Least Action Routing: Identifying the Optimal Path in a Wireless Relay Network

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***Abstract*—Consider a dense wireless network of nodes, which can be used to transfer data between arbitrary sources and destinations. In this paper we develop a methodology based on variational calculus to optimize a number of path metrics, such as the success probability or the total power consumed of a packet delivery in the presence of external interference. We then extend the approach to a two-packet transmission, in which the relaying of each packet causes interference to the other. In both cases, we show that the optimal path may differ significantly from a straight line. We then discuss the consequences of these deviations in the context of network design.**

I. Introduction

Future wireless networks are expected to become increasingly dense, as a way to satisfy growing traffic demands, and combat intereference by transmitting at lower local power. As a result, nodes in the wireless network are expected to act not only as sources and destinations but also as relays. Traffic will then need to be routed in a way so as to optimize certain metrics, in the presence of constraints and/or external interference from exogenous factors, such as macro-base stations. Such objectives include the reception probability over a number of many hops, the total routing time, the number of hops, or the trasmitted power.

One way to analyze this optimization problem is to take the continuous limit, i.e. to treat the routing process as a continuous information flow, treating each hop as an infinitesimal step. Underlying this approximation is the observation that one is interested in the behavior of quantities over lengthscales much larger than a characteristic length. Since the pathloss is typically assumed as a power law and hence scale-free, the length is typically introduced either through the average node density and its associated internode distance, or through a transmitted power scale, necessary to obtain a target SNR over any given hop.

This approach has been successfully applied in the past, e.g. providing macroscopic predictions of the network capacity limits in a wireless network [1]. In addition, the author of [2] has applied it to find the optimal route for a packet in the presence of a variable node density, by making the analogy to the brachistochrone problem. Building on this result, [3] found the conditions for the minimal node number necessary to transmit information from one region to another by taking into account the interference effects of the node density on the routes themselves. They obtain the optimal node solution by showing that the optimal flow is irrotational. More recently, [4] used the continuum approximation to find an optimality criterion for transmission through a static interference field, assuming a fixed hopping distance.

In this paper we extend the ideas of [4], focusing on a specific optimization criterion for the path, namely the maximization of the total reception probability for fixed average power and maximum delay time. We assume a static interference field, which may be present and makes the straight (closest) route not necessarily optimal. In contrast to previous approaches, which assumed a constant relay distance, we allow for a variable distance, which in fact may be spatially varying and hence may be optimized. By employing the continuum approximation, we apply variational calculus to obtain a differential equation, characterizing the optimal routing path. We map the logarithm of the success probability to a quantity that has been called *action* in classical mechanics [5]. We then optimize subject to a number of constraints, such as the average power or the packet delivery deadline. We also generalize the methodology to the case of two packets being relayed simultaneously in the network, while interfering each other.

In the next section we properly define the system model. In Section III we analyze the behavior of single packet transmission in the presence of interference. We derive the Euler-Lagrange equations for this case and provide details for their solution. In Section IV we generalize the approach to two packets interfering with each other, while in Section V we discuss a number of representative examples.

II. System Model

Consider a wireless network with relays distributed uniformly throughout the plane with density λ. Suppose that a packet is transmitted at position $\mathbf{r}_o$ with destination a given node located at position $\mathbf{r}_d$. This packet can arrive at its destination through a number of relays, that receive and re-transmit it along a particular path. Hence an M-relay path is the set of positions $\mathcal{P} = \{\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_M\}$, where $\mathbf{r}_0 = \mathbf{r}_o$ and $\mathbf{r}_M = \mathbf{r}_d$. The distance of the kth relay retransmission ("hop") is $|\delta\mathbf{r}_k|$, where $\delta\mathbf{r}_k = \mathbf{r}_k - \mathbf{r}_{k-1}$.

The received signal at the kth relay can be expressed as

$$r_k = h_k t_k + z_k \tag{1}$$

This research was partially supported by the Huawei HIRP FLAGSHIP project ULTRON and the French National Research Agency (ANR) grant ORACLESS (contract no. ANR–16–CE33–0004–01).

