Fast Routing under Uncertainty: Adaptive Learning in Congestion Games with Exponential Weights

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Abstract

We examine an adaptive learning framework for nonatomic congestion games where the players' cost functions may be subject to exogenous fluctuations (e.g., due to disturbances in the network, variations in the traffic going through a link, etc.). In this setting, the popular multiplicative / exponential weights algorithm enjoys an $\mathcal{O}(1/\sqrt{T})$ equilibrium convergence rate; however, this rate is suboptimal in *static* environments – i.e., when the network is not subject to randomness. In this static regime, accelerated algorithms achieve an $\mathcal{O}(1/T^2)$ convergence speed, but they fail to converge altogether in *stochastic* problems. To fill this gap, we propose a novel, *adaptive exponential weights* method – dubbed AdaWeight – that seamlessly interpolates between the $\mathcal{O}(1/T^2)$ and $\mathcal{O}(1/\sqrt{T})$ rates in static and stochastic environments respectively. Importantly, this "best-of-both-worlds" guarantee does not require *any* prior knowledge of the problem's parameters or any tuning by the optimizer; in addition, the method's convergence speed depends subquadratically on the size of the network (number of vertices and edges), so it scales gracefully to large, real-life urban networks.

1 Introduction

Navigation apps like Google Maps and Waze have user bases numbering in the hundreds of millions, and they may receive upwards of 10^4 routing requests per second; in fact, Google Maps alone exceeded one billion monthly active users in 2019, and its interface routinely receives up to 10^5 requests during rush hour in major metropolitan centers [7]. This vast number of users must be routed efficiently, in real-time, and without causing any "ex-post" regret at the user end; otherwise, if a user could have experienced better travel times along a non-recommended route, they would have no incentive to follow the app recommendation in the first place.

In the language of congestion games [34], this requirement is known as a "Wardrop equilibrium", and it is typically represented as a high-dimensional vector describing the traffic flow along each path in the network [43]. Ideally, this equilibrium should be computed *before* making a recommendation, so as to minimize the number of disgruntled users. In practice however, this is rarely possible: the state of the network typically depends on random factors that vary from one epoch to the next (weather conditions, traffic accidents, fluctuations in the total number of commuters in the system, etc.), so it is generally unrealistic to expect that such a recommendation can be made in advance.

Our paper takes a learning approach to this problem: routing recommendations are provided in an online manner, and they are subsequently updated "on the fly" once the state of the network has

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been observed. In more detail, motivated by applications to GPS routing and navigation apps, we consider an adaptive recommendation paradigm that unfolds as follows:

1. At each epoch $t = 1, 2, \ldots$, a centralized control interface – such as Google Maps – determines a routing flow for its users and provides a recommendation accordingly.
2. After making a recommendation, the interface observes the travel times of the network's users; based on this feedback, it updates the routing recommendation and the process repeats.

Main challenges. There are several key challenges that arise in this setting. First and foremost, learning methods that are well-suited to rapidly fluctuating environments may be highly suboptimal in static networks and vice versa. Second, the problem's dimensionality – the number P of available paths – is exponential in the size of the underlying network, so it is crucial to propose learning methods that remain efficient in large networks. Finally, methods that require prior knowledge of the problem's parameters – e.g., the smoothness modulus of the network's latency functions – are beyond reach because such knowledge cannot be realistically obtained by the optimizer.

In view of all this, our paper seeks to answer the following question:

Is it possible to design an adaptive, parameter-agnostic algorithm that is simultaneously optimal in static and stochastic networks, and whose convergence rate is polynomial in the network's size?

Our contributions in the context of related work. Our paper proposes a novel, *adaptive exponential weights* algorithm – dubbed AdaWeight – which enjoys the following desirable properties:

1. In static networks, the method converges to a Wardrop equilibrium at a rate of $\mathcal{O}((\log P)^{\frac{3}{2}}/T^2)$.
2. In stochastic networks, it converges to a mean Wardrop equilibrium at an $\mathcal{O}((\log P)^{\frac{3}{2}}/\sqrt{T})$ rate.
3. These rates are attained without any prior tuning by the optimizer.

In the above, T denotes the learning horizon (number of epochs) and P is the number of paths used to route traffic in the network. Thus, even though P may grow exponentially, the logarithmic dependence on P ensures that the algorithm's runtime remains *polynomial*, and in fact, subquadratic, in the size of the network. To the best of our knowledge, AdaWeight is the first method that simultaneously achieves these desiderata; to provide the necessary context, we give below a detailed account of the related work on the topic.

The *static* regime of our learning model matches the standard framework of Blum et al. [5] who showed that a variant of the classic *exponential weights* (EW) algorithm [2, 3, 27, 42] converges to a Wardrop equilibrium at an $\mathcal{O}(1/\sqrt{T})$ rate (in the Cesàro, time-averaged sense). This result was subsequently extended to *stochastic* congestion games by Krichene et al. [18, 20], who showed that the EW algorithm also enjoys an $\mathcal{O}(1/\sqrt{T})$ convergence rate to *mean* Wardrop equilibria (again, in a Cesàro sense). As we discuss in the sequel, the convergence speed of the EW algorithm of Blum et al. [5] and Krichene et al. [18] is $\mathcal{O}(\log P/\sqrt{T})$ in both cases; however, if the method's learning rate is not chosen appropriately, the EW algorithm may lead to non-convergent, chaotic behavior, even in symmetric congestion games over a 2-link Pigou network [36].

In general equilibrium problems, the $\mathcal{O}(1/\sqrt{T})$ rate cannot be improved without more stringent assumptions – such as strong monotonicity and the like. However, nonatomic congestion games are well known to admit a convex potential – sometimes referred to as the *Beckmann–McGuire–Winsten* (BMW) potential [4] – so the $\mathcal{O}(1/\sqrt{T})$ convergence guarantee of Blum et al. [5] is *not* optimal. In the static regime, the optimal smooth convex minimization rate is $\mathcal{O}(1/T^2)$ [28, 30], and it is achieved by the seminal "accelerated gradient" algorithm of Nesterov [29]. If applied directly to our problem, Nesterov's algorithm has a catastrophic $\Theta(P)$ dependence on the number of paths; however, by coupling it with a "mirror descent" template in the spirit of [32, 44], Krichene et al. [20] proposed an accelerated method with an exponential projection step that is particularly well-suited for congestion problems. In fact, going a step further, it is possible to design an accelerated exponential weights method – AcceleWeight for short – that achieves an $\mathcal{O}(\log(P)/T^2)$ rate in static environments.

Crucially, the learning rate parameter of AcceleWeight must be tuned with prior knowledge of the problem's smoothness parameters; moreover, despite its optimality in the static regime, the method fails to converge altogether in stochastic problems. The universal algorithm of Nesterov [33] provides a method to resolve the former issue, but it relies on a line-search mechanism that cannot be

	EW	ACCELEWEIGHT	UNIXGRAD	UPGD	AC-SA	ADAWEIGHT [ours]
STATIC	$\log P/\sqrt{T}$	$\log P/T^2$	P/T^2	$\log P/T^2$	$\log P/T^2$	$\log P/T^2$
STOCH.	$\log P/\sqrt{T}$	×	$P/\sqrt{T}$	×	$\log P/\sqrt{T}$	$\log P/\sqrt{T}$
ANYTIME	partially	✓	✓	×	×	✓
PAR. AGN.	partially	×	✓	✓	×	✓

Table 1: Overview of related work. All rates are reported in the $\mathcal{O}(\cdot)$ sense; "par. agn." means that the method is *parameter-agnostic*, i.e., it does not require prior tuning or knowledge of the problem's parameters.

adapted to a stochastic framework, so it does not resolve the latter. On the flip side, the accelerated stochastic approximation (AC-SA) algorithm of Lan [22] achieves optimal rates in both the static and stochastic regimes, but it does not provide anytime guarantees (the iteration budget must be fixed as a function of the accuracy threshold required), and it assumes full knowledge of the smoothness modulus of the game's cost functions (which is not realistically available to the optimizer).

Our approach adapts to the problem's smoothness parameters via an "inverse-sum-of-squares" learning rate in the spirit of ADAGRAD [12] – though ADAGRAD itself lacks an acceleration mechanism, so it is suboptimal in static environments [25, 26]. To the best of our knowledge, the first order-optimal interpolation result was achieved by the ACCELEGRAD algorithm of Levy et al. [25] for *unconstrained* problems (and assuming knowledge of a compact set containing a solution of the problem). The UNIXGRAD proposal of Kavis et al. [17] subsequently achieved the desired adaptation in constrained problems, but under the requirement of a bounded Bregman diameter. This requirement rules out the EW template (the simplex has infinite entropic diameter), so the convergence speed of UNIXGRAD is polynomial in the number of paths, and hence unsuitable for large network instances. For convenience, we compare all these works in Table 1 above.

2 Problem setup

2.1. The game. Building on the classical model of Beckmann et al. [4], we will consider a class of routing games defined by three basic primitives: *(i)* the game's *network structure*; *(ii)* the associated set of *traffic demands*; and *(iii)* the network's *cost functions*. The formal definition is as follows:

1. **The network structure:** Consider a directed graph $\mathcal{G} \equiv \mathcal{G}(\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. The focal point of interest is a set of pre-determined *origin-destination* (O/D) pairs $(O_i, D_i) \in \mathcal{V} \times \mathcal{V}$ indexed by $i \in \mathcal{N} = \{1, \dots, N\}$. Each pair $i \in \mathcal{N}$ is associated to a *traffic demand* $M_i > 0$ that is to be routed from O_i to D_i via a fixed set of *paths* (or *routes*) $\mathcal{P}_i$ joining O_i to D_i in $\mathcal{G}$. We denote the set of all such paths in the network by $\mathcal{P} \coloneqq \bigcup_{i\in\mathcal{N}} \mathcal{P}_i$ and, for concision, we denote the corresponding cardinalities as $P_i \coloneqq |\mathcal{P}_i|$ and $P \coloneqq |\mathcal{P}| = \sum_i P_i$. We also write $M_{\text{tot}} \coloneqq \sum_{i\in\mathcal{N}} M_i$ and $M_{\max} \coloneqq \max_{i\in\mathcal{N}} M_i$ for the total and maximum traffic demand associated to network's O/D pairs.
2. **Routing flows:** In order to route the traffic, the set of feasible *flow profiles* is defined as
$$\mathcal{X} \coloneqq \left\{x \in \mathbb{R}_+^{\mathcal{P}} : \textstyle\sum_{p\in\mathcal{P}_i} x_{i,p} = M_i, i = 1, \dots, N\right\} \tag{1}$$
i.e., as the product of scaled simplices $\mathcal{X} = \prod_i M_i \, \Delta(\mathcal{P}_i)$. In turn, each feasible flow profile $x \in \mathcal{X}$ induces on each edge $e \in \mathcal{E}$ a *routing load* $\mu_e(x) = \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} \mathbb{1}_{\{e\in p_i\}} x_{i,p}$, i.e., the accumulated mass of all traffic associated to the focal set of O/D pairs that goes through e.
3. **Congestion cost:** The traffic routed through a given edge $e \in \mathcal{E}$ incurs a *congestion cost* (or *latency*) depending on the total traffic on the edge and/or any other exogenous factors. Formally, we will collectively encode all such factors in a *state variable* $\omega \in \Omega$ taking values in some ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will further assume that each edge $e \in \mathcal{E}$ is endowed with an *edge-cost function* $c_e \colon \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$; in this way, given a flow profile $x \in \mathcal{X}$ and a state $\omega \in \Omega$, the cost to traverse edge $e \in \mathcal{E}$ will be $c_e(\mu_e(x), \omega)$. Analogously, the cost to traverse a path $p \in \mathcal{P}$ will be given by the induced *path-cost function* $c_p \colon \mathcal{X} \times \Omega \to \mathbb{R}_+$ defined as $c_p(x, \omega) = \sum_{e\in p} c_e(\mu_e(x), \omega)$.

The only assumption that we will make for the game's cost functions is as follows:

Assumption 1. Each cost function $c_e(u,\omega)$, $e \in \mathcal{E}$ is measurable in ω and non-decreasing, bounded and Lipschitz continuous in u. Specifically, there exist $H > 0$ and $L > 0$ such that $c_e(u,\omega) \leq H$ and $|c_e(u,\omega) - c_e(u',\omega)| \leq L|u' - u|$ for all $u, u' \in [0, M_{\text{tot}}]$ and all $\omega \in \Omega$, $e \in \mathcal{E}$.

Remark. Assumption 1 is a very mild regularity condition that is satisfied by most congestion models that occur in practice – including BPR, polynomial, or regularly-varying latency functions, cf. [5, 8, 9, 18, 21, 23, 34, 35, 37, 39] and references therein. For this reason, we will treat Assumption 1 as a standing, blanket assumption and we will not mention it explicitly in the sequel.

The key difference between our problem setup and other nonatomic congestion models in the literature [34, 39] is the advent of uncertainty in the game's cost functions (as modeled by the game's ambient state variable $\omega \in \Omega$). For concreteness, we demonstrate below two examples of this type of uncertainty that could arise in practice:

Example 1 (Stochastically perturbed BPR costs)**.** Standard BPR models [9, 23, 35, 37], under the form $c_e^{\text{BPR}}(u) = a_e + b_e(u/\text{cap}_e)^r, \forall u \geq 0$, capture the effects of a link's length, its capacity, free-flow speed, etc., on urban network congestion. However, they neglect other miscellaneous factors such as weather conditions, accidents and other random factors which may cause a temporary increase – or decrease – in congestion. In our notation, if ω_e is an additive random fluctuation for the cost of edge $e \in \mathcal{E}$, the induced cost at a flow $x \in \mathcal{X}$ becomes $c(x,\omega) = c_e^{\text{BPR}}(\mu_e(x)) + \omega_e$. The noise can also appear in cases where the actual latency / congestion can only be measured up to a certain error (a case of vital importance for Internet-like networks). §

Example 2 (BPR models with exogenous loads)**.** In practice, the total traffic is the aggregation of all commuters, irrespective of whether they are using a navigation app or not. With this in mind, consider the problem of a navigation app making a flow recommendation $x \in \mathcal{X}$ for its users. If ω_e denotes the *exogenous* traffic load on edge $e \in \mathcal{E}$ (i.e., commuters *not* using the app), the total load on e will be $\mu_e(x) + \omega_e$. Thus, following the BPR model, the induced cost is $c_e(x,\omega) = c_e^{\text{BPR}}(\mu_e(x) + \omega_e)$, i.e., the randomness is woven *implicitly* in the model. §

2.2. The mean game and equilibrium flows. In the above framework, each exogenous state variable $\omega \in \Omega$ determines an instance of a routing game, defined formally as a tuple $\Gamma_\omega \equiv \Gamma_\omega(\mathcal{G}, \mathcal{N}, \mathcal{P}, c_\omega)$ where c_ω is shorthand for the network's cost functions $\{c_e(\cdot,\omega)\}_{e\in\mathcal{E}}$ instantiated at ω. Of course, in analyzing the game, each individual instance Γ_ω is meaningless by itself, unless $\mathbb{P}$ assigns positive probability only to a *single* ω. For this reason, we will instead focus on the *mean game* $\Gamma \equiv \Gamma(\mathcal{G}, \mathcal{N}, \mathcal{P}, C)$ which has the same network and routing flow structure as every Γ_ω, $\omega \in \Omega$, but whose congestion costs are determined by the *mean cost functions* $C_p(x) = \mathbb{E}[c_p(x,\omega)]$.

