gamma$ where $v_c(\gamma) = 1/\phi^{-1}(1)$ similar to above and α_n is now exactly defined as

$$\alpha_n = \frac{\phi^{-1}(1)}{1-c} \frac{1}{N} \operatorname{tr} D_N C_N^{-1}.$$

The factor of importance is then here the trace $\frac{1}{N} \operatorname{tr} D_N C_N^{-1}$ which, if large, induces a decay in the outlier importance, and vice-versa. Note again that, for D_N and C_N of similar trace, it is of key importance that C_N be as distinct from I_N as possible for outlier rejection to be possible. Note also that, when seen as functions of ε, $\gamma_n(\varepsilon) \to \gamma$ and $\alpha_n(\varepsilon) \to \alpha_n$ continuously with $\varepsilon \to 0$, so that the predicted behavior for $\varepsilon = 0$ is a good approximation of the behavior for all small $\varepsilon > 0$.

We now provide simulation results that shed some more light on the conclusions drawn from Theorem 7.1.1 and Corollary 7.1.

Let us place ourselves first under the setting of Theorem 7.1.1. Taking $N = 100$, $n = 500$, we assume $[C_N]_{ij} = .9^{|i-j|}$ and let $\varepsilon_n n = 2$ with $a_1 = \mathbf{1}$, the N-dimensional vector of all-ones, and a_2 such that $[a_2]_k = \exp(\pi \imath k)$ (a steering vector at 30°). In this setting, $\frac{1}{N} a_1^* C_N^{-1} a_1 \simeq 0.06$ while $\frac{1}{N} a_2^* C_N^{-1} a_2 \simeq 19$. We compare the results obtained for $u_1(x) = (1+t)/(t+x)$ against $u_2(x) = \min\{1, (1+t)/(t+x)\}$ for $t = .1$ and denote v_1, v_2 their corresponding v_c functions.

Numerically, we obtain for the function u_1 the weights

$$\begin{aligned} v_1(\gamma_n) &\simeq .992 \\ v_1(\alpha_{1,n}) &\simeq 6.42 \\ v_1(\alpha_{2,n}) &\simeq .006. \end{aligned}$$

We thus observe an important attenuation of the second outlier, while the first outlier is strongly enhanced. Comparatively, for the Huber-like function u_2, we have

$$\begin{aligned} v_2(\gamma_n) &\simeq .984 \\ v_2(\alpha_{1,n}) &= 1.00 \\ v_2(\alpha_{2,n}) &\simeq .006. \end{aligned}$$

Thus, here, Huber's type estimator prevents, as it should (based on our earlier comment), the outlier a_1 to be enhanced. This however induces a slight loss in the closeness of $v_2(\gamma_n)$ to one, which can only be a finite-n effect.

We now place ourselves under the hypotheses of Corollary 7.1 with, as above $[C_N]_{ij} = .9^{|i-j|}$, $N = 100$, $n = 500$ (thus $c = .2$), while $D_N = I_N$. We also take $u = u_2$ defined previously and an outlier density of $\varepsilon = 0.05$, i.e., a 5% data pollution by outliers. We obtain theoretically in this case

$$\begin{aligned} v_2(\gamma_n) &\simeq 1.00 \\ v_2(\alpha_n) &\simeq .1219 \end{aligned}$$

which leads to a strong attenuation of the outliers, made particularly efficient by the ill-conditioning of C_N. Note that in the limit $\varepsilon \to 0$, $v_2(\gamma_n) \to 1$ while $v_2(\alpha_n) \to .1179$. To visually assess the outlier rejection efficacy of $\hat{C}_N$, we compare for this setting the eigenvalue distribution of the sample covariance matrix $\frac{1}{n} XX^*$ and that of $\hat{C}_N$ against the outlier-free SCM $\frac{1}{n} \sum_{i=1}^{(1-\varepsilon_n)n} x_i x_i^*$. From our earlier discussions, we wish ideally that the eigenvalue distributions of the former two match as closely as possible that of the latter.

Remark 7.1. *Before presenting the results, note that, since* $\frac{1}{N} \operatorname{tr} C_N = \frac{1}{N} \operatorname{tr} D_N = 1$ *for each* N*, the often advertised "robust" technique, that consists in normalizing every column* X_i *of* X *as* $\bar{X}_i = X_i / \|X_i\|$ *and from which the per-column normalized sample covariance matrix* $\frac{1}{n} \bar{X} \bar{X}^*$ *is built, has the same limiting eigenvalue distribution as* $\frac{1}{n} XX^*$ *so performs essentially the same for all large* N, n *in the present comparison study.*

To avoid imprecise Monte Carlo simulations, instead of providing a direct comparison of averaged eigenvalue distributions, we compare here the associated theoretical deterministic equivalent distributions (obtained from the Stieltjes transform of the deterministic equivalent measure associated to the model $\hat{S}_N^{\mathrm{rnd}}$, using e.g., the results of (Couillet et al., 2011a)). This is depicted in Figure 7.1, which shows a tight match between $\hat{C}_N$ and the target distribution, while the sample covariance matrix is strongly affected in its shifting much weight towards the purely-outlier distribution that would be the Marčenko–Pastur distribution (since $D_N = I_N$).

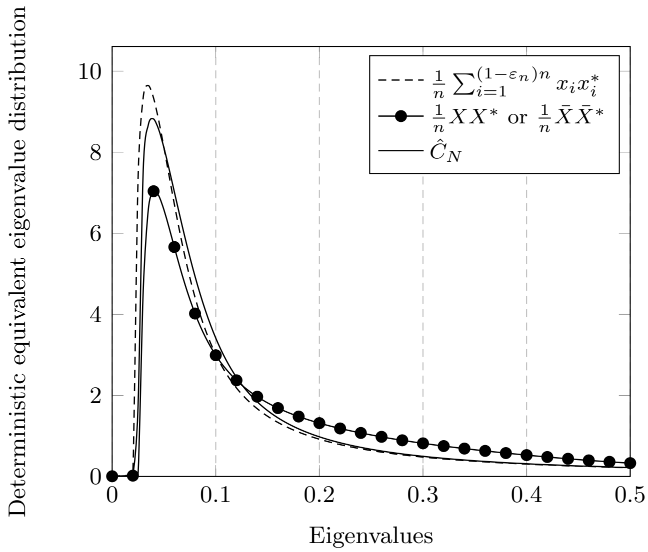


Figure 7.1: Limiting eigenvalue distributions. $[C_N]_{ij} = .9^{|i-j|}$, $D_N = I_N$, $\varepsilon = .05$.

7.2 Proof of the main results

The outline of the proof follows tightly from the proof of Theorem 4.1.2 in Chapter 4, however for a model that is (i) simpler in assuming the model-fitting data to be Gaussian an not elliptical, discarding the complication of the control of the τ_i's parameters, and (ii) made slightly more complex due to the deterministic addition of the vectors $a_1, \ldots, a_{\varepsilon_n n}$. Our way to deal with aspect (ii) is by controlling in parallel the quantities asymptotically approximated by γ_n and those asymptotically approximated by $\alpha_{i,n}$.

Since some parts of the proof are quite redundant with Chapter 4 or even Chapter 5, we shall mainly focus on the aspects that significantly differ from these and will leave some indisputable results without a proof.