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where h_k is the channel gain coefficient, t_k is the transmitted signal with variance $p(\mathbf{r}_k)$, and z_k is the total interference and noise at the receiver. We further assume that the variance of h_k is

$$\mathbb{E}[|h_k|^2] = x_k g(\delta \mathbf{r}_k), \tag{2}$$

where x_k is an independent exponentially distributed random variable representing the fast-fading of the link, and $g(\mathbf{r})$ is the pathloss function. For the latter, we assume the general form $g(\mathbf{r}) = g_0|\mathbf{r}|^{-\alpha}$, where g_0 is a constant and $\alpha > 2$ is the pathloss exponent. Similarly, the variance of z_k is assumed to be of the form

$$\mathbb{E}[|z_k|^2] = 1 + y_k I_k(\mathbf{r}_k), \tag{3}$$

where again y_k is exponentially distributed and $I_k(\mathbf{r})$ denotes the interference function at hop k. The first term corresponds to thermal noise variance, which has been set to unity for simplicity, while the second term corresponds to the interference evaluated at the receiver node of the kth hop, which may be position-dependent, due to the existence of a macro-base station transmitting in the area, or due to the interference of other packets being relayed at the same time in the network.

The success probability q_k for the kth relay hop can be expressed as

$$-\log(q_k) = \frac{\gamma|\delta\mathbf{r}_k|^\alpha}{p(\mathbf{r}_k)} + \log\left(1 + \frac{\gamma I(\mathbf{r}_k)|\delta\mathbf{r}_k|^\alpha}{p(\mathbf{r}_k)}\right) \tag{4}$$

where $\gamma = (e^r - 1)/g_0$ and the desired rate of the transmission is r. Hence, the success probability $\Pi_{\mathcal{P}}$ for a given path $\mathcal{P}$ is the product of the success probabilities over each relay.

From an optimization point of view, the most challenging case is when there are many hops per path. Of course, in this case, for the total success probability to be sufficiently high, the success probability for each hop needs to be close to unity. Hence, both terms proportional to γ in the above equation need to be $O(1/M)$. This observation leads to the following approximation

$$-\log(q_k) \approx \frac{\gamma(1 + I(\mathbf{r}_k))|\delta\mathbf{r}_k|^\alpha}{p(\mathbf{r}_k)} \tag{5}$$

In this case, the success probability for the total path can be expressed as $\log \Pi_{\mathcal{P}} = -\mathcal{S}_1$, where

$$\mathcal{S}_1 = \gamma \sum_{k=1}^{M} \frac{I(\mathbf{r}_k)|\delta\mathbf{r}_k|^\alpha}{p(\mathbf{r}_k)} \tag{6}$$

where for simplicity we have redefined the interference function I to include the thermal noise term, i.e. $I(\mathbf{r}) \leftarrow I(\mathbf{r}) + 1$.

In the same limit, it should be pointed out that $\mathcal{S}_1$ is approximately the average excess delay of a packet being relayed through a given path, if retransmissions are allowed whenever the packet is not transmitted successfully over a given relay. Indeed, for the kth relay, the average number of retransmissions is given by $q_k^{-1} - 1$. Hence, the total number of retransmissions over a given path is

$$\delta\mathcal{N}_{\mathcal{P}} = \sum_{k=1}^{M}\left(q_k^{-1} - 1\right) \approx \mathcal{S}_1 \tag{7}$$

We now proceed to take the continuum limit of (6). For this limit to be meaningful, we need the noise field to vary slowly at the scale of the typical relay distance $\delta\mathbf{r}$, which is of the order of the internode distance, namely $\lambda^{-1/2}$. In addition, and as mentioned above, for the success probability for each path to be appreciable, we need the right-hand-side of (6) to be of order $O(1)$, even for large number of hops M. With these considerations in mind, we rescale γ to $\gamma_0 = \gamma\tau^{\alpha-1}$, where τ is the time needed for any relay process and is assumed independent of the relay distance, depending only on transceiver physical layer specifics. As a result, (6) becomes

$$\mathcal{S}_1 = \gamma_0 \int_0^T dt \frac{I(\mathbf{r}(t))|\dot{\mathbf{r}}(t)|^\alpha}{p(\mathbf{r})} \tag{8}$$

where $\dot{\mathbf{r}}(1) = \delta\mathbf{r}_k/\tau$ is the velocity at position $\mathbf{r}$, while $T = M\tau$ is the total time necessary for the relay. $\mathcal{S}_1$ is known as the *action* in classical mechanics [5].