Now, motivated by the route recommendation problem described in the introduction, we will focus on learning *equilibrium flows* where the controller can guarantee Wardrop's principle on average [43], i.e., that *all traffic is routed along a path with minimal mean cost.* Formally, we have:

Definition 1 (Mean equilibrium flows)**.** We say that $x^* \in \mathcal{X}$ is a *mean equilibrium flow* if and only if, for all $i \in \mathcal{N}$ and all $p, q \in \mathcal{P}_i$ such that $x^*_{i,p} > 0$, we have $C_p(x^*) \leq C_q(x^*)$.

Remark. Definition 1 means that, on average, no user has an incentive to deviate from the recommended route; obviously, when the support of $\mathbb{P}$ is a singleton, we recover the usual definition of a *Wardrop equilibrium* [4, 34, 43]. This special case will be particularly important and we describe it in detail in the next section.

Importantly, the problem of finding an equilibrium flow of a (fixed) routing game Γ_ω admits a *potential function* – often referred to as the Beckmann–McGuire–Winsten (BMW) potential [4, 11]. Specifically, for a given instance $\omega \in \Omega$, the BMW potential is defined as

$$F_\omega(x) \coloneqq \sum\nolimits_{e\in\mathcal{E}} \int_0^{\mu_e(x)} c_e(u,\omega)\, du \quad \text{for all } x \in \mathcal{X}, \tag{BMW}$$

and it has the property that $\arg\min F_\omega$ coincides with the set $\operatorname{Eq}(\Gamma_\omega)$ of equilibrium flows of Γ_ω.

In our stochastic context, a natural question that arises is whether the potential property for each *fixed* $\omega \in \Omega$ can be extended to the mean game Γ when ω is randomly generated. Clearly, the most direct candidate for a potential function in this case is is the averaged BMW potential:

$$F(x) = \mathbb{E}[F_\omega(x)] \coloneqq \mathbb{E}\left[\sum_{e\in\mathcal{E}} \int_0^{\mu_e(x)} c_e(u,\omega)\, du\right]. \tag{2}$$

Indeed, as we show in Appendix B, we have:

Proposition 1. *A flow profile* $x^* \in \mathcal{X}$ *is a* mean equilibrium flow *if and only if it is a minimizer of* F *over* $\mathcal{X}$*; more succinctly,* $\mathrm{Eq}(\Gamma) = \arg\min_{x \in \mathcal{X}} F(x)$.

In view of the above, F provides a natural merit function for examining how close a given flow profile $x \in \mathcal{X}$ is to being an equilibrium; on that account, *all our convergence certificates in the sequel will be stated in terms of* F. Note also that since $c_e, e \in \mathcal{E}$, are continuous and non-decreasing, F is a differentiable, convex function on $\mathcal{X}$. However, since the probability law $\mathbb{P}$ on Ω is not known, we will *not* assume that F (and/or its gradients) can be explicitly computed in general.

2.3. The learning model. The last component of our model is the actual learning process that unfolds over time. The specific sequence of events that we will consider evolves as follows:

1. At each stage $t = 1, 2, \dots$, the navigation interface selects a flow profile $X^t \in \mathcal{X}$ and makes the corresponding routing recommendation to its users.
2. Concurrently, the state ω^t of the network is drawn from Ω (i.i.d. relative to $\mathbb{P}$).
3. The interface observes the realized congestion costs $c_e(\mu_e(X^t), \omega^t)$ along each $e \in \mathcal{E}$ (possibly up to some error); subsequently, the flow recommendation is updated, and the process repeats.

We will refer to this general model as the **stochastic regime**. For concreteness, we discuss below two special cases that have attracted particular interest in the literature:

Example 3 (Static environments)**.** In the absence of randomness, $\mathbb{P}$ is supported on a single instance $\omega \in \Omega$, so we have $\omega^t = \omega$ for all $t = 1, 2, \dots$, and we assume that the navigation interface measures directly $C(X^t) = c(X^t, \omega)$. This setup matches the deterministic model of Blum et al. [5], Fischer and Vöcking [13] and Krichene et al. [19, 21], and we will refer to it as the **static regime**. §

Example 4 (Routing games with noisy observations)**.** Consider the setting where only a stochastic "perturbed" cost $c_e(\mu_e(x), \omega^t) = c_e(\mu_e(x), \omega) + \omega_e^t$ is observed when a flow profile $x \in \mathcal{X}$ is employed at time t (here, $\omega \in \Omega$ is fixed). When ω_e^t is the random noise such that $\mathbb{E}[\omega_e^t] = 0, \forall e, \forall t$, our learning model is reduced to the routing game with noisy observations studied by Krichene et al. [18, 21]. §

For a given learning window, it might not be a priori clear whether the system is in the static or stochastic regime. As we shall see in Section 3, standard stochastic algorithms are suboptimal in the static regime, while order-optimal deterministic algorithms may fail to converge altogether in the stochastic regime. As such, the key question that we aim to answer is as follows: *is it possible to design routing algorithms that automatically adapt to the appropriate setting and provide optimal convergence guarantees in both static and stochastic environments?* We address this in Section 4.

3 Non-adaptive methods

To set the stage for the analysis to come, we begin by presenting the equilibrium convergence properties of two learning methods that are tailor-made for each of the two basic regimes described in the previous section: the "vanilla" exponential weights algorithm for the stochastic case, and an accelerated exponential weights method for the static one. Both algorithms rely crucially on the (rescaled) *logit choice map* $\Lambda\colon \mathbb{R}^P \to \mathcal{X}$, given in components as

$$\Lambda_p(Y) = \frac{M_i \exp(Y_{i,p})}{\sum_{q \in \mathcal{P}_i} \exp(Y_{i,q})} \quad \text{for all } p \in \mathcal{P}_i \text{ and all } i \in \mathcal{N}. \tag{3}$$

For concision, we will also write C_p^t for the total congestion cost measured for path $p \in \mathcal{P}$ at stage t, and $C^t = (C_p^t)_{p \in \mathcal{P}}$ for the profile thereof.

3.1. Exponential weights in the stochastic regime. We begin by presenting the standard exponential weights algorithm in the stochastic regime. In pseudocode form, we have:

Algorithm 1: Exponential weights (EXPWEIGHT)

Initialize $Y^0 \leftarrow 0$
for $t = 1, 2, \dots$ **do**
  set $X^t \leftarrow \Lambda(Y^{t-1})$ and get $C^t \leftarrow c(X^t, \omega^t)$ // route and measure costs
  set $Y^t = Y^{t-1} - \gamma^t C^t$ // update path scores

Then, by applying the classical analysis of exponential weights methods [6, 40], we obtain the following equilibrium convergence guranatee:

Theorem 1. *Let x^* be an equilibrium of Γ. If Algorithm 1 is run with variable learning rate $\gamma^t = 1/\sqrt{t}$, the time-averaged flow profile $\bar{X}^T = (1/T)\sum_{t=1}^{T} X^t$ enjoys the equilibrium convergence rate*

$$\mathbb{E}\big[F\big(\bar{X}^T\big) - F(x^*)\big] \leq \frac{N M_{\max}}{\sqrt{T}}\left[\log\left(\frac{M_{\max} P}{M_{\text{tot}}}\right) + H^2(1+\log T)\right] = \mathcal{O}\left(\frac{\log(PT)}{\sqrt{T}}\right). \tag{4}$$

This result confirms that EXPWEIGHT achieves a speed of convergence that is logarithmic in terms of P, and hence linear in the size $|\mathcal{G}| := |\mathcal{V}| + |\mathcal{E}|$ of the underlying network; for completeness, we provide a full proof in Appendix C. We make two relevant remarks here: First, the convergence rate obtained for Algorithm 1 concerns the empirical average flow, not the actual recommendation, a distinction which is important for practical applications. Second, there is a slight suboptimality in terms of the learning horizon T: the rate presented in Theorem 1 contains a $\log(T)$ term which can be shaved off by switching to the so-called "dual averaging" variant of the exponential weights template (or finetune the method's learning rate in terms of T and subsequently use a doubling trick to obtain an anytime guarantee). For the details, we refer the reader to [32, 40, 44].

3.2. Accelerated exponential weights in static environments. We now turn our attention to the static regime, i.e., when there are no exogenous variations in the game's state ($\omega^t = \omega$ for all $t = 1, 2, \dots$). In this case, it is reasonable to expect that a faster convergence rate should be attainable: in particular, as we show below, the game's potential is Lipschitz smooth (see Proposition 2 below), so the optimal convergence speed in this case is the iconic $\mathcal{O}(1/T^2)$ rate of Nesterov [29]. In more detail, we have:

Proposition 2. *The BMW potential is Lipschitz smooth relative to the L^1 norm on $\mathcal{X}$ and has smoothness modulus $\beta = KL$, where K is the length of the longest path in $\mathcal{P}$.*

Now, to maintain the graceful scaling guarantees of the EW template, our proposal to achieve an $\mathcal{O}(1/T^2)$ rate is an *accelerated exponential weights* algorithm that builds on ideas by Nesterov [29], **?**] and Krichene et al. [20]. In pseudocode form, the method unfolds as follows:

Algorithm 2: Accelerated exponential weights (ACCELEWEIGHT)

Input: Smoothness parameter β
Initialize $Y^0 \leftarrow 0$, $\alpha^0 \leftarrow 0$ and $\gamma^0 \leftarrow \frac{1}{N M_{\max}\beta}$
for $t = 1, 2, \dots$ **do**
  set $Z^t \leftarrow \Lambda(Y^{t-1})$ // exploratory flow obtained from path scores
  set $X^t \leftarrow \alpha^{t-1} X^{t-1} + (1-\alpha^{t-1}) Z^t$ // average with previous state
  set $\gamma^t \leftarrow \frac{1}{2}\left[2\gamma^{t-1} + \gamma^0 + \sqrt{4\gamma^{t-1}\gamma^0 + (\gamma^0)^2}\right]$ // update step-size
  set $\alpha^t \leftarrow \gamma^{t-1}/\gamma^t$ // update moving weight
  set $\bar{Z}^t \leftarrow \alpha^t X^t + (1-\alpha^t) Z^t$ and get $C^t \leftarrow c\big(\bar{Z}^t, \omega^t\big)$ // route and measure costs
  set $Y^t \leftarrow Y^{t-1} - (1-\alpha^t)\gamma^t C^t$ // update path scores

Importantly, the step-size of Algorithm 2 is finetuned relative to the smoothness modulus of F, which is assumed known to the navigation interface. With this in mind, our main convergence guarantee for ACCELEWEIGHT is as follows:

Theorem 2. *Let x^* be an equilibrium of Γ. If Algorithm 2 is run for T epochs in the static regime, the traffic flow profile X^T enjoys the equilibrium convergence rate:*

$$F\big(X^T\big) - F(x^*) \leq \frac{4\beta N^2 M_{\max}^2 \log(M_{\max} P / M_{\text{tot}})}{(T-1)^2} = \mathcal{O}\left(\frac{\log(P)}{T^2}\right). \tag{5}$$

Theorem 2 confirms that, in the static regime, AcceleWeight converges to equilibrium with an optimal $\mathcal{O}\big(\log(P)/T^2\big)$ convergence speed, as desired; the proof is based on techniques that are widespread in the analysis of accelerated methods, so we relegate it to the supplement.

We only note here some limitations of the accelerated exponential weights method. First, the algorithm's convergence rate concerns a sequence of routing flows which is never recommended, a disparity which limits the algorithm's applicability. Second, the algorithm's step-size must be tuned with prior knowledge of the problem's smoothness modulus (which is not realistically available to the optimizer), so it is not adaptive in this regard. In addition, AcceleWeight fails to converge altogether in the stochastic regime, so it does not achieve rate interpolation either.

The per-iteration complexity of Algorithms 1 and 2. In the discussions above, we chose to present simple implementations of EW and AcceleWeight (as Algorithms 1 and 2 respectively), in which every iteration runs in $\mathcal{O}(P)$ time. This is inefficient in large-scale networks where P is typically exponentially large; however, owing to the underlying exponential weights template, both algorithms can be implemented efficiently in $\mathcal{O}(|\mathcal{E}|)$ time and space via a dynamic programming procedure known as "weight-pushing" [15, 41]. The details of this efficient implementation lie beyond the scope of this work, so we do not present them here.

4 AdaWeight: Adaptive learning in the presence of uncertainty

4.1. Statement and discussion of results. To summarize the situation so far, we have seen that ExpWeight attains an $\mathcal{O}(\log(P)/\sqrt{T})$ rate, which is order-optimal in the stochastic case but suboptimal in static environments; by contrast, AcceleWeight attains an $\mathcal{O}(\log(P)/T^2)$ rate in static environment, but has no convergence guarantees in the presence of randomness and uncertainty. Consequently, neither of these algorithms meets our stated objective to concurrently achieve order-optimal guarantees in both the static and stochastic cases (and without requiring prior knowledge of the problem's smoothness modulus).