As in Chapter 4, we may assume without loss of generality that $C_N = I_N$. We therefore take this assumption from now on. Using the definition $v(x) = u(g^{-1}(x))$ where $g(x) = x/(1 - c_n\phi(x))$, similar to Chapter 4, let us write $\hat{C}_N$ as

$$\hat{C}_N = \frac{1}{n}\sum_{i=1}^{(1-\varepsilon_n)n} v\left(d_i\right) x_i x_i^* + \frac{1}{n}\sum_{i=1}^{\varepsilon_n n} v\left(b_i\right) a_i a_i^*$$

where $d_i \triangleq \frac{1}{N} x_i^* \hat{C}_N^{-1} x_i$ and $b_i = \frac{1}{N} a_i^* \hat{C}_N^{-1} a_i$. We now define

$$e_i \triangleq \frac{v(d_i)}{v(\gamma_n)}$$
$$f_i \triangleq \frac{v(b_i)}{v(\alpha_{i,n})}$$

with γ_n and $\alpha_{i,n}$ as in the theorem statement (but for $C_N = I_N$).

The core of the proof is to show that

$$\max_{1\leq i\leq (1-\varepsilon_n)n} |e_i - 1| \xrightarrow{\text{a.s.}} 0$$
$$\max_{1\leq i\leq \varepsilon_n n} |f_i - 1| \xrightarrow{\text{a.s.}} 0.$$

Let us first relabel e_i and f_i such that $e_1 \leq \ldots \leq e_{(1-\varepsilon_n)n}$ and $f_1 \leq \ldots \leq f_{\varepsilon_n n}$, and let us further denote $\delta_n = \max(e_{(1-\varepsilon_n)n}, f_{\varepsilon_n n})$. Using classical inequalities as in Chapter 4, we have

$$\begin{aligned}
e_i &= \frac{v\left(\frac{1}{N}x_i^*\left(\frac{1}{n}\sum_{j\neq i}^{(1-\varepsilon_n)n} v\left(d_j\right) x_j x_j^* + \frac{1}{n}\sum_{j=1}^{\varepsilon_n n} v\left(b_j\right) a_j a_j^*\right)^{-1} x_i\right)}{v(\gamma_n)} \\
&= \frac{v\left(\frac{1}{N}x_i^*\left(\frac{1}{n}\sum_{j\neq i}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) e_j x_j x_j^* + \frac{1}{n}\sum_{j=1}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) f_j a_j a_j^*\right)^{-1} x_i\right)}{v(\gamma_n)} \\
&\leq \frac{v\left(\frac{1}{\delta_n N}x_i^*\left(\frac{1}{n}\sum_{j\neq i}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) x_j x_j^* + \frac{1}{n}\sum_{j=1}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^*\right)^{-1} x_i\right)}{v(\gamma_n)}.
\end{aligned}$$

We now exploit the following random matrix result, which we obtain similarly as Lemma 4.2 in Chapter 4

$$\max_{1\leq i\leq (1-\varepsilon_n)n} \left| \frac{1}{N}x_i^*\left(\frac{1}{n}\sum_{j\neq i}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) x_j x_j^* + \frac{1}{n}\sum_{j=1}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^*\right)^{-1} x_i - \gamma_n \right| \xrightarrow{\text{a.s.}} 0.$$

Then, taking $\zeta > 0$ we have for all large n almost surely

$$e_{(1-\varepsilon_n)n} \leq \frac{v\left(\frac{1}{\delta_n}(\gamma_n - \zeta)\right)}{v(\gamma_n)}. \tag{7.2}$$

We proceed similarly to upper-bound f_i as

$$f_i \leq \frac{v\left(\frac{1}{\delta_n N}a_i^*\left(\frac{1}{n}\sum_{j=1}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) x_j x_j^* + \frac{1}{n}\sum_{j\neq i}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^*\right)^{-1} a_i\right)}{v(\alpha_{i,n})}$$

and we now use the random matrix identity

$$\max_{1\leq i\leq \varepsilon_n n} \left| \frac{1}{N}a_i^*\left(\frac{1}{n}\sum_{j=1}^{(1-\varepsilon_n)n} v\left(\gamma_n\right) x_j x_j^* + \frac{1}{n}\sum_{j\neq i}^{\varepsilon_n n} v\left(\alpha_{j,n}\right) a_j a_j^*\right)^{-1} a_i - \alpha_{i,n} \right| \xrightarrow{\text{a.s.}} 0.$$

Therefore, for the same $\zeta > 0$ as above and for all large n almost surely,

$$f_{\varepsilon_n n} \leq \frac{v\left(\frac{1}{\delta_n}(\alpha_{i,n} - \zeta)\right)}{v(\alpha_{i,n})}. \tag{7.3}$$

We now need to consider study both cases where either $e_{(1-\varepsilon_n)n} \geq f_{\varepsilon n}$ or $e_{(1-\varepsilon_n)n} < f_{\varepsilon n}$.

Consider a subsequence over which $e_{(1-\varepsilon_n)n} \geq f_{\varepsilon n}$. On this subsequence, (7.2) becomes

$$e_{(1-\varepsilon_n)n} \leq \frac{v\left(\frac{1}{e_{(1-\varepsilon_n)n}}(\gamma_n - \zeta)\right)}{v(\gamma_n)}$$

or equivalently

$$1 \leq \frac{\Psi\left(\frac{\gamma_n}{e_{(1-\varepsilon)n}}\left(1 - \frac{\zeta}{\gamma_n}\right)\right)}{\Psi(\gamma_n)\left(1 - \frac{\zeta}{\gamma_n}\right)}.$$

As usual, we want to prove that, given $\ell > 0$, $e_{(1-\varepsilon_n)n} \leq 1 + \ell$ for all large n a.s. Let us assume the opposite, i.e., $e_{(1-\varepsilon_n)n} > 1 + \ell$ infinitely often, and let us restrict ourselves to a (further) subsequence where this always holds. Then

$$1 \leq \frac{\Psi\left(\frac{\gamma_n}{1+\ell}\left(1 - \frac{\zeta}{\gamma_n}\right)\right)}{\Psi(\gamma_n)\left(1 - \frac{\zeta}{\gamma_n}\right)} \leq \frac{\Psi\left(\frac{\gamma_n}{1+\ell}\right)}{\Psi(\gamma_n)\left(1 - \frac{\zeta}{\gamma_n}\right)}.$$