The objective of this paper is to minimize the above quantity over all paths connecting the origin of the packet to its destination, thus maximizing the success probability. It should be pointed out that this maximization is optimal in the sense that the interference field $I(\mathbf{r})$ needs to be known over the whole region of interest. In the Section III, we derive a second differential equation that characterizes this optimal path subject to a number of different constraints on the power. In Section IV we explore the case where the paths of two packets are simultaneously optimized in the presence of mutual interference.

III. Analysis for Single Packet Routing

In this section we analyze the optimal path minimizing $\mathcal{S}_1$ in the single origin-destination case.

A. Maximum Success Probability for Fixed Power and Time

The simplest case is to assume that the power over each transmission is fixed at $p(\mathbf{r}) = p_0$. In this case we simply have

$$\mathcal{S}_1 = \frac{\gamma_0}{p_0} \int_0^T dt I(\mathbf{r})|\dot{\mathbf{r}}|^\alpha \tag{9}$$

To proceed, it is convenient to introduce the so-called Lagrangian functional

$$L_1(\mathbf{r}, \dot{\mathbf{r}}) = I(\mathbf{r})|\dot{\mathbf{r}}|^\alpha \tag{10}$$

We are now in a position to state the Euler-Lagrange equations that are satisfied by the path that minimizes $\mathcal{S}_1$, subject to required initial and final conditions.

Theorem 1. *Let $\mathbf{r}(0) = \mathbf{r}_o$ and $\mathbf{r}(T) = \mathbf{r}_d$. Then, the path that minimizes $\mathcal{S}_1$ satisfies the second-order differential equations:*

$$\frac{d}{dt}\left(\alpha I(\mathbf{r})|\dot{\mathbf{r}}|^{\alpha-2}\dot{\mathbf{r}}\right) = |\dot{\mathbf{r}}|^\alpha \nabla I(\mathbf{r}) \tag{11}$$

Proof. See Appendix A. □

1) Optimal path and success probability: The solution of the above differential equations is greatly facilitated by the existence of constants of motion. In particular, the fact that L_1 does not depend explicitly on t implies that the so-called Hamiltonian

$$H_1 = \dot{\mathbf{r}} \cdot \frac{\partial L_1}{\partial \dot{\mathbf{r}}} - L_1 = \left(\frac{\alpha}{2} - 1\right) L_1 \tag{12}$$

remains constant along the optimal path. Therefore its value can be obtained directly from the initial location of the packet and its initial velocity, implying in turn that $\mathcal{S}_1 = L(\mathbf{r}_o, \dot{\mathbf{r}}_o)T$. However, this is not enough, since we need to express this in terms of packet's initial and final position; in particular we still need to integrate the above differential equations.

An additional constant of motion exists, if $I(\mathbf{r})$ only depends on the modulus of the location $\mathbf{r}$, i.e. if $I(\mathbf{r}) \equiv I(|\mathbf{r}|)$. To find this invariant, we first express the velocity in polar coordinates, as $\dot{\mathbf{r}} = \dot{r}(\cos\theta, \sin\theta) + r\dot{\theta}(-\sin\theta, \cos\theta)$ so $|\dot{\mathbf{r}}|^2 = (\dot{r}^2 + r^2\dot{\theta}^2)$. As a result, optimal routing paths conserve *angular momentum*

$$p_\theta = \frac{\partial L_1}{\partial \dot{\theta}} = \alpha I(r) r^2 |\dot{\mathbf{r}}|^{\alpha-2}\dot{\theta} \tag{13}$$

Taking into account the above invariants, we obtain the following first order differential equation, which can be integrated directly to obtain the evolution of the radius of the optimal path $r(t)$:

$$\dot{r} = \pm \left(\frac{L_1}{I(r)}\right)^{\frac{1}{\alpha}} \sqrt{1 - \frac{p_\theta^2 L_1^{\frac{2}{\alpha}-2}}{\alpha^2 r^2 I(r)^{\frac{2}{\alpha}}}} \tag{14}$$

In the above equation, the quantities L_1, p_θ are constants, while the $\pm$ sign indicates the original sign of the radial velocity $\dot{r}$. If at some point the quantity under the square-root sign vanishes, the radial velocity re-emerges with the opposite sign. The time-dependence of the angle $\theta(t)$ can be obtained by integrating over (13).