To resolve this gap, we propose below an *adaptive exponential weights* method – AdaWeight for short – which achieves these objectives by mixing the acceleration template of AcceleWeight with the dual extrapolation method of Nesterov [31]. We present the pseudocode of AdaWeight below:

Algorithm 3: adaptive exponential weights (AdaWeight)

1 **Initialize** $Y^1 \leftarrow 0$, $\alpha^0 \leftarrow 0$ and $\eta^1 \leftarrow 1$
2 **for** $t = 1, 2, \dots$ **do**
3 set $Z^t \leftarrow \Lambda\big(\eta^t Y^t\big)$ // exploratory flow obtained from path scores
4 set $\bar{Z}^t \leftarrow \Big(\alpha^t Z^t + \sum_{s=0}^{t-1} \alpha^s Z^{s+\frac{1}{2}}\Big) \Big/ \sum_{s=0}^{t} \alpha^s$ and get $\bar{C}^t \leftarrow c(\bar{Z}^t, \omega^t)$ // reweigh + explore
5 set $Y^{t+\frac{1}{2}} \leftarrow Y^t - \alpha^t \bar{C}^t$ // exploratory score update
6 set $Z^{t+\frac{1}{2}} \leftarrow \Lambda\Big(\eta^t Y^{t+\frac{1}{2}}\Big)$ // exploratory flow update
7 set $X^t \leftarrow \Big(\sum_{s=0}^{t} \alpha^s Z^{s+\frac{1}{2}}\Big) \Big/ \sum_{s=0}^{t} \alpha^s$ and get $C^t \leftarrow c(\bar{Z}^t, \omega^t)$ // route and measure costs
8 set $Y^{t+1} \leftarrow Y^t - \alpha^t C^t$ // update path scores
9 set $\eta^{t+1} \leftarrow \eta^t \Big/ \sqrt{1 + \sum_{s=0}^{t} \big\|\alpha^s (C^s - \bar{C}^s)\big\|_\infty^2}$ // update learning rate

The main novelty in the definition of the AdaWeight algorithm is the introduction of two "extrapolation" sequences, $Z^{t+\frac{1}{2}}$ and $Y^{t+\frac{1}{2}}$, that venture outside the convex hull of the generated primal (flow) and dual (score) variables respectively. These leading states are subsequently averaged, and the method proceeds with an adaptive step-size rule. In more details, AdaWeight relies on three key components:

a) A dual extrapolation mechanism for generating the leading sequences in Lines 3 and 6; these sequences are central for anticipating the loss landscape of the problem.

b) An acceleration mechanism obtained from the (α^t)-*weighted average* steps in Lines 4 and 7; in the analysis, α^t will grow as t, so almost all the weight will be attributed to the state closest to the current one.

c) An *adaptive sequence of learning rates* (cf. Line 9) in the spirit of Rakhlin and Sridharan [38], Kavis et al. [17] and Antonakopoulos et al. [1]. This choice is based on the ansatz that, if the algorithm encounters coherent gradient updates (which can only occur in static environments), it will eventually stabilize to a strictly positive value; otherwise, it will decrease to zero at a $\Theta(1/\sqrt{t})$ rate. This property is crucial to interpolate between the stochastic and static regimes.

The combination of the weighted average iterates and adaptive learning rate in AdaWeight is shared by the UnixGrad algorithm proposed by Kavis et al. [17], which also provides rate interpolation in constrained problems. However, UnixGrad requires the problem's domain to have a finite Bregman diameter – and, albeit compact, the set of feasible flows $\mathcal{X}$ has an *infinite* diameter under the entropic regularizer that generates the EW template. Therefore, UnixGrad is not applicable to our routing games. This is the reason for switching gears to the "primal-dual" approach offered by the dual extrapolation template; this primal-dual interplay provides the missing link that allows AdaWeight to simultaneously enjoy order-optimal convergence guarantees in both settings while maintaining the desired polynomial dependency on the problem's dimension. Finally, as in the case of ExpWeight and AcceleWeight, we note that AdaWeight can also be implemented efficiently via the weight-pushing technique of Takimoto and Warmuth [41] (with linear space and time complexity in the size of the underlying graph).

In light of the above, our main convergence result for AdaWeight is as follows:

Theorem 3. *Let x^* be an equilibrium of Γ. If Algorithm 3 is run for T epochs with $\alpha^t = t$, $t = 1, 2, \ldots$, the recommended flow profile X^T enjoys the equilibrium convergence rate:*

$$\mathbb{E}\big[F(X^T) - F(x^*)\big] \leq \frac{2\sqrt{2}NM_{\max}A}{\sqrt{T}} + \frac{12\beta N^2 M_{\max}^2 A^{3/2} + B}{T^2} = \mathcal{O}\left(\frac{(\log P)^{3/2}}{\sqrt{T}}\right). \tag{6a}$$

Moreover, in the static case, Algorithm 3 enjoys the sharper rate:

$$F(X^T) - F(x^*) \leq \frac{B}{T^2} = \mathcal{O}\left(\frac{(\log P)^{3/2}}{T^2}\right). \tag{6b}$$

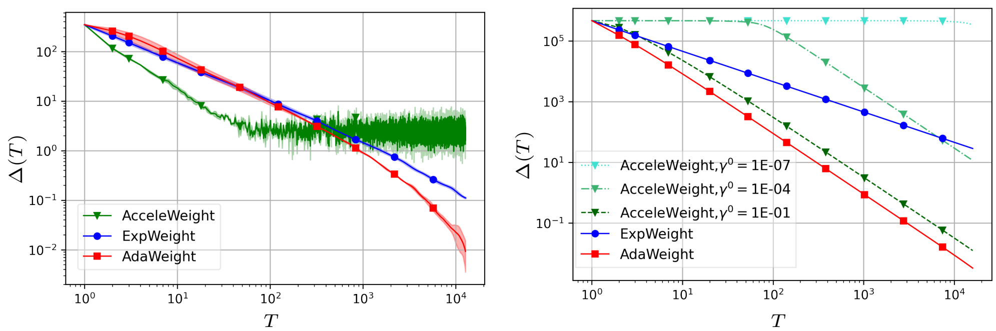
In the above expressions, A and B are positive constants given by $A \coloneqq 2\log(PM_{\max}/M_{\text{tot}}) + 13 = \mathcal{O}(\log P)$ and $B \coloneqq 4\beta N^2 M_{\max}^2 A^{3/2} + 2M_{\text{tot}}\log(M_{\max}P/M_{\text{tot}}) = \mathcal{O}\Big((\log P)^{3/2}\Big)$.

Theorem 3 confirms that AdaWeight enjoys all of the desired features: (*i*) it achieves *simultaneously optimal guarantees* in both stochastic and static environments (i.e., $\mathcal{O}(1/\sqrt{T})$ and $\mathcal{O}(1/T^2)$ respectively); (*ii*) the derived rates maintain a *polynomial dependency* in terms of the network' s combinatorial primitives; and (*iii*) it requires *no prior tuning* by the learner. Moreover, unlike ExpWeight, the convergence of AdaWeight corresponds to an actual traffic flow profile that is implemented in epoch t and not the average flow.

The proof of Theorem 3 is technically involved and worth discussing. We analyze its main ideas here and give a detailed proof in Appendix D.

4.2. Sketch of proof of Theorem 3. We focus on finding an upper-bound of the "weighted regret"—defined as $\hat{\Delta}_T(x^*) = \sum_{t=1}^T \alpha^t \langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x^* \rangle$, which will be directly translated into an upper-bound of the left-hand-side in (6a) and (6b) via the inequality $F(X^T) - F(x^*) \leq \hat{\Delta}_T(x^*)/T^2$ (extracted from [10]). To do this, we take the advantage of the dual-extrapolation template to obtain the following "energy inequality":

$$\begin{aligned}\hat{\Delta}_T(x^*) \leq &\frac{h(x^*) - \min_{x\in\mathcal{X}} h(x)}{\eta^{T+1}} + \frac{\sigma}{2}\sum_{t=1}^T (\alpha^t)^2 \eta^{t+1} \big\|\nabla F(X^t) - \nabla F(\bar{Z}^t)\big\|_\infty^2 \\ &+ \frac{2M_{\text{tot}}^2}{\sigma}\sum_{t=1}^T\left(\frac{1}{\eta^{t+1}} - \frac{1}{\eta^t}\right) - \sum_{t=1}^T \frac{1}{\eta^t}\mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big).\end{aligned} \tag{7}$$



(a) Convergence in a stochastic environment **(b)** Convergence in a static environment

Figure 1: The convergence speed of ExpWeight, AcceleWeight and AdaWeight in routing games.

Here, $\sigma := M_{\max}N$, h denotes the entropy regularizer $h(x) = \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}} x_{i,p}\log(x_{i,p})$ (which is σ-strongly convex on $\mathcal{X}$ w.r.t the L^1 norm), and $\mathrm{KL}(\cdot\|\cdot)$ denotes the Kullback–Leibler divergence.

The key idea to handle the terms in the right-hand-side of (7) is our *special choice of learning rates* η^t that makes these terms summable. This induces an upper-bound in the form of $\sum_{t=1}^{T}(\alpha^t)^2 g(\eta^{t+1})\|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|^2_\infty$ where $g(\cdot)$ is a function of η^t. Interestingly, we can also prove that there exists $T_0 < \infty$ such that $g(\eta^t) < 0$ for any $t \geq T_0$; therefore, the above term is always bounded by a sum of only T_0 elements as $T \to \infty$. Finally, in the static case, we can use the smoothness of F (cf. Proposition 2) to bound $\|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|_\infty$, and in the stochastic regime, we can do this by using the fact that ∇F is bounded due to Assumption 1. This provides the desired results. □

5 Numerical Experiments

5.1. Experiments in the stochastic regime. First, we conduct an experiment on a routing game with noisy observations (as described in Example 4) with the following setup: on a randomly generated network with 24 vertices and 276 edges, we assign to each edge a BPR cost function and choose $N = 6$ origin-destination pairs, each of which comes with a traffic demand; the traffic will be routed via the set of $P = 3366$ paths. Particularly, at each time T, after the algorithms decide their flow profiles, we perturb the induced costs on each edge e by a noise ω_e^T that is drawn independently from the normal distribution $\mathcal{N}(0, 10)$. The algorithms only observe these stochastically perturbed costs and use them to update the next iterations.

For validation purposes, we run ExpWeight, AcceleWeight and AdaWeight in 5 instances (of noises' layout) and report, in Figure 1a, the averaged results across these instances. Particularly, we plot out the evolution of the following values: $\Delta_{\textsc{ExpWeight}}(T) := F\left(\bar{X}^T\right) - F(x^*)$ where $\bar{X}^T$ is the time-averaged of outputs of Algorithm 1, $\Delta_{\textsc{AcceleWeight}}(T) := F\left(X^T\right) - F(x^*)$ where X^T is output by Algorithm 2 and $\Delta_{\textsc{AdaWeight}}(T) := F\left(X^T\right) - F(x^*)$ where X^T is output by Algorithm 3. Here, x^* is chosen such that it induces the minimum F-value among all flow profiles computed by these algorithms after 15000 iterations; thus, x^* represents the equilibrium flow.

Figure 1a shows that $\Delta_{\textsc{ExpWeight}}(T)$ and $\Delta_{\textsc{AdaWeight}}(T)$ tend to zero as T increases; this confirms that ExpWeight and AdaWeight converge toward equilibrium. To this end, we observe that AcceleWeight fails to converge altogether. We also plot out $\sqrt{T}\cdot\Delta_{\textsc{AcceleWeight}}(T)$ and $\sqrt{T}\cdot\Delta_{\textsc{AdaWeight}}(T)$ (see Appendix E), and we observe that these terms approach horizontal lines as T increases. This reaffirms the fact that the speed of convergence of ExpWeight and AdaWeight is $\mathcal{O}(1/\sqrt{T})$ in the stochastic regime which is consistent with our theoretical analyses.

5.2. Experiments in static regime. To illustrate the performance of the algorithms in the static case, we take advantage of a real data set collected and provided by [14] (with free license). This data set contains no personally identifiable information or offensive content. In this section, we present the experimental results on one instance in this data set: the SiouxFalls network (originated by [24]). Results corresponding to other network instances are presented in Appendix E. The Sioux-

Falls network has 24 vertices and 76 edges. We choose $N = 30$ origin-destination pairs and the total number of paths in use is $P = 3000$. We extract from the data set the BPR cost function corresponding to each edge and the traffic demand corresponding to each O/D pair. We run ExpWeight, AcceleWeight and AdaWeight in 16000 iterations and report the results thereof in Figure 1b.

Figure 1b confirms that in this static environment, all three algorithms converge towards equilibrium. Particularly, AcceleWeight and AdaWeight converges at an $\mathcal{O}(1/T^2)$ rate while ExpWeight fails to achieve this speed. These convergence rates are reaffirmed by observing that $T^2 \cdot \Delta_{\text{AcceleWeight}}(T)$ and $T^2 \cdot \Delta_{\text{AdaWeight}}(T)$ approach horizontal lines as T increases (cf. Appendix E). We also run AcceleWeight with several initial step-size options and observe in Figure 1b that with badly-tuned parameters (i.e., with a wrongly estimation of the smoothness level of F), AcceleWeight might have a slow "warm-up" phase or might even diverge. By contrast, AdaWeight retains its appealing fast convergence properties throughout our experiments.

References

[1] Kimon Antonakopoulos, E. Veronica Belmega, and Panayotis Mertikopoulos. Adaptive extra-gradient methods for min-max optimization and games. In *ICLR '21: Proceedings of the 2021 International Conference on Learning Representations*, 2021.

[2] Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. Gambling in a rigged casino: The adversarial multi-armed bandit problem. In *Proceedings of the 36th Annual Symposium on Foundations of Computer Science*, 1995.

[3] Amir Beck and Marc Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. *Operations Research Letters*, 31(3):167–175, 2003.

[4] Martin Beckmann, C. B. McGuire, and C.B. Winsten. *Studies in the Economics of Transportation*. Yale University Press, 1956.

[5] Avrim Blum, Eyal Even-Dar, and Katrina Ligett. Routing without regret: on convergence to Nash equilibria of regret-minimizing in routing games. In *PODC '06: Proceedings of the 25th annual ACM SIGACT-SIGOPS symposium on Principles of Distributed Computing*, pages 45–52, 2006.

[6] Sébastien Bubeck and Nicolò Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Foundations and Trends in Machine Learning*, 5(1):1–122, 2012.

[7] Théophile Cabannes, Marco Sangiovanni, Alexander Keimer, and Alexandre M Bayen. Regrets in routing networks: Measuring the impact of routing apps in traffic. *ACM Transactions on Spatial Algorithms and Systems (TSAS)*, 5(2):1–19, 2019.

[8] Nicolò Cesa-Bianchi and Gábor Lugosi. Combinatorial bandits. In *COLT '08: Proceedings of the 22nd Annual Conference on Learning Theory*, 2009. URL http://www.cs.mcgill.ca/%7Ecolt2009/papers/024.pdf#page=1.

[9] Riccardo Colini-Baldeschi, Roberto Cominetti, Panayotis Mertikopoulos, and Marco Scarsini. When is selfish routing bad? The price of anarchy in light and heavy traffic. *Operations Research*, 68(2):411–434, March 2020.