By definition, it is now easily proved that $0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < \infty$ (this exploits in particular the important fact that $c < 1 - \varepsilon$) so that, considering yet a further subsequence over which $\gamma_n \to \gamma_0$, we obtain, taking the limits

$$\Psi_c(\gamma_0)\left(1 - \frac{\zeta}{\gamma_0}\right) \leq \Psi_c\left(\frac{\gamma_0}{1+\ell}\right)$$

with $\Psi_c(x) = xv_c(x)$. This being valid for each $\zeta > 0$, we raise a contradiction in the limit $\zeta \to 0$. Therefore, either there exists no sequence over which $e_{(1-\varepsilon_n)n} \geq f_{\varepsilon_n n}$ or $e_{(1-\varepsilon_n)n} \leq 1 + \ell$ for all large n a.s. Assuming the former, then over a subsequence we in fact have $e_{(1-\varepsilon_n)n} < f_{\varepsilon_n n}$. Starting now from (7.3), we have over this subsequence

$$f_{\varepsilon_n n} \leq \frac{v\left(\frac{1}{f_{\varepsilon_n n}}(\alpha_{\varepsilon_n n,n} - \zeta)\right)}{v(\alpha_{\varepsilon_n n,n})}$$

for all large n, or equivalently

$$1 \leq \frac{\Psi\left(\frac{\alpha_{\varepsilon_n n,n}}{f_{\varepsilon_n n}}\left(1 - \frac{\zeta}{\alpha_{\varepsilon_n n,n}}\right)\right)}{\Psi(\alpha_{\varepsilon_n n,n})\left(1 - \frac{\zeta}{\alpha_{\varepsilon_n n,n}}\right)}.$$

Again, with, say, the same $\ell > 0$ as above, we wish to show that $f_{\varepsilon_n n,n} \leq 1 + \ell$ for all large n. We instead assume the opposite, i.e., $f_{\varepsilon_n n} > 1 + \ell$ infinitely often and restrict ourselves to a further subsequence satisfying this identity for all n. Then, as above for $e_{(1-\varepsilon_n)n}$, we have this time

$$1 \leq \frac{\Psi\left(\frac{\alpha_{\varepsilon_n n,n}}{1+\ell}\right)}{\Psi(\alpha_{\varepsilon_n n,n})\left(1 - \frac{\zeta}{\alpha_{\varepsilon_n n,n}}\right)}.$$

Using the fact that $0 < \min_i \liminf_n \|a_i\| \leq \max_i \limsup_n \|a_i\| < \infty$, it is easily shown that $0 < \liminf_n \alpha_{\varepsilon_n n,n} \leq \limsup_n \alpha_{\varepsilon_n n,n} < \infty$, so that we can take a further subsequence over which $\alpha_{\varepsilon_n n,n} \to \alpha_0$. In this limit, we have

$$\Psi_c(\alpha_0)\left(1 - \frac{\zeta}{\alpha_0}\right) \leq \Psi_c\left(\frac{\alpha_0}{1+\ell}\right)$$

which is contradictory for sufficiently small ζ. Thus, necessarily $f_{\varepsilon_n n} \leq 1+\ell$ for all large n almost surely, unless we have $e_{(1-\varepsilon_n n)} \geq f_{\varepsilon_n n}$ and then $e_{(1-\varepsilon_n n)} \leq 1+\ell$. In any case, we necessarily have

$$\max\left\{e_{(1-\varepsilon_n n)}, f_{\varepsilon_n n}\right\} \leq 1+\ell$$

for all large n a.s., which we wanted to prove. All the same, by reverting the inequalities, we prove the converse identity that, for all large n a.s.

$$\min\left\{e_1, f_1\right\} \geq 1+\ell.$$

This establishes the main result, from which Theorem 7.1.1 unfolds easily.

The proof of Corollary 7.1 follows easily from applying standard random matrix identities (a further deterministic equivalent) to the random model of Theorem 7.1.1. We do not detail these classical steps.

Chapter 8

Conclusion and Perspectives: Random Matrix Theory for Big Data

8.1 Summary and open avenues to robust statistics analysis

Thanks to a systematic analysis of the behavior of robust estimators of scatter for large dimensional datasets, we showed in this report that these estimators behave similar to mathematically tractable random matrix models. The theoretical study of these asymptotically equivalent models for various input data statistics revealed a lot of practical information on the robust M-estimators, the most fundamental of which can be shortly summarized as follows:

- (Chapter 4) When impulsiveness in the data is modelled through the random norm of an elliptical distribution, sample covariance matrices tend to have a large eigenvalue spectrum spread, which robust estimators shrink down to a provably bounded spectrum; in the particular case of spiked models, this induces the possibility to recover isolated eigenvalues that sample covariance matrices keep hidden in the spectrum, leading to improved statistical inference techniques.

- (Chapter 5) To cope with the scenarios of few observations of large dimensional impulsive data, the hybrid robust shrinkage estimators benefit both from the regularization of the ill-conditioned sample covariance matrix and from the robustness of M-estimators of scatter; the extra degree of freedom induced by the regularization parameter allows for further improved statistical estimators applicable to various fields of research.

- (Chapter 6) Bilinear forms built upon robust M-estimators for elliptical data are so well approximated by the same bilinear forms but built upon the asymptotic random matrix approximations that they share the same second-order statistics (i.e., central limit results). This makes it possible, on top of the first-order statistical methods described in the previous two items, to design second-order improved estimators by a mere application of classical random matrix methods.

- (Chapter 7) Finally, when modelling data impulsiveness through deterministic outliers, robust M-estimators manage to harness the impact of some specific outliers, while not being able to control others; unlike the previous items which suggested an overall advantage of estimators close to Tyler's (or simply per-data normalized sample covariance matrices), against deterministic outliers the latter may be hazardous choices which are more adequately replaced by estimators of the Huber type.

The results studied thus far however only target the subfield of robust statistics that is concerned with robust estimators of scatter for centered data. Robust statistics however go beyond these considerations. Still on order statistics estimation, robust estimation of location (that is the estimation of the mean of median of random impulsive data) is sometimes a more important concern than robust estimation of scatter. Joint robust estimation of location and scatter may also be considered which however often lead to not always solvable implicit equations (Maronna et al., 2006). A more complete analysis of robust M-estimators in the large dimensional regime would therefore demand to take this location aspect into account. The closest result we have on these aspects concerns the application of robust shrinkage estimates of scatter to financial returns datasets for which an empirical mean was discarded from each datum (see Section 5.2); the elliptical modelling of the data induces already here some complications, although no actual location estimate was made.

But robust statistics are also concerned with improved regression from impulsive readings. Recent considerations were made towards this direction, for instance in (El Karoui, 2013) where the theoretical performance of robust regressors based on n independent readings of N dimensional signals is studied in the large N, n regime. It is precisely shown that, although inconsistent in this regime, the estimated (large dimensional) regression vector asymptotically departs from the sought for vector by a computable (although not explicit) quantity. This allows for an analysis of the power of various regressors for various data models. The analysis of (El Karoui, 2013) is however restricted to input data with i.i.d. entries, although results for general elliptical data could be easily achieved in a similar manner. Surprisingly, no follow-up on this work was made in spite of the major consequences such a result may bring to large dataset regression at large. This is surely an area of important future exploration.

Recently, considerations of "robust" estimation have re-emerged under the umbrella of compressive sensing. Underlying this robust terminology is the regression analysis of large dimensional but sparse vectors. That is, unlike the works of (El Karoui, 2013) where the data observations are possibly impulsive and lead, due to the large N, n assumption, to inconsistent regressors, in compressive sensing the regressor is assumed to be identically null on most of its entries. This structure allows for a perfect reconstruction of the sought-for vector under some conditions, which is stable against background noise, hence the robustness naming. Given that the sparsity assumption cannot be fully met in most practical cases and that impulsiveness (or the presence of outliers) in real-life datasets is more a rule than an exception, it would be important to further investigate the connections between both fields and to bridge the gap between robust statistics and the very different "robustness" considerations of compressive sensing.