2) Additional constraint on packet delivery deadlines: In the previous subsection we showed how to obtain the path that maximizes the success probability for fixed power per hop and a fixed number of hops. However, the total relay duration is a variable of the problem as well, which, depending on the application may need to be constrained to be less then a given maximum delay $T_{\max}$. Rather than adding an additional inequality constraint in the problem, we observe that by rescaling time to $t = sT$ with $0 < s < 1$, the action $\mathcal{S}_1$ is rescaled as

$$\mathcal{S}_1(T) = T^{1-\alpha}\mathcal{S}_1(1) \tag{15}$$

As a result of this rescaling, the dependence of the optimal success probability on time can thus be obtained directly. The above time-dependence indicates that the success probability always decreases with increasing time. As a result, a maximum time delay constraint will be exhausted with equality. This indicates that we can trade probability of success with the time duration depending on the urgency or aggression of the information to be sent.

B. Minimum Total Power for Fixed Success Probability

A related optimization setting has to do with the case where each relay is always able to provide enough power to ensure a certain fixed success probability per link given by $q = e^{-c_0\tau}$. In this situation the total path success probability along the path is fixed to $\Pi = e^{c_0 T}$ irrespective of the path choice (the path can be further optimized to minimize the total power consumed). It is then easy to see that the power per hop is given by

$$p(\mathbf{r}) = \frac{\gamma_0 I(\mathbf{r})|\dot{\mathbf{r}}|^\alpha}{c_0} \tag{16}$$

Hence, minimizing the total power is exactly analogous to the analysis of the previous section by considering instead the action

$$\mathcal{S}_1 = \frac{\gamma_0}{c_0}\int_0^T dt I(\mathbf{r})|\dot{\mathbf{r}}|^\alpha \tag{17}$$

Therefore the optimal path for fixed instantaneous power and variable success probability is identical to the case of fixed success probability and minimum power. It is worth pointing out that the same functional also appears in the case where the time is to be minimized for a fixed success probability.

C. Minimum Success Probability for Maximum Average Power

If the inversion of the channel is not possible, one may consider to fix an average power budget of the form

$$\int_0^T p(\mathbf{r}(t))dt \leq p_0 T \tag{18}$$

Since $\mathcal{S}_1$ is a decreasing function of all powers, its minimum will exhaust all available power, therefore making the above expression an equality. To enforce this constraint, we modify the objective function by adding a Lagrange multiplier, as follows:

$$\mathcal{S}_{1p} = \mathcal{S}_1 + \mu \int_0^T dt\,(p(\mathbf{r}(t)) - p_0) \tag{19}$$

We may now directly minimize the above expression over $\{p(\mathbf{r})\}$, to obtain the expression

$$p(\mathbf{r}) = \frac{\sqrt{I(\mathbf{r}(t))}|\dot{\mathbf{r}}(t)|^{\frac{\alpha}{2}}}{\mu} \tag{20}$$

where μ can be evaluated using the total power constraint in (18). As a result we have

$$\mathcal{S}_{1p} = \frac{\gamma_0}{p_0 T}\left(\int_0^T dt\,\sqrt{I(\mathbf{r}(t))}|\dot{\mathbf{r}}(t)|^{\frac{\alpha}{2}}\right)^2 \tag{21}$$

As we can see, the action functional in the above equation is, up to a square root of exactly the same form as $\mathcal{S}_1$, so least action paths are of the same in both cases.

IV. Analysis for Two Packet Routing

Let us now generalize the model of the previous section to allow for two packets to be simultaneously transmitting origin-destination pairs, in such a way that the interference on each relay is due to the signal transmitted at the other. Note that the analysis includes the case where two packets are routed through different paths with the same origin and destination pair. For simplicity, we assume constant power in each relay transmission. The aim of the analysis will be to find the joint paths of the packets that maximize the product of success probabilities.

In this case the Lagrangian function can be expressed as

$$L_2 = \frac{1}{p_0}\left[\gamma_2 I(\mathbf{r}_{12} + \delta\mathbf{r}_1)|\dot{\mathbf{r}}_2|^\alpha + \gamma_1 I(\mathbf{r}_{12} + \delta\mathbf{r}_2)|\dot{\mathbf{r}}_1|^\alpha\right] \tag{22}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. The first term is due to the relaying of packet 2 and vice-versa. The extra factor of $\delta\mathbf{r}_1$ in the argument of the first term is due to the fact that the interference at the receiver of packet 2 is due to the transmitter of packet 1, which is $\delta\mathbf{r}_1$ away from its corresponding receiver. Similarly, $\delta\mathbf{r}_2$ has to appear in the second interference term. However, since $\delta\mathbf{r}_1 = \dot{\mathbf{r}}_1\tau$ is of order $O(1/M)$ when the number of hops grows large, we will ignore them in the analysis below. As a result, the joint $-\log P_{success}$ can be expressed as

$$\mathcal{S}_2 = \frac{1}{p_0}\int_0^T dt I(\mathbf{r}_{12})\left(\gamma_1|\dot{\mathbf{r}}_1|^\alpha + \gamma_2|\dot{\mathbf{r}}_2|^\alpha\right) \tag{23}$$