[10] Ashok Cutkosky. Anytime online-to-batch, optimism and acceleration. In *International Conference on Machine Learning*, pages 1446–1454. PMLR, 2019.

[11] S. C. Dafermos and F. T. Sparrow. The traffic assignment problem for a general network. *Journal of Research of the National Bureau of Standards*, 73B(2):91–118, 1969.

[12] John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *The Journal of Machine Learning Research*, 12:2121–2159, 2011.

[13] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Proceedings of the 12th European Symposium on Algorithms*, 2004.

[14] Transportation Networks for Research Core Team. Transportation networks for research project. https://github.com/bstabler/TransportationNetworks, 2021.

[15] András György, Tamás Linder, Gábor Lugosi, and György Ottucsák. The online shortest path problem under partial monitoring. *Journal of Machine Learning Research*, 8:2369–2403, 2007.

[16] Ali Kavis, Kfir Y Levy, Francis Bach, and Volkan Cevher. Unixgrad: A universal, adaptive algorithm with optimal guarantees for constrained optimization. In *Advances in Neural Information Processing Systems*, pages 6260–6269, 2019.

[17] Ali Kavis, Kfir Yehuda Levy, Francis Bach, and Volkan Cevher. UnixGrad: A universal, adaptive algorithm with optimal guarantees for constrained optimization. In *NeurIPS '19: Proceedings of the 33rd International Conference on Neural Information Processing Systems*, 2019.

[18] Syrine Krichene, Walid Krichene, Roy Dong, and Alexandre Bayen. Convergence of heterogeneous distributed learning in stochastic routing games. In *Allerton '15: Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing*, 2015.

[19] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *ICML '14: Proceedings of the 31st International Conference on Machine Learning*, 2014.

[20] Walid Krichene, Alexandre Bayen, and Peter L. Bartlett. Accelerated mirror descent in continuous and discrete time. In *NIPS '15: Proceedings of the 29th International Conference on Neural Information Processing Systems*, 2015.

[21] Walid Krichene, Benjamin Drighès, and Alexandre M. Bayen. Online learning of Nash equilibria in congestion games. *SIAM Journal on Control and Optimization*, 53(2):1056–1081, 2015.

[22] Guanghui Lan. An optimal method for stochastic composite optimization. *Mathematical Programming*, 133(1-2):365–397, June 2012.

[23] L. J. LeBlanc, E. K. Morlok, and W. P. Pierskalla. An efficient approach to solving the road network equilibrium traffic assignment problem. *Transportation Research*, 9:309–318, 1975.

[24] Larry J LeBlanc and Mustafa Abdulaal. An efficient dual approach to the urban road network design problem. *Computers & Mathematics with Applications*, 5(1):11–19, 1979.

[25] Kfir Yehuda Levy, Alp Yurtsever, and Volkan Cevher. Online adaptive methods, universality and acceleration. In *NeurIPS '18: Proceedings of the 32nd International Conference of Neural Information Processing Systems*, 2018.

[26] Xiaoyu Li and Francesco Orabona. On the convergence of stochastic gradient descent with adaptive stepsizes. In *AISTATS '19: Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics*, 2019.

[27] Nick Littlestone and Manfred K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

[28] Arkadi Semen Nemirovski and David Berkovich Yudin. *Problem Complexity and Method Efficiency in Optimization*. Wiley, New York, NY, 1983.

[29] Yurii Nesterov. A method for unconstrained convex minimization problem with the rate of convergence $\mathcal{O}(1/k^2)$. *Proceedings of the USSR Academy of Sciences*, 269(543-547), 1983.

[30] Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Number 87 in Applied Optimization. Kluwer Academic Publishers, 2004.

[31] Yurii Nesterov. Dual extrapolation and its applications to solving variational inequalities and related problems. *Mathematical Programming*, 109(2):319–344, 2007.

[32] Yurii Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120 (1):221–259, 2009.

[33] Yurii Nesterov. Universal gradient methods for convex optimization problems. *Mathematical Programming*, 152(1-2):381–404, 2015.

[34] Noam Nisan, Tim Roughgarden, Éva Tardos, and V. V. Vazirani, editors. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[35] Steven J. O'Hare, Richard Connors, and David P. Watling. Mechanisms that govern how the Price of Anarchy varies with travel demand. *Transportation Research*, 84:55–80, February 2016.

[36] Gerasimos Palaiopanos, Ioannis Panageas, and Georgios Piliouras. Multiplicative weights update with constant step-size in congestion games: Convergence, limit cycles and chaos. In *NIPS '17: Proceedings of the 31st International Conference on Neural Information Processing Systems*, 2017.

[37] Michael Patriksson. *The traffic assignment problem: models and methods*. Courier Dover Publications, 2015.

[38] Alexander Rakhlin and Karthik Sridharan. Optimization, learning, and games with predictable sequences. In *NIPS '13: Proceedings of the 27th International Conference on Neural Information Processing Systems*, 2013.

[39] Tim Roughgarden and Éva Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.

[40] Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.

[41] Eiji Takimoto and Manfred K. Warmuth. Path kernels and multiplicative updates. *Journal of Machine Learning Research*, 4(773-818), 2003.

[42] Vladimir G. Vovk. Aggregating strategies. In *COLT '90: Proceedings of the 3rd Workshop on Computational Learning Theory*, pages 371–383, 1990.

[43] John Glen Wardrop. Some theoretical aspects of road traffic research. In *Proceedings of the Institute of Civil Engineers, Part II*, volume 1, pages 325–378, 1952.

[44] Lin Xiao. Dual averaging methods for regularized stochastic learning and online optimization. *Journal of Machine Learning Research*, 11:2543–2596, October 2010.

A Auxiliary results and notions

The gradient of the function F is important for our sequel analyses, we present its explicit computation in Appendix A.1. We also introduce several useful technical notions that will be used throughout the remainders of this work in Appendix A.2.

A.1. Gradient of F. We notice that under Assumption 1, $c_e, e \in \mathcal{E}$ are continuous and non-decreasing functions. As a trivial consequence, F is a convex function. Moreover, it is differentiable and when ω is generated randomly, for any $i \in \mathcal{N}$ and any path $p \in \mathcal{P}_i$, by recalling that $\mu_e(x) = \sum_{i\in\mathcal{N}} \sum_{\substack{p\in\mathcal{P}_i \\ p\ni e}} x_{i,p}$ and applying the dominated convergence theorem, we have:

$$\begin{aligned}\frac{\partial F}{\partial x_{i,p}}(x) = \mathbb{E}\left[\frac{\partial F_\omega}{\partial x_{i,p}}(x)\right] = \mathbb{E}\left[\frac{\sum_{e\in\mathcal{E}} \int_0^{\mu_e(x)} c_e(u,\omega)\, du}{\partial x_{i,p}}\right] &= \mathbb{E}\left[\sum_{e\in\mathcal{E}} \frac{\partial \mu_e(x)}{\partial x_{i,p}} c_e(\mu_e(x),\omega)\right] \\ &= \mathbb{E}\left[\sum_{e\in p} c_e(\mu_e(x),\omega)\right] \\ &= C_p(x). \end{aligned} \tag{A.1}$$

Using the notation $c(x,\omega)$ to present the (random) vector of costs at flow x and ω, we can also rewrite that $\nabla F(x) = \mathbb{E}[c(x,\omega)]$ for any $x \in \mathcal{X}$.

A.2. Auxiliary notions. Throughout the remainder of this work, we use the notation $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product of two vectors. We also denote $\mathbb{P}(E)$ the probability of an event E. We say that h is σ-*strongly* convex over $\mathcal{K}$ w.r.t $\|\cdot\|$ if and only if for $x, x' \in \mathcal{K}$, we have $h(x) - h(x') - \langle W, x - x'\rangle \geq \frac{\sigma}{2}\|x - x'\|^2$ for any $W \in \partial h(x')$ (i.e., W is a sub-gradient of h at x').

Entropy regularizer. In the remainders of this paper, we recurrently work with the function

$$h(x) = \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x_{i,p} \log(x_{i,p}), \forall x \in \mathcal{X}. \tag{A.2}$$

This function is often known as the *entropy regularizer*. It has several properties of interests as follows:

- For any $x \in \mathcal{X}$, $h(x) \leq \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}} x_{i,p} \log(M_i) \leq M_{\text{tot}} \log(M_{\max})$.
- h is a $\frac{1}{\sigma}$-strongly convex function on $\mathcal{X}$ w.r.t the $\|\cdot\|_1$ norm where we denote $\sigma := M_{\max} N$.[1]
- The Fenchel conjugate of h—denoted by h^*—and its gradient at an arbitrary $Y \in \mathbb{R}^P$ are:

$$h^*(Y) := \max_{x\in\mathcal{X}} \langle x, Y\rangle - h(x) = \sum_{i\in\mathcal{N}} M_i \log\left(\sum_{q\in\mathcal{P}_i} \exp(Y_{i,q})\right),$$

$$\nabla h^*(Y) = \left(M_i \frac{\exp(Y_{i,p})}{\sum_{q\in\mathcal{P}_i} \exp(Y_{i,q})}\right)_{\substack{i\in\mathcal{N} \\ p\in\mathcal{P}_i}}.$$

[1]Consider the functions $h_i((x_{i,p})_{p\in\mathcal{P}_i}) = \sum_{p\in\mathcal{P}_i} x_{i,p}\log(x_{i,p})$, it is trivial to see that $h_i(\cdot)$ is $\frac{1}{M_i}$-strongly convex on $\mathcal{X}_i = \left\{(x_{i,p})_{p\in\mathcal{P}_i} : \sum_{p\in\mathcal{P}_i} x_{i,p} = M_i\right\}$ w.r.t to the norm $\|\cdot\|_1$. Therefore, for any $x, x' \in \mathcal{X}$ and a sub-gradient $W \in \partial h(x')$, we have $h(x) = \sum_{i\in\mathcal{N}} h_i((x_{i,p})_{p\in\mathcal{P}_i}) \geq h(x') + \langle W, x - x'\rangle + \sum_{i\in\mathcal{N}} \frac{1}{2M_i}\left(\sum_{p\in\mathcal{P}}\left|x_{i,p} - x'_{i,p}\right|\right)^2 \geq h(x') + \langle W, x - x'\rangle + \frac{1}{2M_{\max}N}\|x - x'\|_1^2$ due to Cauchy-Schwarz inequality.

KL divergence. Induced from the entropy regularizer h, we also define the Kullback–Leibler (KL) divergence between any flows $x, x' \in \mathcal{X}$ as follows:

$$\mathrm{KL}(x\|x') := \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i} x_{i,p}\log\left(\frac{x_{i,p}}{x'_{i,p}}\right).$$

In other words, $\mathrm{KL}(x\|x') = h(x) - h(x') - \langle\nabla h(x'), x - x'\rangle$. In the remainders of this work, we will also regularly use the following trivial equality regarding the KL-divergence:

$$\mathrm{KL}(x\|x'') = \mathrm{KL}(x\|x') + \mathrm{KL}(x'\|x'') + \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\left(x'_{i,p} - x_{i,p}\right)\log\left(\frac{x''_{i,p}}{x'_{i,p}}\right). \tag{A.3}$$

Fenchel Coupling. The *Fenchel coupling* between $x \in \mathcal{X}$ and $Y \in \mathbb{R}^P$ w.r.t the regularizer h is defined as follows:

$$\mathcal{F}(x, Y) = h(x) + h^*(Y) - \langle x, Y\rangle. \tag{A.4}$$

Smoothness alternative definition. In previous sections, we use the definition that $g : \mathcal{K} \to \mathbb{R}$ is a β-smooth convex function over $\mathcal{K} \subset \mathbb{R}^d$ w.r.t the $\|\cdot\|$ norm if and only if $\|\nabla g(x) - \nabla g(x')\|_\infty \leq \beta\|x - x'\|, \forall x, x' \in \mathcal{K}$. This is also equivalent to the following conditions:

$$0 \leq g(x) - g(x') - \langle\nabla g(x'), x - x'\rangle \leq \frac{\beta}{2}\|x - x'\|^2, \forall x, x' \in \mathcal{K}. \tag{A.5}$$

Proof. While the first inequality in (A.5) is trivial from the convexity of g, the second inequality can be proved by applying the mean value theorem as follows:

$$\begin{aligned}
& g(x) - g(x') - \langle\nabla g(x'), x - x'\rangle \\
=& \int_0^1 \langle\nabla g(x' + u(x - x')), x - x'\rangle\, du - \int_0^1 \langle\nabla g(x'), x - x'\rangle\, du \\
\leq& \int_0^1 \|\nabla g(x' + u(x - x')) - \nabla g(x')\|_\infty \cdot \|x - x'\|\, du \qquad \text{(by Cauchy-Schwarz inequality)} \\
\leq& \|x - x'\| \int_0^1 \beta\|(x' + u(x - x')) - x'\|\, du \\
=& \frac{\beta}{2}\|x - x'\|^2.
\end{aligned}$$

□

B Proofs of Results in Section 2

B.1. Proof of Proposition 1.

($\Rightarrow$) Let $x^* \in \mathcal{X}$ be an equilibrium flow of the mean game Γ. For each $i \in \mathcal{N}$, let us define $C_i := \min_{\{q\in\mathcal{P}_i : x^*_{i,q} > 0\}} C_q(x^*)$. By definition of the equilibrium flow, for any $p \in \mathcal{P}_i$, we have

$$C_p(x^*) > C_i \text{ if } x^*_{i,p} = 0,$$
$$\text{and } C_p(x^*) = C_i \text{ if } x^*_{i,p} > 0.$$

Combining this with (A.1), for any arbitrary $x \in \mathcal{X}$, we have:

$$\langle\nabla F(x^*), x\rangle = \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\frac{\partial F}{\partial x_{i,p}}(x^*)x_{i,p} = \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i} C_p(x^*)x_{i,p} \geq \sum_{i\in\mathcal{N}} C_i \sum_{p\in\mathcal{P}_i} x_{i,p} = \sum_{i\in\mathcal{N}} C_i M_i,$$

Moreover, for x^*, we also obtain:

$$\langle \nabla F(x^*), x^* \rangle = \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} \frac{\partial F}{\partial x_{i,p}}(x^*) x^*_{i,p} = \sum_{i\in\mathcal{N}} \sum_{\substack{p\in\mathcal{P}_i \\ x^*_{i,p}>0}} C_p(x^*) x^*_{i,p} = \sum_{i\in\mathcal{N}} C_i \sum_{\substack{p\in\mathcal{P}_i \\ x^*_{i,p}>0}} x^*_{i,p} = \sum_{i\in\mathcal{N}} C_i M_i.$$

In conclusion, we have $\langle \nabla F(x^*), x - x^* \rangle \geq 0$ for any $x \in \mathcal{X}$. Therefore, x^* is a minimizer of F.