8.2 Beyond robust estimation: the Big Data paradigm

In recollection of the ten year-progress of random matrix theory applied to signal processing, it appears that the rule-of-thumb has so far been to study sample covariance matrices to extract from them the necessary information one seeks; this is obtained by the study of various sample covariance matrix models in the random matrix regime (spiked models, separable covariance models, etc.) and by means of smart complex analysis tools. Although this is quite reasonable an approach, it may nonetheless seem quite arbitrary to use the sample covariance matrix in the first place, as it is anyhow no longer a good estimator for the population covariance matrix. The only reasonable explanation for its use has to do with its being a maximum likelihood estimator for the population covariance matrix when observations are Gaussian and when no further information about the population model is known.

In the present report, we observed (more than we actually proved) that robust estimators of scatter are often more helpful than sample covariance matrices when the observed vector data contain outliers or have heavy-tailed distributions, and are particularly suitable to elliptically distributed data for which they are the maximum-likelihood estimators. An important outcome of our study on robust estimation of scatter as a whole is that the latter may be used in place of sample covariance matrices as the building block for powerful algorithms of statistical inference.

In fact, other studies have recently considered other types of random matrix models as a starting point matrix for statistical inference, in particular when side information about the system model is a priori known. This is the case of a recent work of ours (Vinogradova et al., 2014) in which the performance of detection schemes under time-correlated noise is analyzed, based on a Toeplitzified version of the sample covariance matrix rather than based on the sample covariance matrix itself. Another line of work concerns the block-Hankel stacking of observation vectors (Loubaton, 2014) of important use for the coherent detection of temporal data with memory. These various objects, in general more challenging to study than the mere sample covariance matrix, constitute in my opinion the future of random matrix research in signal processing.

Generally speaking, mathematicians also have started to consider models that disrupt from the traditional Wigner and i.i.d. random matrices, for which about everything is known by now. Their focus is now steering towards other types of random matrix models, less “pure” in their mathematical model simplicity but which, while remaining rather simple, have important relations to engineering objects at large. This is the case for instance of the spectrum analysis of random graphs which, from a random matrix perspective, is the random eigenvalue distribution of the Laplacian or adjacency matrix of the graph under consideration. Various teams have made some recent breakthroughs in this direction, as with (Bordenave and Lelarge, 2010b,a) for the spectrum of sparse non-directed graphs, or (Bordenave et al., 2013) for the Laplacian matrix of a random possibly sparse directed graph, or finally (El Karoui, 2010) for the spectrum of random kernel matrices such as the adjacency matrix of Euclidean-weighted graphs.

This recent move from the mathematical side, especially on random graphs, along with the increasing understanding of the strong potential behind other classes of random matrix models for signal processing applications, brought us to considering a new direction of exploration:

that of signal processing on large dimensional graphs and particularly the study of machine learning algorithms in the large dimensional regime. Machine learning is indeed an area of important focus today, especially since the advent of big data processing, which encompasses many subspace methods (e.g., for classification or clustering) based on the so-called affinity matrix of the observed data, also referred to as the kernel matrix where the implicit kernel is the affinity function. Machine learning also deals with more heuristic approaches such as neural networks which are on the verge today of being more clearly understood than ever before. Let us detail these two specific examples some more.

8.2.1 Kernel random matrices

8.2.1.1 Theoretical aspects

Let $x_1, \ldots, x_n$ be n vectors of $\mathbb{R}^N$ and $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that $f(x,y) = f(y,x)$. We define the kernel matrix W with kernel function f as

$$W = [W_{i,j}]_{1 \leq i,j \leq n}, \text{ with } W_{i,j} = f(x_i, x_j).$$

As such, W is a symmetric matrix. Generalizing to complex valued vectors $x_1, \ldots, x_n$ and to the constraint $f(x,y) = \overline{f(y,x)}$, we would get W to be Hermitian.

Classical examples of kernel functions f are $f(x,y) = x^\mathsf{T} y$ or $f(x,y) = g(\|x-y\|)$ for some function g. In the latter example with $g(t) = t^2$, the matrix W is precisely a Euclidean matrix. This denomination follows from assuming that $x_1, \ldots, x_n$ are geographically located nodes of a network in $\mathbb{R}^N$ (for instance in the plane for $N = 2$), in which case W is the adjacency matrix of the network whose entry (i,j) is directly related to the geometrical (Euclidean) distance between nodes i and j in the graph.

Let us assume that the vectors $x_1, \ldots, x_n$ are random and sufficiently independent. Then, for N fixed, taking n large leads W to be a random matrix of $O(n)$ degrees of freedom, as it is only defined through the n vectors x_i. However, if N is also taken large enough to be comparable to n and the entries of the vectors x_i are sufficiently independent, then W becomes a matrix constituted of $O(nN) = O(n^2)$ degrees of freedom, which is the classical setting of large dimensional random matrix theory. In this case, it was shown in the simplest setting of independent vectors x_i with independent and identically distributed zero mean entries, and for smooth enough functions f, that the matrix W has the peculiar asymptotic behavior to be equivalent to a simple random matrix model or even to a deterministic matrix. It is in particular proved in (El Karoui, 2010) that W is essentially a rank-one matrix in the large N, n regime with leading eigenvector the all-one vector. The proof considered in (El Karoui, 2010) however relies on rather involved combinatorial calculus and in the quite loose bound of the spectral norm by the fourth-moment trace norm, which we believe can be simplified a lot in practical settings. Also, it seems that the proper setting for which W does not degenerate into a rank-one matrix consists in taking $N = O(\sqrt{n})$ rather than $N = O(n)$, in which case more interesting properties should appear. There is therefore room for much theoretical progress to be made on purely mathematical grounds. Technical considerations set aside, the powerful convergence of W to a matrix with isolated eigenvalues has important consequences in practice as it implies the possibility to read deterministic information on the system model straightforwardly from the matrix W and not solely from functionals of its eigenvalue distribution. That is, if one seeks statistical information concerning $x_1, \ldots, x_n$ defined as deterministic features (e.g., order statistics) of their affinity relation, this can be obtained relatively easily from understanding the deterministic structure in W.

Of particular interest in practice, and notably for machine learning, is the possibly to read out data clustering from the leading eigenvectors of the affinity matrix W when $x_1, \ldots, x_n$ are no longer i.i.d. but fall in k classes. We provide a short introduction to this problem below.

8.2.1.2 Spectral clustering

Clustering is probably one of the most typical examples of machine learning methods for arbitrary data collections. The principle of data clustering is to group a set of data $x_1, \ldots, x_n$ into k similarity classes of indexes $\mathcal{S}_1, \ldots, \mathcal{S}_k$, disjoint but exhaustively covering the set $\mathcal{S} = \{1, \ldots, n\}$, with k a parameter sometimes determined beforehand while $\mathcal{S}_i$ is unknown. We shall assume here that the n data are N-dimensional vectors, i.e., $x_1, \ldots, x_n \in \mathbb{R}^N$.