Note that in the above equation we have assumed reciprocity in the interference between the two packets. The Euler-Lagrange equations for the jointly optimal paths that minimize $\mathcal{S}_2$ are summarized in the following theorem.

Theorem 2. *Let $\mathbf{r}_1(0) = \mathbf{r}_{1,o}$, $\mathbf{r}_1(T) = \mathbf{r}_{1,d}$ and $\mathbf{r}_2(0) = \mathbf{r}_{2,o}$, $\mathbf{r}_2(T) = \mathbf{r}_{2,d}$ be the two origin and destination pairs. Then the paths $\mathbf{r}_i(t)$, for $i = 1, 2$, that minimizes $\mathcal{S}_2$ satisfy the following differential equations:*

$$\alpha\gamma_i\frac{d}{dt}\left(I(\mathbf{r}_{12})|\dot{\mathbf{r}}_i|^{\alpha-2}\dot{\mathbf{r}}_i\right) = \left(\gamma_1|\dot{\mathbf{r}}_1|^\alpha + \gamma_2|\dot{\mathbf{r}}_2|^\alpha\right)\nabla I(\mathbf{r}) \tag{24}$$

where the index $i = 1, 2$ represents the corresponding packet.

Proof. The proof follows the lines of the proof of Theorem 1 and is therefore omitted. □

1) Optimal path and success probability: The above equations constitute a set of four coupled second order differential equations. As in the case of a single origin-destination pair, constants of motion are very useful to obtain analytic solutions of these equations. Once again, the absence of explicit time dependence of the integrand in (23) implies that the Hamiltonian H_2 below is constant

$$H_2 = \dot{\mathbf{r}}_1\cdot\frac{\partial L_2}{\partial\dot{\mathbf{r}}_1} + \dot{\mathbf{r}}_2\cdot\frac{\partial L_2}{\partial\dot{\mathbf{r}}_2} - L_2 = \left(\frac{\alpha}{2} - 1\right)L_2 \tag{25}$$

In turn, this implies the invariance of the Lagrangian L_2, which will be useful in the solution for the optimal paths. In addition, the fact that the location of the packets enters in the problem only in terms of the difference of their position $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ provides two additional constants of motion. To find them, we change velocity variables from $\dot{\mathbf{r}}_1$, $\dot{\mathbf{r}}_2$ to $\mathbf{u} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$ and $\mathbf{v}_c = \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2$, where the latter is the velocity of the barycenter of the two packet locations. Then the vector quantity

$$\mathbf{p}_c = \frac{\partial L_2}{\partial\mathbf{v}_c} = I(\mathbf{r}_{12})\left(\gamma_1|\dot{\mathbf{r}}_1|^{\alpha-2}\dot{\mathbf{r}}_1 + \gamma_2|\dot{\mathbf{r}}_2|^{\alpha-2}\dot{\mathbf{r}}_2\right) \tag{26}$$

Finally, as in the case of a single origin-destination pair, if $I(\mathbf{r}_{12})$ is only a function of the modulus of the distance between the two packet positions, then an additional constant of motion exists.

2) Additional constraint on packet delivery deadline: As in the previous section, the dependence of the success probability on time can be immediately extracted without solving the Euler-Lagrange equations, by simply rescaling the time in L_2. As a result, we again obtain that

$$\mathcal{S}_2(T) = T^{1-\alpha}\mathcal{S}_2(1) \tag{27}$$

V. Results and Analysis

In this section we will analyze the equations derived in the previous sections and discuss the results.