($\Leftarrow$) Let $x^* \in \mathcal{X}$ be a minimizer of F. From the variational inequality corresponding to F, we have

$$\begin{aligned} & \langle \nabla F(x^*), x - x^* \rangle \geq 0, \forall x \in \mathcal{X} \\ \Leftrightarrow & \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} C_p(x^*) x_{i,p} \geq \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} C_p(x^*) x^*_{i,p}, \forall x \in \mathcal{X}. \end{aligned} \tag{B.1}$$

We proceed by proof of contradiction. Assume that x^* is *not* an equilibrium flow of Γ, i.e., there exist $j \in \mathcal{N}$ and $\hat{p}, \hat{q} \in \mathcal{P}_j$ such that

$$x^*_{j,\hat{p}} > 0 \text{ and } C_{\hat{p}}(x^*) > C_{\hat{q}}(x^*). \tag{B.2}$$

Consider the flow $x' \in \mathcal{X}$ that is defined as follows:

$$\begin{cases} x'_{j,p} = x^*_{j,p}, \forall p \in \mathcal{P}_j \backslash \{\hat{p}, \hat{q}\}, \\ x'_{j,\hat{p}} = x^*_{j,\hat{p}} - \delta, \\ x'_{j,\hat{q}} = x^*_{j,\hat{q}} + \delta, \\ x'_{i,p'} = x^*_{i,p'}, \forall i \in \mathcal{N} \backslash \{j\}, p' \in \mathcal{P}_i. \end{cases}$$

Here δ is taken such that $0 < \delta < x^*_{j,\hat{p}}$. Intuitively, x' is the flow obtained by moving a δ amount of j-type traffic from the path $\hat{p}$ to the path $\hat{q}$ in the flow x^*.

By this definition, we have

$$\begin{aligned} & \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} C_p(x^*) x'_{i,p} - \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} C_p(x^*) x^*_{i,p} \\ =& C_{\hat{p}}(x^*) \cdot (-\delta) + C_{\hat{q}}(x^*) \cdot (\delta) \\ =& \delta (C_{\hat{q}}(x^*) - C_{\hat{p}}(x^*)) \\ <& 0. && \text{(due to (B.2))} \end{aligned}$$

This contradicts (B.1); therefore, x^* is a mean equilibrium flow of Γ. □

C Proofs of results in Section 3

We start this section by presenting the proof of Theorem 1 concerning the convergence rate of EXPWEIGHT Algorithm (see Appendix C.1). Next, we provide the proof of Proposition 2 showing the smooth-continuity of function F in Appendix C.2, then in Appendix C.3, we prove Theorem 2 showing the convergence properties of the ACCELEWEIGHT algorithm .

C.1. Convergence properties of the EXPWEIGHT algorithm: Proof of Theorem 1. Let h be the entropy regularizer defined in (A.2) and x^* be any equilibrium flow of the mean game Γ, we have:

$$M_{\text{tot}} \log(M_{\max}) + \sum_{t=1}^{T} \gamma^t \langle x^*, c(X^t, \omega^t) \rangle \tag{C.1}$$

$$= \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x^*_{i,p} \log(M_{\max}) + \sum_{t=1}^{T} \gamma^t \langle x^*, c(X^t, \omega^t) \rangle$$

$$= \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x^*_{i,p} \log(x^*_{i,p}) - \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x^*_{i,p} \log\left(\frac{x^*_{i,p}}{M_{\max}}\right) + \sum_{t=1}^{T} \gamma^t \langle x^*, c(X^t, \omega^t) \rangle$$

$$
\begin{aligned}
&\geq h(x^*) + \sum_{t=1}^{T} \gamma^t \langle x^*, c(X^t, \omega^t)\rangle && \left(\text{since } \log\left(\tfrac{x^*_{i,p}}{M_{\max}}\right) \leq 0, \forall i \in \mathcal{N}, \forall p \in \mathcal{P}_i \right)\\
&= h(x^*) - \left\langle x^*, -\sum_{t=1}^{T} \gamma^t c(X^t, \omega^t) \right\rangle\\
&= h(x^*) - \langle x^*, Y^T \rangle && \text{(due to Line 4 of Algorithm 1)}\\
&\geq -h^*(Y^T) && \text{(due to Fenchel-Young inequality)}\\
&= -h^*(Y^0) - \sum_{t=1}^{T} \left[h^*(Y^t) - h^*(Y^{t-1})\right]\\
&= -h^*(Y^0) - \sum_{t=1}^{T} \sum_{i\in\mathcal{N}} M_i \log\left(\frac{\sum_{p\in\mathcal{P}_i} \exp\left(Y^t_{i,p}\right)}{\sum_{q\in\mathcal{P}_i} \exp\left(Y^{t-1}_{i,q}\right)}\right) && \text{(by definition of } h^* \text{)}\\
&= -h^*(Y^0) - \sum_{t=1}^{T} \sum_{i\in\mathcal{N}} M_i \log\left(\frac{M_i \sum_{p\in\mathcal{P}_i} \exp\left(Y^{t-1}_{i,p}\right) \exp(-\gamma^t c_p(X^t, \omega^t))}{\sum_{q\in\mathcal{P}_i} \exp\left(Y^{t-1}_{i,q}\right)} \cdot \frac{1}{M_i}\right)\\
&= -h^*(Y^0) - \sum_{t=1}^{T} \left[\sum_{i\in\mathcal{N}} M_i \log\left(\sum_{p\in\mathcal{P}_i} X^t_{i,p} \exp\left(-\gamma^t c_p(X^t, \omega^t)\right)\right) - \sum_{i\in\mathcal{N}} M_i \log(M_i)\right]. && \text{(C.2)}
\end{aligned}
$$

We look for lower-bounds of the terms in the right-hand-size of (C.2). First, since we chose $Y^0 = \mathbf{0}$, we have $-h^*(Y^0) = -\max_{x\in\mathcal{X}}(\langle \mathbf{0}, x\rangle - h(x)) = \min_{x\in\mathcal{X}} h(x) = -\min_{x\in\mathcal{X}} \sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x_{i,p} \log(1/x_{i,p})$. Moreover, apply Jensen's inequality on the function $\log(\cdot)$ (that is a concave function), for any $x \in \mathcal{X}$, we have $\sum\limits_{i\in\mathcal{N}} \sum\limits_{p\in\mathcal{P}_i} x_{i,p} \log\left(\frac{1}{x_{p,i}}\right) \leq \left(\sum\limits_{i\in\mathcal{N}} \sum\limits_{p\in\mathcal{P}_i} x_{i,p}\right) \log\left(\frac{\sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} \frac{x_{i,p}}{x_{i,p}}}{\sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} x_{i,p}}\right) = M_{\text{tot}} \log(P/M_{\text{tot}})$. Therefore, we obtain

$$
-h^*(Y^0) = \min_{x\in\mathcal{X}} h(x) \geq -M_{\text{tot}} \log(P/M_{\text{tot}}). \tag{C.3}
$$

To derive a bound for the second term in (C.2), we apply the inequality $\exp(-\alpha) \leq 1 - \alpha + \alpha^2$ for all $\alpha \geq 0$ to obtains the following inequality

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}} M_i \log\left(\sum_{p\in\mathcal{P}_i} X^t_{i,p} \exp\left(-\gamma^t c_p(X^t, \omega^t)\right)\right) - \sum_{i\in\mathcal{N}} M_i \log(M_i)\\
&\leq \sum_{i\in\mathcal{N}} M_i \log\left(\sum_{p\in\mathcal{P}_i} X^t_{i,p} \left[1 - \gamma^t c_p(X^t, \omega^t) + (\gamma^t c_p(X^t, \omega^t))^2\right]\right) - \sum_{i\in\mathcal{N}} M_i \log(M_i)\\
&\leq \sum_{i\in\mathcal{N}} M_i \log\left(M_i - \sum_{p\in\mathcal{P}_i} X^t_{i,p} \left[\gamma^t c_p(X^t, \omega^t) - (\gamma^t c_p(X^t, \omega^t))^2\right]\right) - \sum_{i\in\mathcal{N}} M_i \log(M_i)\\
&= \sum_{i\in\mathcal{N}} M_i \log(M_i) + \sum_{i\in\mathcal{N}} M_i \log\left(1 - \frac{\sum_{p\in\mathcal{P}_i} X^t_{i,p} \left[\gamma^t c_p(X^t, \omega^t) - (\gamma^t c_p(X^t, \omega^t))^2\right]}{M_i}\right) - \sum_{i\in\mathcal{N}} M_i \log(M_i)\\
&\leq -\sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}_i} X^t_{i,p} \left[\gamma^t c_p(X^t, \omega^t) - (\gamma^t c_p(X^t, \omega^t))^2\right] && \text{(C.4)}\\
&= -\left\langle X^t, \gamma^t c(X^t, \omega^t)\right\rangle + \left\langle X^t, (\gamma^t c(X^t, \omega^t))^2\right\rangle. && \text{(C.5)}
\end{aligned}
$$

Here, (C.4) comes from applying the inequality $\log(1-\alpha) \leq -\alpha$ for any $\alpha < 1$.

Now, combining (C.3) and (C.5) with (C.2), we have

$$\sum_{t=1}^{T}\left[\left\langle X^t, \gamma^t c(X^t,\omega^t)\right\rangle - \langle x^*, \gamma^t c(X^t,\omega^t)\rangle\right] \le M_{\text{tot}} \log(PM_{\max}/M_{\text{tot}}) + \sum_{t=1}^{T}\left\langle X^t, (\gamma^t c(X^t,\omega^t))^2\right\rangle. \tag{C.6}$$

On the other hand, from Assumption 1,

$$\left\langle X^t, (\gamma^t c(X^t,\omega^t))^2\right\rangle = \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i} X^t_{i,p}\left[\gamma^t c_p(X^t,\omega^t)\right]^2 \le \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i} X^t_{i,p}(\gamma^t H)^2 = (\gamma^t H)^2 M_{\text{tot}}.$$

Combining this with (C.6), we easily obtain that:

$$\sum_{t=1}^{T}\gamma^t\left\langle X^t - x^*, c(X^t,\omega^t)\right\rangle \le M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + MH^2\sum_{t=1}^{T}(\gamma^t)^2. \tag{C.7}$$

Now, for any t, let us denote $v^t := \mathbb{E}\left[c(X^t,\omega^t)\middle| X^{t-1},\omega^{t-1},\dots,X^1,\omega^1\right]$. From the law of total expectation, we obtain that:

$$\mathbb{E}\left[\gamma^t\left\langle X^t - x^*, c(X^t,\omega^t)\right\rangle\right] = \mathbb{E}\left[\gamma^t\left\langle X^t - x^*, v^t\right\rangle\right]. \tag{C.8}$$

On the other hand, since ω^t is drawn randomly from an i.i.d and due to the computation of the gradient of F (see (A.1)), we have $v^t = \nabla F(X^t)$ which induces that

$$\left\langle X^t - x^*, v^t\right\rangle \ge F(X^t) - F(x^*) \ge 0. \tag{C.9}$$

Here, the last inequality comes from the fact that $x^* = \arg\min_{x\in\mathcal{X}} F(x)$ since x^* is an equilibrium flow of Γ (cf. Proposition 2). Finally, take the expectation on two sides of (C.7) and apply (C.8) and (C.9), we have:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{T}\gamma^t\big(F(X^t)-F(x^*)\big)\right] \le& \mathbb{E}\left[\sum_{t=1}^{T}\gamma^t\left\langle X^t-x^*, c(X^t,\omega^t)\right\rangle\right]\\ \le& M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + MH^2\sum_{t=1}^{T}(\gamma^t)^2.\end{aligned} \tag{C.10}$$

Now, by choosing $\gamma^t = 1/\sqrt{t}$ for any t, we rewrite (C.10) as:

$$\begin{aligned}&\mathbb{E}\left[\sum_{t=1}^{t}\frac{1}{\sqrt{t}}\big(F(X^T)-F(x^*)\big)\right] \le M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + MH^2\sum_{t=1}^{T}\frac{1}{t}\\ \Rightarrow &\mathbb{E}\left[\sum_{t=1}^{t}\frac{1}{\sqrt{T}}\big(F(X^T)-F(x^*)\big)\right] \le M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + MH^2(\log(T)+1)\\ \Rightarrow &\mathbb{E}\left[\sum_{t=1}^{t}\frac{1}{T}\big(F(X^T)-F(x^*)\big)\right] \le \frac{M_{\text{tot}}}{\sqrt{T}}\left[\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + H^2(\log(T)+1)\right]\\ \Rightarrow &\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}F(X^T)-F(x^*)\right] \le \frac{M_{\text{tot}}}{\sqrt{T}}\left[\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + H^2(\log(T)+1)\right].\end{aligned} \tag{C.11}$$

Finally, since $F(\cdot)$ is a convex function on $\mathcal{X}$, we have $F(\bar{X}^T) = F\left(\frac{1}{T}\sum_{t=1}^{T}X^t\right) \le \frac{1}{T}\sum_{t=1}^{T}F(X^t)$. Combining this with (C.11) and the fact that $M_{\text{tot}} = \sum_{i\in\mathcal{N}}M_i \le NM_{\max}$, we conclude the proof. □

C.2. Smoothness of F: Proof of Proposition 2. From (A.1) for any $x, x' \in \mathcal{X}$, we have:

$$
\begin{aligned}
&\|\nabla F(x) - \nabla F(x')\|_\infty \\
=&\max_{\substack{i\in\mathcal{N}\\ p\in\mathcal{P}_i}} \left| \mathbb{E}\left[\sum_{e\in p} c_e(\mu_e(x),\omega)\right] - \mathbb{E}\left[\sum_{e\in p} c_e(\mu_e(x'),\omega)\right]\right| \\
\le&\max_{\substack{i\in\mathcal{N}\\ p\in\mathcal{P}_i}} \mathbb{E}\left[\left|\sum_{e\in p} c_e(\mu_e(x),\omega) - c_e(\mu_e(x'),\omega)\right|\right] \\
\le&\max_{\substack{i\in\mathcal{N}\\ p\in\mathcal{P}_i}} \mathbb{E}\left[\sum_{e\in p} L|\mu_e(x) - \mu_e(x')|\right] && \text{(due to Assumption 1)} \\
\le&\max_{\substack{i\in\mathcal{N}\\ p\in\mathcal{P}_i}} \mathbb{E}\left[L \sum_{e\in p}\sum_{j\in\mathcal{N}}\sum_{\substack{q\in\mathcal{P}_j\\ q\ni e}} |x_{j,q} - x'_{j,q}|\right] \\
\le&\mathbb{E}\left[L \sum_{e\in\mathcal{E}}\sum_{j\in\mathcal{N}}\sum_{\substack{q\in\mathcal{P}_j\\ q\ni e}} |x_{j,q} - x'_{j,q}|\right]. && \text{(C.12)}
\end{aligned}
$$