To this purpose, one needs to start by defining an affinity (or proximity) metric between any pair of data, which shall measure how similar these data are. This choice is usually up to the system setting under consideration. Since the proximity notion is a symmetric one (i.e., the proximity between x_i and x_j should be the same as that between x_j and x_i, for any given pair of vectors x_i, x_j), we shall assume it to be the symmetric function f defined earlier. A classical choice is to take the Gaussian kernel

$$f(x, y) = \exp\left(-\frac{\|x - y\|^2}{2\sigma^2}\right)$$

for some parameter $\sigma > 0$. Note that in this setting, two data vectors are extremely similar if the similarity function affected to these vectors is close to one, and are extremely dissimilar if the similarity function is close to zero. From there, we then define the kernel matrix W as before.

Clustering the set $x_1, \ldots, x_n$ in k classes may now be defined as solving the following graph cut problem

$$(\text{MinCut}) \qquad \min_{\substack{\mathcal{S}_1, \ldots, \mathcal{S}_k \\ \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k = \mathcal{S} \\ \forall i \neq j,\ \mathcal{S}_i \cap \mathcal{S}_j = \emptyset}} \sum_{i=1}^{k} \sum_{j \in \mathcal{S}_i, \bar{j} \in \mathcal{S}_i^c} f(x_j, x_{\bar{j}}).$$

The objective in this minimization problem is to ensure that the components of each given subset $\mathcal{S}_i$ are maximally dissimilar to the components of the union of all other subsets $\mathcal{S}_{\bar{i}}$, $\bar{i} \neq i$. This minimization problem however has some limitations in that it may lead to graph cuts in which many sets are singletons, which is often an unsatisfactory outcome. To ensure, as is often desired, that the sets $\mathcal{S}_i$ are well balanced, i.e., have cardinalities of the same order of magnitude, the MinCut problem is often substituted by the following RatioCut alternative

$$(\text{RatioCut}) \quad \min_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_k \\ \mathcal{S}_1\cup\ldots\cup\mathcal{S}_k=\mathcal{S} \\ \forall i\neq j,\ \mathcal{S}_i\cap\mathcal{S}_j=\emptyset}} \sum_{i=1}^{k} \sum_{j\in\mathcal{S}_i,\bar{j}\in\mathcal{S}_i^c} \frac{f(x_j,x_{\bar{j}})}{|\mathcal{S}_i|}$$

with $|\cdot|$ the cardinality of the set. This problem, or similar generalizations to other metrics of set sizes, is however much more difficult to solve than MinCut with standard algorithms. It can in fact be proved to be a necessarily NP hard problem.

The spectral clustering approach provides a method, by means of a relaxation of the RatioCut (or similar) problem, which is polynomially solvable, although not necessarily finding the minimum of RatioCut. The approach consists in noting, after some simple algebraic considerations, that RatioCut is equivalently written as

$$(\text{RatioCut}) \quad \min_{M\in\mathcal{M},\ M^{\mathsf{T}}M=I_k} \operatorname{tr}\left(M^{\mathsf{T}}LM\right)$$

where $\mathcal{M}$ is the set of matrices $M=[m_{ij}]_{1\leq i\leq n,1\leq j\leq k}$ with $m_{ij}=|\mathcal{S}_j|^{-\frac{1}{2}}\boldsymbol{\delta}_{i\in\mathcal{S}_j}$, and L is the Laplacian matrix

$$L=[L_{ij}]_{1\leq i,j\leq n}=[-W+\operatorname{diag}(W\cdot 1)]_{1\leq i,j\leq n}=\left[-f(x_i,x_j)+\boldsymbol{\delta}_{i,j}\sum_{l=1}^{n} f(x_i,x_l)\right]_{1\leq i,j\leq n}.$$

With this notation, observe that RatioCut consists in finding a matrix M made of orthogonal columns in the very specific class of isometric matrices $\mathcal{M}$. The key of the spectral clustering method is to realize that, if $\mathcal{M}$ were to be replaced by the larger set $\mathcal{O}$ of $n\times N$ isometric matrices, then RatioCut would consist in retrieving the eigenvectors corresponding to the smallest eigenvalues of L.

The spectral clustering algorithms precisely consist in determining these k smallest eigenvalues, extract their corresponding eigenvectors and, up to some further manipulation to turn the continuous entries into discrete ones, read off the eigenvectors the precise partition of the set $\mathcal{S}=\{1,\ldots,n\}$ into k disjoint subsets of low mutual similarity.

It is clear, from the details above, that spectral clustering methods are intimately linked to the spectrum and eigenspaces of the Laplacian matrix L of the affinity matrix W. In order to understand the performance of the algorithm in the context of big data processing, it is therefore fundamental to quantify, from a proper statistical representation of the set $x_1,\ldots,x_n$, the spectral attributes of L in terms of eigenvalues and eigenvectors. We believe that in this setting, the result from (El Karoui, 2010) generalizes to W being well approximated by a rank-k matrix in the large dimensional limit, as long as $N=O(n)$. If so, then optimal clustering should naturally relate to how good the largest k eigenvalues of W isolate from each other (so that their respective eigenvectors be sufficiently uncorrelated). The appropriate choice of the kernel function f as a function of the data distributions, which so far is quite empirical in the literature, should be understandable from this simple setting. To further understand the impact for lower dimensional datasets, the study of the regime $N = O(\sqrt{n})$ should then reveal a refined model for W, which we believe could fall within the spiked model setting.

Aside spectral clustering, machine learning contains many other tools and techniques which are mostly understood from hand-wavy considerations rather than profound theoretical conclusions. One of such methods is the extensively used technique of neural networks which has recently known a new surge of interest with the introduction of recurrent echo-state networks.

8.2.2 Echo-state neural networks

Echo-state neural networks consist of artificial neural networks which, unlike traditional sequential neural networks, contain active inter-neuronal connections. This feature has multiple advantages over sequential neural networks and in particular it importantly helps conveying memory capacities to the network in a similar manner that memorization mechanisms presumably work in biological neural networks. The idea here is that the information carried by input stimuli to the network, instead of being forwarded sequentially towards an output sensor neuron, may remain trapped within the network due to neuronal interconnections.

The simplest model of a (discrete time) echo-state neural network is described as a set of N neurons connected together via a connectivity matrix $W \in \mathbb{R}^{N\times N}$, a scalar input (stimulus) source $s_t \in \mathbb{R}$ indexed by the time t, and a scalar output $y_t \in \mathbb{R}$ also indexed by time t. In particular, the (i,j)-entry $W(i,j)$ of W models the excitation or inhibition effect of neuron i over neuron j. The connection between the input source and the neural network is given by a vector $m \in \mathbb{R}^N$ with $m(i)$ the connection between the source and neuron i. Similarly, the connection between the neurons and the output is characterized by a vector $w \in \mathbb{R}^N$. The specificity of echo-state neural networks is that W is composed of random entries which are fixed once and for all, as opposed to classical neural networks where a training phase is dedicated to build W based on input-output pilots. The training phase in echo-state neural networks will only set up the output weight vector w based also on input-output pilot sequences.