A. Single Packet Analysis

1) Constant Interference: To provide a baseline case in the case of the single packet routing, we start setting the interference to a constant $I(\mathbf{r}) = 1$. From (11) we see that

$$\frac{d^2\mathbf{r}(t)}{dt^2} = 0 \tag{28}$$

This means that the velocity is constant, indicating a straight line from origin to destination. This result is of course expected, but it provides an additional validation of the method. The corresponding value of the exponent of the success probability is

$$\mathcal{S}_1 = T^{1-\alpha}\frac{\gamma_0|\mathbf{r}_d - \mathbf{r}_o|^\alpha}{p_0} \tag{29}$$

2) Interference due to Macro Base-Station: To analyze the effects of the interference to the path, we choose a specific dependence of $I(\mathbf{r})$ on distance, namely $I(\mathbf{r}) = g_0 r^{-\alpha}$. This is the interference due e.g. to a static macro base station in the vicinity of the transmitting O/D pair. In this case, the optimal path solution of (14) can be expressed in polar coordinates as

$$r(t) = r_o\exp\left[\frac{t}{T}\log\left(\frac{r_d}{r_o}\right)\right] \tag{30}$$
$$\theta(t) = \frac{(\theta_d - \theta_o)t}{T}$$

where r_o, θ_o and r_d, θ_d are the radial distance and angle of the origin and destination locations of the packet with respect to the location of the interferer. The corresponding success probability exponent can be expressed as

$$\mathcal{S}_1 = T^{1-\alpha}\left((\theta_d - \theta_o)^2 + \log\left[\frac{r_d}{r_o}\right]^2\right)^{\frac{\alpha}{2}} \tag{31}$$

B. Two Packet Analysis

To obtain closed form results, we will make a number of simplifying assumptions, which however do not diminish the relevance of the results. First, we introduce a cutoff in the interference between packets, namely

$$I(r) = \min\left(r_0^{-\alpha}, r^{-\alpha}\right) \tag{32}$$

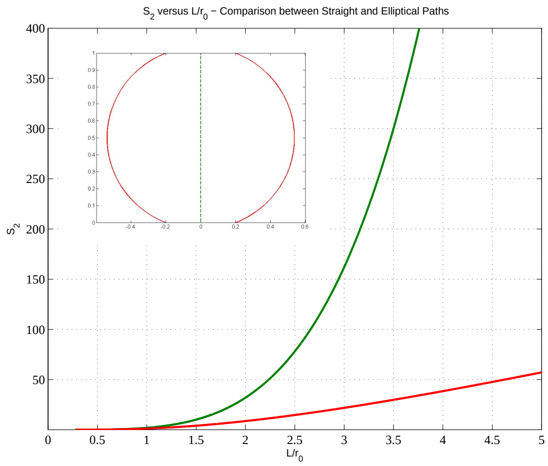

Fig. 1: Comparison between Linear and Elliptical paths when the both origin and destination pairs are r_0 apart, while each origin is L apart from its destination. The red curve corresponds to the optimal path in which the two origin and destination pairs are separated by r_0. Optimal Paths for 2 Packets traveling from the origin to $0, L$. We have also drawn the straight path for comparison. The circles on the paths represent the location where the path switches from linear to elliptical. Parameter values $r_0 = 0.25L$ and $\gamma_1 = \gamma_2$.

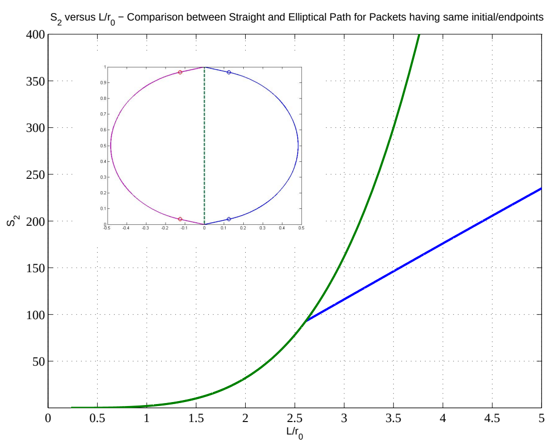

Fig. 2: Comparison between Linear and Elliptical paths when the two packets have the same origin and destination, which are L apart. The pathloss here is given by $\max(r_0^{-\alpha}, r^{-\alpha})$. The blue curve represents $\mathcal{S}_2$ for the elliptical path, while the green curve is $\mathcal{S}_2$ for the straight path. In the inset we have included the shape of the elliptical paths. We have also drawn the straight path for comparison. The circles on the paths represent the location where the path switches from linear to elliptical. Parameter values $r_0 = 0.25L$ and $\gamma_1 = \gamma_2$.