Now, for any $i \in \mathcal{N}$, we denote the length a path $p \in \mathcal{P}_i$ by K_p (i.e., the number of edges containing in p). Recall that K denotes the length of the longest paths in $\mathcal{P} = \bigcup_{i\in\mathcal{N}} \mathcal{P}_i$, we have

$$
\begin{aligned}
\|x - x'\|_1 = \sum_{j\in\mathcal{N}}\sum_{q\in\mathcal{P}_j} |x_{j,q} - x'_{j,q}| &= \sum_{j\in\mathcal{N}}\sum_{q\in\mathcal{P}_j}\sum_{e\in q} \frac{|x_{j,q} - x'_{j,q}|}{K_q} \\
&\ge \sum_{j\in\mathcal{N}}\sum_{q\in\mathcal{P}_j}\sum_{e\in q} \frac{|x_{j,q} - x'_{j,q}|}{K} \\
&= \frac{1}{K}\sum_{e\in\mathcal{E}}\sum_{j\in\mathcal{N}}\sum_{\substack{q\in\mathcal{P}_j\\ q\ni e}} |x_{j,q} - x'_{j,q}|.
\end{aligned}
$$

Combine this with (C.12), we have $\|\nabla F(x) - \nabla F(x')\|_\infty \le KL\|x - x'\|_1$. This shows that F is a β-smooth function where $\beta := KL$ and we conclude the proof of this proposition. □

C.3. Convergence properties of the ACCELEWEIGHT algorithm: Proof of Theorem 2. Theorem 2 considers the static regime where we consider a fixed game Γ_ω with a fixed $\omega \in \Omega$. In this setting, from (A.1), we also have $\nabla F(x) = c(x, \omega)$ for any flow $x \in \mathcal{X}$. Due to this reason, in this section, *we will use these notations interchangeably without further explanations*. Note also that it is trivial to verify that ACCELEWEIGHT outputs a valid flow $X^t \in \mathcal{X}$ at each time t.[2]

We will prove Theorem 2 in the following 3 steps.

Step 1: Define $\Delta^t := \gamma^{t-1}[F(X^t) - F(x^*)] + KL(x^*\|Z^t)$ and prove that Δ^t is a decreasing sequence as t increases. Here, X^t, Z^t, γ^{t-1} are defined as in Algorithm 2 and x^* is any equilibrium of the game.

Fix an iteration $t \ge 1$. Since F is a β-smooth function on $\mathcal{X}$ w.r.t the $\|\cdot\|_1$ norm (cf. Proposition 2), we have:

$$
\begin{aligned}
&F\big(X^{t+1}\big) \\
\le&F\big(\bar{Z}^t\big) + \langle \nabla F(\bar{Z}^t), X^{t+1} - \bar{Z}^t\rangle + \frac{\beta}{2}\|X^{t+1} - \bar{Z}^t\|_1^2 && \text{(due to (A.5))}
\end{aligned}
$$

[2] Indeed, by its definition in Line 3 of Algorithm 2, $Z^t \in \mathcal{X}, \forall t$. Moreover, due to the choice of α^0, $X^1 = Z^1 \in \mathcal{X}$. Assume that $X^{t-1} \in \mathcal{X}$; then, we have $X^t = \alpha^{t-1}X^{t-1} + (1 - \alpha^{t-1})Z^t$ also belongs to $\mathcal{X}$ (since $\mathcal{X}$ is a convex set). By proof of induction, we have that $X^t \in \mathcal{X}, \forall t$.

$$
\begin{aligned}
&= F(\bar{Z}^t) + \langle \nabla F(\bar{Z}^t), X^{t+1} - \bar{Z}^t \rangle + \frac{\beta}{2}(1-\alpha^t)^2 \|Z^{t+1} - Z^t\|_1^2 \\
&\hspace{20em} \text{(from Lines 4 and 7 of Algorithm 2)} \\
&\leq F(\bar{Z}^t) + \langle \nabla F(\bar{Z}^t), X^{t+1} - \bar{Z}^t \rangle + \frac{\beta}{2}(1-\alpha^t)^2 \cdot 2\sigma \cdot \mathrm{KL}(Z^{t+1} \| Z^t) \\
&\hspace{20em} \text{(since } h \text{ is } \tfrac{1}{\sigma}\text{-strongly convex)} \\
&= F(\bar{Z}^t) + \langle \nabla F(\bar{Z}^t), X^{t+1} - \bar{Z}^t \rangle + \beta\sigma(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t).
\end{aligned} \tag{C.13}
$$

On the other hand, from the convexity of F, we have

$$
\alpha^t F(X^t) + (1-\alpha^t) F(x^*) \geq F(\alpha^t X^t + (1-\alpha^t)x^*) \geq F(\bar{Z}^t) + \langle \nabla F(\bar{Z}^t), \alpha^t X^t + (1-\alpha^t)x^* - \bar{Z}^t \rangle,
$$

which deduces the following inequality:

$$
\begin{aligned}
& F(X^{t+1}) - F(x^*) - \alpha^t [F(X^t) - F(x^*)] \\
=& F(X^{t+1}) - [\alpha^t F(X^t) + (1-\alpha^t) F(x^*)] \\
\leq& F(X^{t+1}) - F(\bar{Z}^t) + \langle \nabla F(\bar{Z}^t), \bar{Z}^t - \alpha^t X^t - (1-\alpha^t)x^* \rangle \\
\leq& \langle \nabla F(\bar{Z}^t), X^{t+1} - \bar{Z}^t \rangle + \beta\sigma(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t) + \langle \nabla F(\bar{Z}^t), \bar{Z}^t - \alpha^t X^t - (1-\alpha^t)x^* \rangle \\
& \hspace{24em} \text{(due to (C.13))} \\
=& \langle \nabla F(\bar{Z}^t), X^{t+1} - \alpha^t X^t - (1-\alpha^t)x^* \rangle + \beta\sigma(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t) \\
=& (1-\alpha^t) \langle \nabla F(\bar{Z}^t), Z^{t+1} - x^* \rangle + \beta\sigma(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t).
\end{aligned} \tag{C.14}
$$

Here, the last equality comes from Line 4 of Algorithm 2. Multiplying two sides of (C.14) by γ^t, we have

$$
\begin{aligned}
& \gamma^t [F(X^{t+1}) - F(x^*)] - \gamma^t \alpha^t [F(X^t) - F(x^*)] \\
\leq& (1-\alpha^t)\gamma^t \langle \nabla F(\bar{Z}^t), Z^{t+1} - x^* \rangle + \gamma^t \beta\sigma(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t).
\end{aligned} \tag{C.15}
$$

Now, for the sake of brevity, let us denote $W^t = (1-\alpha^t)\gamma^t c(\bar{Z}^t, \omega) = (1-\alpha^t)\gamma^t \nabla F(\bar{Z}^t)$. From the facts that $Z^{t+1} = \Lambda(Y^t)$ (see Line 3 of Algorithm 2) and that $Y^t = \sum_{s=1}^t W^s$ (see Line 7 of Algorithm 2), we can rewrite that $Z^{t+1}_{i,p} = M_i \frac{Z^t_{i,p} \exp(-W^t_{i,p})}{\sum_{q \in \mathcal{P}_i} Z^t_{i,q} \exp(-W^t_{i,q})}$. Therefore, we have:

$$
\begin{aligned}
& \langle \nabla h(Z^t) - \nabla h(Z^{t+1}), Z^{t+1} - x^* \rangle \\
=& \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}} \log\left(\frac{Z^t_{i,p}}{Z^{t+1}_{i,p}}\right)(Z^{t+1}_{i,p} - x^*_{i,p}) \\
=& \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}} \log\left(\frac{Z^t_{i,p}}{M_i \frac{Z^t_{i,p}\exp(-W^t_p)}{\sum_{q \in \mathcal{P}_i} Z^t_{i,q}\exp(-W^t_q)}}\right)(Z^{t+1}_{i,p} - x^*_{i,p}) \\
=& \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}} \log\left(\frac{\sum_{q \in \mathcal{P}_i} Z^t_{i,q}\exp(-W^t_q)}{M_i}\right)(Z^{t+1}_{i,p} - x^*_{i,p}) - \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}} \log(\exp(-W^t_{i,p}))(Z^{t+1}_{i,p} - x^*_{i,p}) \\
=& \sum_{i \in \mathcal{N}} \log\left(\frac{\sum_{q \in \mathcal{P}_i} Z^t_{i,q}\exp(-W^t_q)}{M_i}\right) \sum_{p \in \mathcal{P}_i} (Z^{t+1}_{i,p} - x^*_{i,p}) + \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}} W^t_{i,p}(Z^{t+1}_{i,p} - x^*_{i,p}) \\
=& 0 + \langle W^t, Z^{t+1} - x^* \rangle \qquad \text{(since } \textstyle\sum_{p \in \mathcal{P}_i} Z^{t+1}_{i,p} = \sum_{p \in \mathcal{P}_i} x^*_{i,p} = M_i, \forall i \in \mathcal{N}\text{)} \\
=& (1-\alpha^t)\gamma^t \langle \nabla F(\bar{Z}^t), Z^{t+1} - x^* \rangle.
\end{aligned} \tag{C.16}
$$

Combining (C.16) with (C.15), we obtain that:

$$
\begin{aligned}
& \gamma^t [F(X^{t+1}) - F(x^*)] - \gamma^t \alpha^t [F(X^t) - F(x^*)] \\
\leq& \langle \nabla h(Z^t) - \nabla h(Z^{t+1}), Z^{t+1} - x^* \rangle + \gamma^t \sigma\beta(1-\alpha^t)^2 \mathrm{KL}(Z^{t+1} \| Z^t)
\end{aligned}
$$

$$\leq \mathrm{KL}\big(x^*\big\|Z^t\big) - \mathrm{KL}\big(x^*\big\|Z^{t+1}\big) - \mathrm{KL}\big(Z^{t+1}\big\|Z^t\big) + \gamma^t\sigma\beta\big(1-\alpha^t\big)^2\mathrm{KL}\big(Z^{t+1}\big\|Z^t\big)$$
(by applying (A.3))
$$= \mathrm{KL}\big(x^*\big\|Z^t\big) - \mathrm{KL}\big(x^*\big\|Z^{t+1}\big) + \Big(\gamma^t\sigma\beta\big(1-\alpha^t\big)^2 - 1\Big)\mathrm{KL}\big(Z^{t+1}\big\|Z^t\big). \tag{C.17}$$

From the update rule of γ^t in Line 5, the choice of γ^0 in Line 1 and the update rule of α^t in Line 6 of Algorithm 2, we can trivially obtain that

$$\gamma^t\sigma\beta\big(1-\alpha^t\big)^2 = \gamma^t\frac{1}{\gamma^0}\left(1-\frac{\gamma^{t-1}}{\gamma^t}\right)^2 = \frac{\big(\gamma^{t-1}-\gamma^t\big)^2}{\gamma^t\gamma^0} = 1. \tag{C.18}$$

Therefore, the last term in (C.17) equals zero and we can rewrite (C.17) as:

$$\gamma^t\big[F(X^{t+1}) - F(x^*)\big] + \mathrm{KL}\big(x^*\big\|Z^{t+1}\big) \leq \gamma^{t-1}\big[F(X^t) - F(x^*)\big] + \mathrm{KL}\big(x^*\big\|Z^t\big).$$

This shows precisely that $\Delta^T \leq \Delta^{T-1} \leq \ldots \leq \Delta^1$ (by convention, we set $\gamma^{-1} = 0$). Particularly, we have:

$$\gamma^{T-1}\big[F(X^T) - F(x^*)\big] \leq \gamma^{T-1}\big[F(X^T) - F(x^*)\big] + \mathrm{KL}\big(x^*\big\|Z^T\big) := \Delta^T \leq \Delta^{T-1} \leq \ldots \leq \Delta^1 = \mathrm{KL}\big(x^*\big\|Z^1\big). \tag{C.19}$$

Step 2: Prove the following inequality for any equilibrium flow x^* of the mean game Γ and the output X^T of Algorithm 2 at any arbitrary time $T > 0$:

$$F\big(X^T\big) - F(x^*) \leq \frac{4\beta\mathrm{KL}\big(x^*\big\|Z^1\big)}{(T-1)^2} = \frac{4\beta\mathrm{KL}\big(x^*\big\|X^1\big)}{(T-1)^2}. \tag{C.20}$$

To prove (C.20), we first look for a lower-bound of γ^{T-1}. From (C.18), we have

$$\sqrt{\gamma^{t-1}\sigma\beta} = \sqrt{\gamma^t\sigma\beta}\sqrt{1 - \frac{1}{\sqrt{\gamma^t\sigma\beta}}}. \tag{C.21}$$

Now, consider the function $l(u) : (0,1] \to \mathbb{R}_{\geq 0}$ such that $l(u) = (1-u)^{1/2}, \forall u \in (0,1]$. Note that, from Line 5 of Algorithm 2, γ^t is an increasing sequence and thus, for any $t \geq 1$, $\gamma^t \geq \gamma^0 = 1/(\sigma\beta)$ which implies that $0 < \frac{1}{\sqrt{\sigma\beta\gamma^t}} \leq 1$. From the fact that l is a concave function, we have:

$$\begin{aligned} & l\left(\frac{1}{\sqrt{\sigma\beta\gamma^t}}\right) \leq l(0) - l'(0)\left(\frac{1}{\sqrt{\sigma\beta\gamma^t}} - 0\right) \\ \Rightarrow & \left(1 - \frac{1}{\sqrt{\sigma\beta\gamma^t}}\right)^{1/2} \leq 1 - \frac{1}{2\sqrt{\sigma\beta\gamma^t}} \end{aligned}$$

Combining this with (C.21), we obtain $\sqrt{\sigma\beta\gamma^t} \geq \sqrt{\sigma\beta\gamma^{t-1}} + 1/2$. Therefore,

$$\sqrt{\sigma\beta\gamma^{T-1}} \geq \sqrt{\sigma\beta\gamma^{T-2}} + \frac{1}{2} \geq \ldots \geq \sqrt{\sigma\beta\gamma^0} + \frac{T-1}{2} = \frac{T+1}{2} \geq \frac{T}{2}.$$

This implies that $\gamma^{T-1} > \frac{T^2}{4\sigma\beta} > 0$; combining this with (C.19), we have

$$F\big(X^T\big) - F(x^*) \leq \frac{4\sigma\beta\mathrm{KL}\big(x^*\big\|Z^1\big)}{T^2} = \frac{4\sigma\beta\mathrm{KL}\big(x^*\big\|X^1\big)}{T^2}.$$

We finish the proof of (C.20) here.