To better understand the behavior of echo-state neural networks, we shall analyze the state evolution of the neurons in a discrete time setting. Letting $x_t \in \mathbb{R}^n$ be the state of the neural network at time t, with $x_t(i)$ the value taken by neuron i at time t, the dynamical state evolution of the echo-state neural network is generally described as

$$x_0 = ms_0 \tag{8.1}$$
$$x_t = S\left(Wx_{t-1} + ms_{t-1}\right),\ t = 1, 2, \dots \tag{8.2}$$

in which $[Wx_{t-1}+ms_{t-1}]_i$ is the summation of the inter-neuronal actions and of the input source action on neuron i, and S is a non-linear function applied entry-wise, generally a sigmoid function, which translates the fact that neurons have a minimal (negative) and maximal (positive) activation and inhibition levels. Note that S may ensure in particular that the dynamical system remains stable in the long run. Sometimes, it is also assumed that the neuronal connections may be subject to external noise sources, not linked to the source of interest, leading $Wx_{t-1} + ms_{t-1}$

to be changed into $Wx_{t-1} + ms_{t-1} + b_{t-1}$ for some noise vector b_{t-1} independent of x_{t-1} and s_{t-1}.

8.2.2.1 Stability of the training phase

To run the neural network, a training phase is necessary by which the weights $w(i)$ from neuron i to the network output y_t will be enforced. For this, couples (s_t, y_t), for $t = 1, \ldots, n$ with n the training period, are defined to be pilot sequences. Given these pilots, the vector w is taken as follows

$$w = \arg\min_{\tilde{w}} \sum_{t=1}^{T} \rho\left(y_t - \tilde{w}^\mathsf{T} x_t\right)$$

for some positive function ρ cancelling at zero. A classical choice is to take $\rho(x) = x^2$, in which case w is taken to be the least square estimate

$$w = (XX^\mathsf{T})^{-1} Xy \tag{8.3}$$

where we defined $X = [x_1, \ldots, x_T] \in \mathbb{R}^{N\times n}$ $(n > N)$ and $y = [y_1, \ldots, y_T]^\mathsf{T}$.

For mathematical tractability, the non-linearity of S may be problematic so that S is in general taken to be the identity function in a first approximation. In this case, to ensure the stability of the system, one makes the additional assumption that the spectral norm of W is less or equal than one. In fact, again stimulated by biological networks, and simultaneously by mathematical tractability (and by the expected blessings of dimensionality), it is generally considered that W is the realization of a random matrix with $O(N^2)$ degrees of freedom. The simplest scenario consists in particular in letting the entries of W be independent and identically distribution with zero mean, variance α/N with $\alpha < 1$ and finite fourth order moment. This ensures, according to the full-circle law theorem (Tao and Vu, 2008) that, asymptotically, the spectral radius of W is strictly less than one with overwhelming probability, leading to a stable network.

As is classical in stability questions of dynamical networks, to characterize the dynamical system x_t, it is essential to understand certain metrics involving the matrix XX^T. In particular, the smallest eigenvalue of XX^T conditions the stability of w from the defining equation (8.3). However, XX^T is not a simple matrix to analyze since each column of X is a recursive combination of the previous columns. Assuming some ergodicity properties in the training sequence $(s_t)_{t=1}^n$ and that $n \to \infty$, it naturally appears that the system stability boils then down to characterizing the random matrix

$$M = \sum_{i=1}^{\infty} W^i mm^\mathsf{T} (W^\mathsf{T})^i.$$

This matrix, if well-defined, appears to be the sum of nonnegative rank-one matrices, however with successive powers of W involved.

One objective may be to study the spectrum of the matrix M. Technically speaking, the matrix M remains in the well-studied realm of Gram matrices formed out of the sum of rank-one nonnegative matrices. However, classically, for Gram matrices studied in random matrix theory, the rank-one elements are usually independent or linearly dependent. Here, the strong dependence between the entries of M creates an original difficulty. Simulation results in fact suggest that M is quite singular in that it only contains a few large eigenvalues, while the other eigenvalues are very small. Simulations also reveal that truncating the sum defining M to an arbitrary large upper integer K provides an accurate approximation of M itself, irrespective of the size N of M. This forcefully suggests the approximation of M by a rank-K matrix for K large but finite. Identifying these eigenvalues and their associated eigenvectors in the large N regime will allow for a proper understanding of the various performance metrics of the neural network. Also, since the stability of the network behavior is mostly parametrized by the scaled variance α of the entries of W, this performance analysis will allow for performance optimization by a proper selection of the parameter α.

However, it appears that defining w as a least square solution induces a problem of over-fitting, in the sense that w will be perfectly fit to the trained data, but may possibly not be flexible enough to smoothly handle untrained data. One of the key features of neural networks is precisely to have an implicit innovation capability to cope with unknown stimuli. To enhance such features, an option is to introduce a noise vector b_t in the neural transition dynamics. A proper control of the noise variance then allows one to establish a trade-off between accuracy in the training phase and innovation in the interpolation phase (i.e., when new data are fed in the network).

Another approach, which turns out to be essential equivalent, is to relax the constraining least-square minimization by enforcing an ℓ_1 or ℓ_2 constraint on w, that is, by setting w to now be defined as

$$w = \arg\min_{\tilde{w}} \sum_{t=1}^{T} \rho\left(y_t - \tilde{w}^{\mathsf{T}} x_t\right) + \beta \|\tilde{w}\|^{\gamma}$$

for some $\beta \geq 0$ and $\gamma \in \{1, 2\}$. By increasing β, this minimization effectively reduces the degrees of freedom of w and therefore does not let w perfectly meet the constraint $y_t = w^{\mathsf{T}} x_t$, therefore allowing for some degree of innovation. In this setting, it is fundamental to clearly understand the effect of β on the trade-off between accuracy and innovation in the network. This mathematically boils down to characterizing a slightly more involved version of the matrix M defined above.

To properly optimize the neural network, one needs investigate the dual impact of the terms α and β on the performance, both in terms of accuracy and innovation. This precise characterization shall then allow for improved solutions by appropriate choices of (α, β) to achieve optimal trade-offs.

8.2.2.2 Stability of the network as a whole

An important question that comes prior to the stability and the accuracy of the training and innovation phases is the stability of the dynamical system (8.1) itself. With S still considered to be the identity function, the stability of the network is then related to the spectral radius of W. In the simplest scenario where W has i.i.d. entries of zero mean and variance α/N, taking $\alpha < 1$ ensures the spectral radius of W to be less than one asymptotically. However, for each finite N, W may have spectral radius greater than one with low but positive probability. To avoid this inappropriate situation, other normalizations may be performed such as replacing W by $W/\|W\|$ ensuring then that the spectral radius of W is equal to one. This then naturally leads to considerations of topological complexities of the dynamical network, such as studied in (Wainrib and Touboul, 2013), where the topological complexity relates to the cardinality of those eigenvalues of W of real part greater than one.

A more practical normalization is to assume that all row-sums of W equal zero. This normalization ensures that each neuron is in an equilibrium state between exhitation and inhibition from connected neurons. This is theoretically done by setting the diagonal elements W_{ii} of W to be $W_{ii} = -\sum_{j\neq i} W_{ij}$, with the W_{ij} in general independent. This therefore turns W into the Laplacian matrix of the random graph whose adjacency matrix is the matrix W with diagonal discarded. This model was recently studied in (Bordenave et al., 2013), where it is shown that, as far as extreme eigenvalues are concerned, the spectrum of W thus defined is fundamentally different from that of W without row normalization. This induces a strikingly different behavior in terms of stability of the neural network, which it is fundamental to investigate.