for some appropriately chosen distance r_0 to control the unphysical divergences when the packets are too close to each other. Second we will take the symmetric transmission case, namely assuming $\gamma_1 = \gamma_2 = \gamma_0$ and with the positions of the packets given by $\mathbf{r}_1 = (x, y)$ and $\mathbf{r}_2 = (-x, y)$. In this case we can express the differential equations of x, y as

$$\dot{x} = \frac{v_0}{x_0} z(x) \sqrt{1 - \frac{v_{y0}^2 z(x)^2}{v_0^2 x_0^2}} \qquad (33)$$

$$\dot{y} = v_{y0} \frac{z(x)^2}{x_0^2}$$

where $r_0 = 2x_0$, v_{y0}, v_0 are the y-component of the initial velocity and its total modulus respectively, and $z(x) = max(x_0, x)$. The above equations can be integrated directly to give a parabola with center $(0, L/2)$ for $x > x_0$ and $x < L - x_0$ and a linear dependence between x, y elsewhere, as can be seen in Fig. 1. Denoting by $\sin 2\theta = 2r_0/L$, we can obtain the maximum horizontal separation between packets at the center of the path to be

$$R = \frac{r_0}{\sin\theta} \qquad (34)$$

In addition, we can calculate the exponent of the success probability to be

$$\mathcal{S}_2 = \frac{2\gamma_0 T^{1-\alpha}}{p_0}\left(\frac{1}{\cos\theta} + \log\left[\frac{1+\cos\theta}{\sin\theta}\right]\right)^{\alpha} \qquad (35)$$

Interestingly, for $L < 2.609 r_0$ the straight, constant velocity path has a lower value of $\mathcal{S}_2$ and hence is preferable, see Fig. 2.

VI. Conclusions and Perspectives

From the analysis in this paper we may conclude that interference plays an important role in the routing of data through an ad-hoc network. We have seen that the deviation of paths from a straight line may add significant benefits in success probability or power gains for a fixed success probability. We have also explored the possibility of variable range hopping. In fact we have seen that the further away from the interference source, the larger the steps the relaying process takes, suggesting the possibility for different node density in various region of the network. In the future we plan to include a dense set of relay paths in the network.

Appendix A
Euler-Lagrange Equations

In this Appendix we derive the Euler-Lagrange equations that appear in Theorem 1. As mentioned there, we let $\mathbf{r}(0) = \mathbf{r}_o$ and $\mathbf{r}(T) = \mathbf{r}_d$. We now assume that $\mathbf{r}^*(t)$ is the path that extremizes $\mathcal{S}_1$ in (9) and elsewhere. This means that small variations of the path around the optimal one leave the value of $\mathcal{S}_1$ unaffected to leading order. Hence, let

$$\mathbf{r}(t) = \mathbf{r}^*(t) + \delta\mathbf{r}(t) \qquad (36)$$

be such a path, which however has the same initial and final points fixed, namely $\delta\mathbf{r}(0) = \delta\mathbf{r}(T) = 0$. The variation of $\mathcal{S}_1$ for this path is

$$\delta\mathcal{S}_1 = \sum_{i=1}^{2}\int_0^T dt\left(\frac{\partial L_1}{\partial \mathbf{r}_i}\delta\mathbf{r}_i + \frac{\partial L_1}{\partial \dot{\mathbf{r}}_i}\delta\dot{\mathbf{r}}_i\right) \qquad (37)$$

$$= \sum_{i=1}^{2}\int_0^T dt\left(\frac{\partial L_1}{\partial \mathbf{r}_i}\delta\mathbf{r}_i - \frac{d}{dt}\left[\frac{\partial L_1}{\partial \dot{\mathbf{r}}_i}\right]\delta\mathbf{r}_i\right) + \left.\frac{\partial L_1}{\partial \dot{\mathbf{r}}_i}\delta\mathbf{r}_i\right|_0^T$$

In the above equation, the index $i = 1, 2$ signifies the vector component. To go from the first to the second line, we integrate by parts. In the second line, the last term vanishes due to the vanishing variations at the boundaries of the path. Thus demanding that $\delta \mathcal{S}_1 = 0$ for any choice of path variation makes the first term vanish identically, which then produces the equations in (11).

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