Step 3: Conclusion. Now, we only need to find an upper-bound of $\mathrm{KL}\big(x^*\big\|X^1\big)$. To do this, from the choice of Y^0 and α^0 in Line 1 of Algorithm 2, we notice that $X^1_{i,p} = Z^1_{i,p} = M_i/P_i$ for any $i \in \mathcal{N}$ and $p \in \mathcal{P}_i$; therefore,

$$\begin{aligned} & \langle \nabla h(X^1), X^1 - x^* \rangle \\ = & \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\log\big(X^1_{i,p}\big)X^1_{i,p} + \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}X^1_{i,p} - \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\log\big(X^1_{i,p}\big)x^*_{i,p} - \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}x^*_{i,p} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\frac{M_i\log\left(M_i/P_i\right)}{P_i}-\sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\log\left(M_i/P_i\right)x^*_{i,p}\\
&= \sum_{i\in\mathcal{N}}M_i\log\left(M_i/P_i\right)-\sum_{i\in\mathcal{N}}M_i\log\left(M_i/P_i\right)\\
&= 0.
\end{aligned}
\tag{C.22}
$$

Combine this and (C.3), we have:

$$
\begin{aligned}
\mathrm{KL}\left(x^*\middle\|X^1\right) &= h(x^*)-h\left(X^1\right)-\langle\nabla h(X^1),x^*-X^1\rangle\\
&\le M_{\text{tot}}\log(M_{\max})-\min_{x\in\mathcal{X}}h(x)\\
&\le M_{\text{tot}}\log(M_{\max})+M_{\text{tot}}\log(P/M_{\text{tot}})\\
&= M_{\text{tot}}\log(PM_{\max}/M_{\text{tot}}).
\end{aligned}
$$

Combine this and (C.20) then recall the notation $\sigma = M_{\max}N$ (i.e., the strongly-convexity constant of h) and the fact that $M_{\text{tot}} = \sum_{i\in\mathcal{N}}M_i \le NM_{\max}$ we obtain that

$$
F\left(X^T\right)-F(x^*)\le\frac{4NM_{\text{tot}}M_{\max}\beta\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)}{(T-1)^2}\le\frac{4\beta(NM_{\max})^2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)}{(T-1)^2}.
$$

This concludes the proof. □

D Proofs of results in Section 4

In this section, we will provide the proof of Theorem 3 concerning the convergence properties of the ADAWEIGHT algorithm (Algorithm 3). In the statements of Theorem 3, we first present Equation (6a) showing the convergence rate of ADAWEIGHT in the generic setting of our learning model (i.e., including the stochastic regime) and then present Equation (6b) concerning the particular case of the static regime. However, it is more convenient to present the proofs of these results in a reversed order: in Appendix D.2, we present the proof of (6b) and in Appendix D.3, we proceed to prove (6a). Before going into the details of these proofs, in Appendix D.1, we prepare some useful notation and lemmas.

Remark. In the remainder of this section, we let $\alpha^t = t, \forall t$ as chosen in Theorem 3.

D.1. Preliminary results for proving Theorem 3.

ADAWEIGHT outputs a valid flow. We justify that $X^t\in\mathcal{X},\forall t$ for any time epoch t. First, from their update-rules (Lines 3-6 of Algorithm 3), we notice that $Z^t, Z^{t+\frac{1}{2}}\in\mathcal{X},\forall t$. We will prove that $\bar{Z}^t, X^t\in\mathcal{X},\forall t$ by induction. Indeed, we have $\bar{Z}^1 = Z^1\in\mathcal{X}$ and $X^1 = Z^{t+\frac{1}{2}}\in\mathcal{X}$. Assume that $\bar{Z}^s, X^s\in\mathcal{X}$ for any $s = 1,2,\dots,t-1$. By the update-rules in Algorithm 3, $\bar{Z}^t$ is a convex combination of Z^t and $Z^{s+\frac{1}{2}}, s = 1,\dots,t-1$. Similarly, X^t is a convex combination of $Z^{t+\frac{1}{2}}$ and $Z^{s+\frac{1}{2}}, s = 1,\dots,t-1$. Combining these with the fact that $\mathcal{X}$ is a convex set, we have that $\bar{Z}^t, X^t\in\mathcal{X},\forall t$.

Next, we present two trivial lemmas:

Lemma D.1. *Let $(u^t)_{t\in\mathbb{N}}$ be a sequence of non-negative numbers, for any $T\in\mathbb{N}$, we have:*

$$
\sqrt{\sum_{t=0}^{T}u^t}\le\sum_{t=0}^{T}\frac{u^t}{\sqrt{\sum_{s=0}^{t}u^s}}\le 2\sqrt{\sum_{t=0}^{T}u^t}
\tag{D.1}
$$

A proof of Lemma D.1 can be extracted from Lemma 2 of [16].

Lemma D.2. *For any $x, x', Y, Y'\in\mathbb{R}^d$ and any norm $\|\cdot\|$, the following identity holds:*

$$
\|x-x'\|_1\|Y-Y'\|_\infty = \min_{u>0}\left\{\frac{u}{2}\|x-x'\|_1^2+\frac{1}{2u}\|Y-Y'\|_\infty^2\right\}.
\tag{D.2}
$$

Proof. For any $x, x', Y, Y' \in \mathbb{R}^d$, let us define the corresponding function $\psi : \mathbb{R} \to \mathbb{R}$ such that for any $u > 0$, $\psi(u) = \frac{u}{2}\|x - x'\|_1^2 + \frac{1}{2u}\|Y - Y'\|_\infty^2$. Then, (D.2) comes from the fact that $\psi'(u^*) = 0$ and $\psi''(u^*) > 0$ where $u^* = \|x - x'\|_1 \|Y - Y'\|_\infty$. $\square$

Finally, we prove the following important lemma=:

Lemma D.3. *Let X^T be the output of Algorithm 3 at iteration T, x^* be any equilibrium flow and denote $\hat{\Delta}_T(x^*) = \sum_{t=1}^{T} \alpha^t \langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x^* \rangle$ we have*

$$\mathbb{E}\big[F(X^T) - F(x^*)\big] \leq 2\,\mathbb{E}\Big[\hat{\Delta}_T(x^*)\Big]/T^2. \tag{D.3}$$

Proof. Let us denote $A^t := \sum_{s=1}^{t} s = \frac{t(t+1)}{2}$. By Line 7 of Algorithm 3, for any t, we have: $Z^{t+\frac{1}{2}} = \frac{A^t}{t}X^t - \frac{A^{t-1}}{t}X^{t-1}$.

Therefore, we have:

$$\begin{aligned}
&\sum_{t=1}^{T} t\langle Z^{t+\frac{1}{2}} - x^*, \nabla F(X^t)\rangle \\
=&\sum_{t=1}^{T} t\left\langle \frac{A^t}{t}X^t - \frac{A^{t-1}}{t}X^{t-1} - x^*, \nabla F(X^t)\right\rangle \\
=&\sum_{t=1}^{T} t\left\langle \frac{A^t}{t}\big(X^t - x^*\big) - \frac{A^{t-1}}{t}\big(X^{t-1} - x^*\big), \nabla F(X^t)\right\rangle && \text{(since } A^t - A^{t-1} = t\text{)} \\
=&\sum_{t=1}^{T} \big[A^t\big\langle X^t - x^*, \nabla F(X^t)\big\rangle - A^{t-1}\big\langle X^{t-1} - x^*, \nabla F(X^t)\big\rangle\big] \\
=&\sum_{t=1}^{T} \big[A^t\big\langle X^t - x^*, \nabla F(X^t)\big\rangle - A^{t-1}\big\langle X^t - x^*, \nabla F(X^t)\big\rangle + A^{t-1}\big\langle X^t - X^{t-1}, \nabla F(X^t)\big\rangle\big] \\
=&\sum_{t=1}^{T} \big[t\big\langle X^t - x^*, \nabla F(X^t)\big\rangle + A^{t-1}\big\langle X^t - X^{t-1}, \nabla F(X^t)\big\rangle\big] \\
\geq&\sum_{t=1}^{T} t\big[F(X^t) - F(x^*)\big] + \sum_{t=1}^{T} A^{t-1}\big[F(X^t) - F(X^{t-1})\big] \\
=&\sum_{t=1}^{T} t\big[F(X^t) - F(x^*)\big] + \sum_{t=1}^{T-1} t\big[F(X^T) - F(X^t)\big] && \text{(via telescopic-sum)} \\
=&T\big[F(X^T) - F(x^*)\big] + \sum_{t=1}^{T-1} t\big[F(X^T) - F(x^*)\big] \\
=&\sum_{t=1}^{T} t\big[F(X^T) - F(x^*)\big].
\end{aligned} \tag{D.4}$$

Divide two sides of (D.4) by $A^t > 0$ and notice that $A^t > \frac{T^2}{2}$, we obtain that:

$$\begin{aligned}
&\frac{1}{A^t}\sum_{t=1}^{T} t\langle Z^{t+\frac{1}{2}} - x^*, \nabla F(X^t)\rangle \geq \frac{1}{A^t}\sum_{t=1}^{T} t\big[F(X^T) - F(x^*)\big] \\
\Rightarrow&\frac{2}{T^2}\sum_{t=1}^{T} t\langle Z^{t+\frac{1}{2}} - x^*, \nabla F(X^t)\rangle \geq F(X^T) - F(x^*).
\end{aligned} \tag{D.5}$$

Taking the expectation of two sides of (D.5), we conclude the proof. $\square$

D.2. Convergence of AdaWeight in the static regime: proof of (6b). In this section, we work with the static regime, i.e., we consider the game Γ_ω where $\omega \in \Omega$ is fixed but unknown. Recall that in this setting, $\nabla F(x) = c(x,\omega)$; therefore, in this section, we will use these notations interchangeably without further explanation. We arrange the proof into 5 steps as follows.

Step 1: Upper-bounding the term $\hat{\Delta}_T(x)$ (defined as in Lemma D.3).

First, we introduce a new notation:

$$\mathcal{Q}^t := \Lambda\big(\eta^t Y^{t+1}\big). \tag{D.6}$$

Note that this notion is only used in our analysis in the sequel and is *not* computed by Algorithm 3. One can easily justify that $\mathcal{Q}^t = \arg\max_{x\in\mathcal{X}} \big\langle x, \eta^t Y^{t+1}\big\rangle - h(x)$ where the function h is defined as in (A.2).

Fix an arbitrary $t \in \mathbb{N}$ and an arbitrary $x \in \mathcal{X}$, from the update rule in Line 8 of Algorithm 3, we have:

$$t\big\langle \nabla F(X^t), \mathcal{Q}^t - x\big\rangle = t\big\langle c(X^t,\omega), \mathcal{Q}^t - x\big\rangle = \frac{1}{\eta^t}\langle \eta^t Y^t - \eta^t Y^{t+1}, \mathcal{Q}^t - x\rangle. \tag{D.7}$$

On the other hand, from the definition of Fenchel coupling (see (A.4)), we have that

$$\begin{aligned}
&\mathcal{F}(\mathcal{Q}^t, \eta^t Y^t)\\
=&h^*(\eta^t Y^t) + h(\mathcal{Q}^t) - \big\langle \mathcal{Q}^t, \eta^t Y^t\big\rangle\\
\geq&\big\langle Z^t, \eta^t Y^t\big\rangle - h(Z^t) + h(\mathcal{Q}^t) - \big\langle \mathcal{Q}^t, \eta^t Y^t\big\rangle && \text{(by the definition of } h^*)\\
=&h(\mathcal{Q}^t) - h(Z^t) - \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle\\
=&\sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}} \mathcal{Q}^t_{i,p}\log\big(\mathcal{Q}^t_{i,p}\big) - \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}} Z^t_{i,p}\log\big(Z^t_{i,p}\big) - \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle\\
=&\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big) + \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i} \log\big(Z^t_{i,p}\big)\big(\mathcal{Q}^t_{i,p} - Z^t_{i,p}\big) - \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle\\
=&\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big) + \sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}} \log\left(M_i \frac{\exp(\eta^t Y^t_{i,p})}{\sum\limits_{q\in\mathcal{P}_i}\exp(\eta^t Y^t_{i,q})}\right)\big(\mathcal{Q}^t_{i,p} - Z^t_{i,p}\big) - \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle\\
=&\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big) + \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle + \sum_{i\in\mathcal{N}} \log\left(\frac{M_i}{\sum\limits_{q\in\mathcal{P}_i}\exp(\eta^t Y^t_{i,q})}\right)\sum_{p\in\mathcal{P}_i}\big(\mathcal{Q}^t_{i,p} - Z^t_{i,p}\big) - \big\langle \mathcal{Q}^t - Z^t, \eta^t Y^t\big\rangle\\
=&\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big). && \text{(since } \textstyle\sum_{p\in\mathcal{P}_i}\mathcal{Q}^t_{i,p} = \sum_{p\in\mathcal{P}_i} Z^t_{i,p} = M_i)
\end{aligned}$$

From this, we can deduce that

$$\begin{aligned}
&\mathcal{F}(x, \eta^t Y^t) - \mathcal{F}(x, \eta^t Y^{t+1}) - \mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big)\\
\geq&\mathcal{F}(x, \eta^t Y^t) - \mathcal{F}(x, \eta^t Y^{t+1}) - \mathcal{F}(\mathcal{Q}^t, \eta^t Y^t)\\
=&-h(\mathcal{Q}^t) - h^*\big(\eta^t Y^{t+1}\big) + \big\langle x, \eta^t Y^{t+1} - \eta^t Y^t\big\rangle + \big\langle \mathcal{Q}^t, \eta^t Y^t\big\rangle\\
=&-h(\mathcal{Q}^t) - h^*\big(\eta^t Y^{t+1}\big) + \big\langle x, \eta^t Y^{t+1} - \eta^t Y^t\big\rangle + \big\langle \mathcal{Q}^t, \eta^t Y^t - \eta^t Y^{t+1}\big\rangle + \big\langle \mathcal{Q}^t, \eta^t Y^{t+1}\big\rangle\\
=&\langle \eta^t Y^t - \eta^t Y^{t+1}, \mathcal{Q}^t - x\rangle.
\end{aligned} \tag{D.8}$$