More advanced, and in fact more biologically realistic, models of connectivity matrices W are also fundamental to understand. For instance, in order to account for the locally higher connectivity of close-by neurons in the network, we may induce a variance profile in the matrix W, i.e., by making the individual variances of W_{ij} depending on i, j. For instance, letting $\mathrm{E}[|W_{ij}|^2]$ be proportional to $|i-j|$ enforces a proximity constraint very classical in actual networks. Studying the limiting eigenvalue distribution of such non-Hermitian models therefore has deep implications in the practical analysis of both artificial and biological neural networks.

8.3 General conclusion

Going back to where we started this report, present and future works of random matrix theory provide promising avenues for the exploitation of the dimensionality blessings brought forward by Donoho. In essence it appears that many techniques used in signal processing and beyond (in fact in statistics at large) can benefit from exploiting the many degrees of freedom of the raw data both in space and time, and that the extent of these techniques is largely not limited to those based on the sample covariance matrix. This report brought this state-of-fact to light by providing a rather complete study of a specific tool: robust M-estimators of scatter.

Precisely, similar to the works in the nineties on the limiting spectral behavior of sample covariance matrices, we conducted the analysis of the limiting behavior of these robust estimators. We then derived consistent estimates for the specific example of a spiked-model extension of these estimators, which mimics the equivalent studies performed since the second half of the 2 000's. We finally developed central limit theorems for specific functionals, which again finds a strong connection with the works on this topic for sample covariance matrices in the late nineties. All these works however came along with a set of new tools, dedicated to the M-estimators, and which allowed us to fill the gap between robust statistics and the random matrix theory. This was essentially as far as we had projected (by then, idealized) the work to bring us when we started. In the meantime though, we discovered along the way that our random matrix analysis could attest of some powerful (maybe so far considered magical) features of these M-estimators. It in particular appeared very clear that, while Tyler's robust estimate shows in simulations outstanding performances in elliptical data settings, making it often more favorable than any of Maronna's estimator, Tyler's estimator suffers from some specific outliers. In this setting, Huber's estimator is much more adequate, and generally estimators of the Maronna class convey appropriate trade-offs between the resilience of Tyler's estimate to scale-free inputs and the appropriateness of Huber's estimator against all sorts of outliers.

Tomorrow's research on applied random matrix theory should provide a large panel of tools beyond sample covariance matrices and robust estimators, among which the aforementioned Toeplitzified sample covariance matrices, block-Hankel matrices, kernel matrices, etc. When those tools are in place, they will then allow for the analysis of conventional techniques built under the large n alone assumption but exploited in more practical large N, n systems. In turn, this analysis will provide new enhanced algorithms that shall enrich the existing panoply of sample covariance matrix-based and now robust estimator-based methods. This overall methodology will remain valid as long as the sought-for information is contained within the eigen-structure of the involved matrix models. Other considerations than eigen-structure related, such as data sparsity explored recently within the scope of compressive sensing – that, in passing, often relies on random matrix considerations – are yet other methods which, alongside random matrix theory, constitute beyond any doubt tomorrow's toolbox of the big data paradigm.

Appendix A

Basic properties and important lemmas

We introduce in this appendix some classical random matrix results used throughout the report but worth recalling, along with some new results of independent interest.

We start with classical well-known random matrix results.

Lemma A.1 (A matrix-inversion lemma). *Let $x \in \mathbb{C}^N$, $A \in \mathbb{C}^{N\times N}$, and $t \in \mathbb{R}$. Then, whenever the inverses exist*

$$x^* \left(A + txx^*\right)^{-1} x = x^* A^{-1} x (1 + tx^* A^{-1} x)^{-1}.$$

Lemma A.2 (Rank-one perturbation). *(Silverstein and Bai, 1995, Lemma 2.6) Let $v \in \mathbb{C}^N$, $A, B \in \mathbb{C}^{N\times N}$ nonnegative definite, and $x > 0$. Then*

$$\left| \operatorname{tr} B \left(A + vv^* + xI_N\right)^{-1} - \operatorname{tr} B \left(A + xI_N\right)^{-1} \right| \leq x^{-1} \|B\|.$$

Lemma A.3 (Trace lemma). *(Bai and Silverstein, 2009, Lemma B.26) Let $A \in \mathbb{C}^{N\times N}$ be non-random and $y = [y_1, \ldots, y_N]^\mathsf{T} \in \mathbb{C}^N$ be a vector of independent entries with $\mathrm{E}[y_i] = 0$, $\mathrm{E}[|y_i|^2] = 1$, and $\mathrm{E}[|y_i|^\ell] \leq \zeta_\ell$ for all $\ell \leq 2p$, with $p \geq 2$. Then,*

$$\mathrm{E}\left[|y^* A y - \operatorname{tr} A|^p \right] \leq C_p \left((\zeta_4 \operatorname{tr} AA^*)^{\frac{p}{2}} + \zeta_{2p} \operatorname{tr}(AA^*)^{\frac{p}{2}} \right)$$

for C_p a constant depending on p only.

The subsequent lemma is a quite expected result unfolding from Lemma A.3 with a technical difficulty due to the statistical unboundedness of the smallest eigenvalue of the resolvent matrix involved.

Lemma A.4. *Let $z_1, \ldots, z_n \in \mathbb{C}^N$ be independent unitarily invariant vectors with $\|z_i\|^2 = N$. Then, if $0 < \liminf_n N/n \leq \limsup_n N/n < 1$,*

$$\max_{1\leq j\leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i=1}^n z_i z_i^* \right)^{-1} z_j - 1 \right| \xrightarrow{\text{a.s.}} 0$$

or equivalently

$$\max_{1\leq j\leq n}\left|\frac{1}{N}z_j^*\left(\frac{1}{n}\sum_{i\neq j}^{n}z_iz_i^*\right)^{-1}z_j-\frac{1}{1-\frac{N}{n}}\right|\xrightarrow{\text{a.s.}}0.$$

Moreover, there exists $\varepsilon>0$ such that, for all large n a.s.

$$\lambda_1\left(\frac{1}{n}\sum_{i=1}^{n}z_iz_i^*\right)\geq\min_{1\leq j\leq n}\left\{\lambda_1\left(\frac{1}{n}\sum_{\substack{1\leq i\leq n\\ i\neq j}}z_iz_i^*\right)\right\}>\varepsilon.$$

Proof. For readability, we denote $F=\frac{1}{n}\sum_{i=1}^n z_iz_i^*$, $F_{(j)}=F-\frac{1}{n}z_jz_j^*$, $\tilde F=\frac{1}{n}\sum_{i=1}^n\tilde z_i\tilde z_i^*$, and $\tilde F_{(j)}=\tilde F-\frac{1}{n}\tilde z_j\tilde z_j^*$, where we recall the relation $z_i=\sqrt{\bar N}\tilde z_i/\|\tilde y_i\|$ for $\tilde z_i$ zero mean I_N-covariance Gaussian and $\tilde y_i$ zero mean $I_{\bar N}$-covariance Gaussian (non-independent).

By (Bai and Yin, 1993), letting $\varepsilon>0$ small enough, the probability of the event $\lambda_1(\tilde F_{(j)})<\varepsilon$ is $o(n^{-\ell})$ for each integer ℓ. As such, by Markov inequality and the Borel Cantelli lemma,

$$\lambda_1(\tilde F)\geq\min_{1\leq j\leq n}\lambda_1(\tilde F_{(j)})>\varepsilon. \tag{A.1}$$

From Lemma A.3 and the same approach as followed in the proof of Theorem 4.1.2 consisting in writing $\frac{1}{N}\tilde z_j^*\tilde F^{-1}\tilde z_j$ as the product $\kappa_j\frac{1}{N}\tilde z_j^*\tilde F^{-1}\tilde z_j$ times κ_j^{-1}, with $\kappa_j=1_{\{\lambda_1(\tilde F_{(j)})>\varepsilon\}}$, we easily obtain that

$$\max_{1\leq j\leq n}\left|\frac{1}{N}\tilde z_j^*\tilde F^{-1}\tilde z_j-1\right|\xrightarrow{\text{a.s.}}0.$$

Now,

$$\min_{1\leq j\leq n}\lambda_1(F_{(j)})\geq\frac{\min_{1\leq j\leq n}\lambda_1(\tilde F_{(j)})}{\max_{1\leq j\leq n}\bar N^{-1}\|\tilde y_j\|^2}$$

Since $\max_{1\leq j\leq n}N^{-1}\|\tilde z_j\|^2\xrightarrow{\text{a.s.}}1$ a.s. from standard probability results, we have that for all large n a.s.

$$\lambda_1(F)\geq\min_{1\leq j\leq n}\lambda_1(F_{(j)})>\varepsilon/2$$

which already gives the second part of the lemma. Using only the outer inequality of (A.1), we now have, for all large n a.s.

$$\begin{aligned}\max_{1\leq j\leq n}\left|\frac{1}{N}\tilde z_j^*F^{-1}\tilde z_j-\frac{1}{N}\tilde z_j^*\tilde F^{-1}\tilde z_j\right|&=\max_{1\leq j\leq n}\left|\frac{1}{N}\tilde z_j^*F^{-1}\left(\tilde F-F\right)\tilde F^{-1}\tilde z_j\right|\\&\leq\max_{1\leq j\leq n}\left\{\frac{1}{n}\sum_{k=1}^{n}\left|1-\frac{\bar N}{\|\tilde y_k\|^2}\right|\left|\frac{1}{N}\tilde z_j^*\tilde F^{-1}\tilde z_k\right|^2\right\}\\&\leq\max_{1\leq k\leq n}\left|1-\frac{\bar N}{\|\tilde y_k\|^2}\right|\frac{1}{N}\left(\max_{1\leq k\leq n}\|\tilde z_k\|\right)^2\frac{4}{\varepsilon^2}\\&\xrightarrow{\text{a.s.}}0.\end{aligned}$$

Finally, for all large n a.s.

$$\begin{aligned}\max_{1\le j\le n}\left|\frac{1}{N}\tilde{z}_j^*F^{-1}\tilde{z}_j-\frac{1}{N}z_j^*F^{-1}z_j\right| &= \max_{1\le j\le n}\left\{\left|\frac{1}{N}\tilde{z}_j^*F^{-1}\tilde{z}_j\right|\left|1-\frac{\bar{N}}{\|\tilde{y}_k\|^2}\right|\right\}\\ &\le \frac{2}{\varepsilon}\max_{1\le k\le n}\left|1-\frac{\bar{N}}{\|\tilde{y}_k\|^2}\right|\max_{1\le j\le n}\frac{1}{N}\|\tilde{z}_j\|^2\\ &\xrightarrow{\text{a.s.}} 0.\end{aligned}$$

The proof is concluded by putting these results together. □

We now introduced some results on martingale's theory for random matrices, which will be required in Chapter 6. We shall denote E_j the conditional expectation with respect to the understood $\sigma-$field $\mathcal{F}_j$ generated by the vectors, say $\{z_\ell, 1\le \ell\le j\}$, with conventionally $\mathrm{E}_0=\mathrm{E}$.

Lemma A.5 (Jensen Inequality, (Boyd and Vandenberghe, 2004))**.** *Let $\mathcal{I}$ be a discrete set of elements of $\{1,\dots,n\}$ with finite cardinality denoted by $|\mathcal{I}|$. Let $(\theta_i)_{i\in\mathcal{I}}$ be a sequence of complex scalars indexed by the set $\mathcal{I}$. Then, for any $p\ge 1$,*

$$\left|\sum_{i\in\mathcal{I}}\theta_i\right|^p\le|\mathcal{I}|^{p-1}\sum_{i=1}^n|\theta_i|^p$$

Lemma A.6 (Generalized Hölder inequality,(Karoui, 2008))**.** *Let $X_1,\cdots,X_k$ be k complex random variables with finite moments of order k. Then,*

$$\left|\mathrm{E}\left[\prod_{i=1}^k X_i\right]\right|\le\prod_{i=1}^k\left(\mathrm{E}\left[|X_i|^k\right]\right)^{\frac{1}{k}}.$$

It remains to introduce the Burkhölder inequalities on which the proof relies.

Lemma A.7 (Burkhölder inequality (Burkholder, 1973))**.** *Let $(X_k)_{k=1}^n$ be a sequence of complex martingale differences sequence. For every $p\ge 1$, there exists K_p dependent only on p such that:*

$$\mathrm{E}\left[\left|\sum_{k=1}^n X_k\right|^{2p}\right]\le K_p n^p\max_k\mathrm{E}\left[|X_k|^{2p}\right].$$

Letting $X_k=(\mathrm{E}_k-\mathrm{E}_{k-1})\,z_k^*A_kz_k$ where A_k is independent of z_k and noting that $\mathrm{E}\left[|X_k|^{2p}\right]\le \mathrm{E}\left[\|A_k\|_{\mathrm{Fro}}^{2p}\right]$, with $\|A\|_{\mathrm{Fro}}\triangleq\sqrt{\mathrm{tr}\,AA^*}$, we get in particular.

Lemma A.8 (Burkhölder inequality for quadratic forms)**.** *Let $z_1,\cdots,z_n\in\mathbb{C}^N$ be n independent random vectors with mean 0 and covariance C_N. Let $(A_j)_{j=1}^n$ be a sequence of $N\times N$ random matrices where for all j, A_j is independent of z_j. Define X_j as*

$$X_j=(\mathrm{E}_j-\mathrm{E}_{j-1})\,z_j^*A_jz_j=z_j^*\mathrm{E}_jA_jz_j-\mathrm{tr}\,\mathrm{E}_{j-1}C_NA_j.$$

Then,

$$\mathrm{E}\left[\left|\sum_{j=1}^{n} X_j\right|^{2p}\right] \leq K_p \left\|C_N\right\|_{\mathrm{Fro}}^{2p} n^p \max_j \mathrm{E}\left[\left\|A_j C_N\right\|_{\mathrm{Fro}}^{2p}\right].$$

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