Here, the last equality comes from the fact that $h^*\big(\eta^t Y^{t+1}\big) = \max\limits_{x\in\mathcal{X}}\big\langle x, \eta^t Y^{t+1}\big\rangle - h(x)$ (by definition) and that $\mathcal{Q}^t = \arg\max_{x\in\mathcal{X}}\big\langle x, \eta^t Y^{t+1}\big\rangle - h(x)$ thus, $h^*(\eta^t Y^{t+1}) = \big\langle \mathcal{Q}^t, \eta^t Y^{t+1}\big\rangle - h(\mathcal{Q}^t)$. Combine (D.7) and (D.8), we have

$$\begin{aligned}
&t\big\langle \nabla F(X^t), \mathcal{Q}^t - x\big\rangle \leq \frac{1}{\eta^t}\big[\mathcal{F}(x, \eta^t Y^t) - \mathcal{F}(x, \eta^t Y^{t+1}) - \mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big)\big]\\
\Rightarrow&\frac{1}{\eta^t}\mathcal{F}(x, \eta^t Y^{t+1}) \leq \frac{1}{\eta^t}\mathcal{F}(x, \eta^t Y^t) - t\big\langle \nabla F(X^t), \mathcal{Q}^t - x\big\rangle - \frac{1}{\eta^t}\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big)
\end{aligned}$$

$$\Rightarrow \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1}) \le \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^t) - t\langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x\rangle - t\langle \nabla F(X^t), \mathcal{Q}^t - Z^{t+\frac{1}{2}}\rangle - \frac{1}{\eta^t}\mathrm{KL}\big(\mathcal{Q}^t \big\| Z^t\big). \tag{D.9}$$

Now, we turn our focus to the term $\mathrm{KL}(\mathcal{Q}^t\|Z^t)$ and look for its upper-bound. To do this, we first notice that:

$$\begin{aligned}
&\frac{1}{\eta^t}\mathrm{KL}\big(\mathcal{Q}^t\big\|Z^t\big) - \frac{1}{\eta^t}\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big)) - \frac{1}{\eta^t}\mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big)\\
=&\frac{1}{\eta^t}\langle \nabla h(Z^t) - \nabla h(Z^{t+\frac{1}{2}}), Z^{t+\frac{1}{2}} - \mathcal{Q}^t\rangle && \text{(due to (A.3))}\\
\ge&\frac{1}{\eta^t}\Big[\mathrm{KL}\big(\mathcal{Q}^t\big\|Z^t\big) - \mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big)\Big]. && \text{(D.10)}
\end{aligned}$$

On the other hand, since $Z^{t+\frac{1}{2}} = \Lambda(\eta^t Y^{t+\frac{1}{2}})$, we can justify that $Z^{t+\frac{1}{2}} = \arg\min_{x\in\mathcal{X}}\big\{\big\langle -t\eta^t\nabla F(\bar{Z}^t), Z^t - x\big\rangle + \mathrm{KL}(x\|Z^t)\big\}$ which helps us deduce:

$$\Big\langle -t\eta^t\nabla F(\bar{Z}^t), Z^t - Z^{t+\frac{1}{2}}\Big\rangle + \mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big) \le \big\langle -t\eta^t\nabla F(\bar{Z}^t), Z^t - \mathcal{Q}^t\big\rangle + \mathrm{KL}\big(\mathcal{Q}^t\big\|Z^t\big).$$

This implies that

$$\begin{aligned}
\mathrm{KL}\big(\mathcal{Q}^t\big\|Z^t\big) - \mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big) &\ge \Big\langle -t\eta^t\nabla F(\bar{Z}^t), Z^t - Z^{t+\frac{1}{2}}\Big\rangle - \big\langle -t\eta^t\nabla F(\bar{Z}^t), Z^t - \mathcal{Q}^t\big\rangle\\
&= t\eta^t\Big\langle \nabla F(\bar{Z}^t), Z^{t+\frac{1}{2}} - \mathcal{Q}^t\Big\rangle
\end{aligned}$$

Plug this into (D.10), we get:

$$\frac{1}{\eta^t}\mathrm{KL}\big(\mathcal{Q}^t\big\|Z^t\big) \ge \frac{1}{\eta^t}\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big) + \frac{1}{\eta^t}\mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big) + t\Big\langle \nabla F(\bar{Z}^t), Z^{t+\frac{1}{2}} - \mathcal{Q}^t\Big\rangle,$$

and replace this into (D.9), then rearrange, we obtain:

$$\begin{aligned}
t\Big\langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x\Big\rangle \le&\frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^t) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1}) + t\Big\langle \nabla F(X^t) - \nabla F(\bar{Z}^t), \mathcal{Q}^t - Z^{t+\frac{1}{2}}\Big\rangle\\
&- \frac{1}{\eta^t}\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big) - \frac{1}{\eta^t}\mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big). \qquad \text{(D.11)}
\end{aligned}$$

Finally, apply (D.11) with $t = 1,\dots,T$ and sum them up, we deduce that:

$$\begin{aligned}
&\hat{\Delta}_T(x)\\
\le&\sum_{t=1}^{T}\Big[\frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^t) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1})\Big]\\
&+\Bigg[\sum_{t=1}^{T} t\Big\langle \nabla F(X^t) - \nabla F(\bar{Z}^t), \mathcal{Q}^t - Z^{t+\frac{1}{2}}\Big\rangle - \sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big)\Bigg] - \sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\Big(Z^{t+\frac{1}{2}}\Big\|Z^t\Big).
\end{aligned} \tag{D.12}$$

<u>Step 2:</u> *Upper-bound of the first term in the right-hand-side of* (D.12).

We start by consider:

$$\begin{aligned}
&\frac{1}{\eta^{t+1}}\mathcal{F}(x,\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1})\\
=&\frac{1}{\eta^{t+1}}h(x) + \frac{1}{\eta^{t+1}}h^*(\eta^{t+1}Y^{t+1}) - \big\langle Y^{t+1}, x\big\rangle - \frac{1}{\eta^t}h(x) - \frac{1}{\eta^t}h^*(\eta^t Y^{t+1}) + \big\langle Y^{t+1}, x\big\rangle\\
=&\Big(\frac{1}{\eta^{t+1}} - \frac{1}{\eta^t}\Big)h(x) + \Big[\frac{1}{\eta^{t+1}}h^*(\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}h^*(\eta^t Y^{t+1})\Big].
\end{aligned} \tag{D.13}$$

Now, we look for the upper-bound of $\left[\frac{1}{\eta^{t+1}}h^*(\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}h^*(\eta^t Y^{t+1})\right]$. To do this, we define the function $\psi(\eta) = \frac{1}{\eta}\left[h^*(\eta Y^{t+1}) + \min_{x'\in\mathcal{X}} h(x')\right]$. It is easy to check that ψ is an non-decreasing function (by proving that $\psi'(\eta) \geq 0$) and since $\eta^{t+1} \leq \eta^t$, we have $\psi\big(\eta^{t+1}\big) \leq \psi(\eta^t)$ which implies that:

$$\frac{1}{\eta^{t+1}}h^*(\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}h^*(\eta^t Y^{t+1}) \leq \left(\frac{1}{\eta^t} - \frac{1}{\eta^{t+1}}\right)\min_{x'\in\mathcal{X}} h(x').$$

Combing this with (D.13), we obtain

$$\frac{1}{\eta^{t+1}}\mathcal{F}(x,\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1}) \leq \left(\frac{1}{\eta^{t+1}} - \frac{1}{\eta^t}\right)\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right]. \tag{D.14}$$

Now, we have,

$$\begin{aligned}
&\sum_{t=1}^{T}\left[\frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^t) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1})\right]\\
=&\frac{1}{\eta^1}\mathcal{F}(x,\eta^1 Y^1) + \sum_{t=1}^{T}\left[\frac{1}{\eta^{t+1}}\mathcal{F}(x,\eta^{t+1}Y^{t+1}) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1})\right] - \frac{1}{\eta^{T+1}}\mathcal{F}(x,\eta^{T+1}Y^{T+1})\\
\leq&\frac{1}{\eta^1}\mathcal{F}(x,\eta^1 Y^1) + \sum_{t=1}^{T}\left(\frac{1}{\eta^{t+1}} - \frac{1}{\eta^t}\right)\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right] - \frac{1}{\eta^{T+1}}\mathcal{F}(x,\eta^{T+1}Y^{T+1})\\
&\hfill\text{(due to (D.14))}\\
\leq&\frac{1}{\eta^1}\mathcal{F}(x,\eta^1 Y^1) + \left(\frac{1}{\eta^{T+1}} - \frac{1}{\eta^1}\right)\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right] \qquad \text{(since } \mathcal{F}(x,\eta^{T+1}Y^{T+1}) \geq 0\text{)}\\
=&\frac{1}{\eta^1}\left[h(x) + h^*\big(\eta^1 Y^1\big) - \langle \eta^1 Y^1, x\rangle\right] + \left(\frac{1}{\eta^{T+1}} - \frac{1}{\eta^1}\right)\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right]\\
=&\frac{1}{\eta^1}\left[h(x) + \max_{x'\in\mathcal{X}}\{-h(x')\}\right] + \frac{1}{\eta^{T+1}}\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right] - \frac{1}{\eta^1}\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right]\\
&\hfill\text{(since } Y^1_{i,p} = 0, \forall i\in\mathcal{N}, p\in\mathcal{P}_i\text{)}\\
=&\frac{1}{\eta^{T+1}}\left[h(x) - \min_{x'\in\mathcal{X}} h(x')\right]
\end{aligned}$$

Moreover, note also that $h(x) \leq M_{\text{tot}}\log(M_{\max}), \forall x\in\mathcal{X}$; therefore, we conclude that:

$$\sum_{t=1}^{T}\left[\frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^t) - \frac{1}{\eta^t}\mathcal{F}(x,\eta^t Y^{t+1})\right] \leq \frac{M_{\text{tot}}\log(M_{\max}) - \min_{x'\in\mathcal{X}} h(x')}{\eta^{T+1}}. \tag{D.15}$$

Step 3: Upper-bound of the second term in the right-hand-side of (D.12).

Now, for any T, from the Cauchy-Schwarz inequality and by applying Lemma D.2 (with $u = \sigma\eta^{t+1}$, $\forall t\in\{1,\dots,T\}$), we have:

$$\begin{aligned}
&\sum_{t=1}^{T} t\langle \nabla F(X^t) - \nabla F(\bar{Z}^t), \mathcal{Q}^t - Z^{t+\frac{1}{2}}\rangle\\
\leq&\sum_{t=1}^{T}\left\|t\nabla F(X^t) - t\nabla F(\bar{Z}^t)\right\|_\infty\cdot\left\|\mathcal{Q}^t - Z^{t+\frac{1}{2}}\right\|_1\\
\leq&\sum_{t=1}^{T}\left[\frac{t^2\sigma\eta^{t+1}}{2}\left\|\nabla F(X^t) - \nabla F(\bar{Z}^t)\right\|^2_\infty + \frac{1}{2\sigma\eta^{t+1}}\left\|\mathcal{Q}^t - Z^{t+\frac{1}{2}}\right\|^2_1\right].
\end{aligned} \tag{D.16}$$

On the other hand, recall that the h is a $\frac{1}{\sigma}$-strongly convex function over $\mathcal{X}$ w.r.t the $\|\cdot\|_1$ norm; therefore, we have $\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big) \geq \frac{1}{2\sigma}\left\|\mathcal{Q}^t - Z^{t+\frac{1}{2}}\right\|^2_1$. Combining this with (D.16), we have:

$$\sum_{t=1}^{T} t\Big\langle \nabla F(X^t) - \nabla F(\bar{Z}^t), \mathcal{Q}^t - Z^{t+\frac{1}{2}}\Big\rangle - \sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\Big(\mathcal{Q}^t\Big\|Z^{t+\frac{1}{2}}\Big)$$

$$\leq \frac{\sigma}{2}\sum_{t=1}^{T} t^2\eta^{t+1}\left\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\right\|_\infty^2+\frac{1}{2\sigma}\sum_{t=1}^{T}\left(\frac{1}{\eta^{t+1}}-\frac{1}{\eta^t}\right)\left\|\mathcal{Q}^t-Z^{t+\frac{1}{2}}\right\|_1^2$$

$$\leq \frac{\sigma}{2}\sum_{t=1}^{T} t^2\eta^{t+1}\left\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\right\|_\infty^2+\frac{1}{2\sigma}\sum_{t=1}^{T}\left(\frac{1}{\eta^{t+1}}-\frac{1}{\eta^t}\right)4M_{\mathrm{tot}}^2. \tag{D.17}$$

Here, the first inequality comes from the fact that $\frac{1}{\eta^{t+1}}-\frac{1}{\eta^t}\geq 0, \forall t$ and the last inequality comes from $\|x-x'\|_1\leq 2M_{\mathrm{tot}}$ for any $x,x'\in\mathcal{X}$.[3]

Now, from (D.17), by applying telescopic-sum to the second term of the right-hand-side, we can conclude that:

$$\sum_{t=1}^{T} t\left\langle \nabla F(X^t)-\nabla F(\bar{Z}^t),\mathcal{Q}^t-Z^{t+\frac{1}{2}}\right\rangle-\sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\left(\mathcal{Q}^t\middle\|Z^{t+\frac{1}{2}}\right)$$

$$\leq \frac{\sigma}{2}\sum_{t=1}^{T} t^2\eta^{t+1}\left\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\right\|_\infty^2+\frac{2M_{\mathrm{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}}-1\right)\qquad\text{(note that } \eta^1=1). \tag{D.18}$$

<u>*Step 4:*</u> *Upper-bound of the third term in the right-hand-side of* (D.12).

Since h is $\frac{1}{\sigma}$-strongly convex over $\mathcal{X}$ w.r.t to $\|\cdot\|_1$, for any T, we have

$$-\sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\left(Z^{t+\frac{1}{2}}\middle\|Z^t\right)$$

$$\leq -\sum_{t=1}^{T}\frac{1}{\eta^t}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2$$

$$=-\sum_{t=1}^{T}\frac{1}{\eta^t}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2+\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2$$

$$\leq \frac{1}{2\sigma}\sum_{t=1}^{T}\left(\frac{1}{\eta^{t+1}}-\frac{1}{\eta^t}\right)\|Z^{t+\frac{1}{2}}-Z^t\|_1^2-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2$$

$$\leq \frac{2M_{\mathrm{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}}-1\right)-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2. \tag{D.19}$$

Here, the last inequality comes from the fact that $\|x-x'\|_1\leq 2M_{\mathrm{tot}}, \forall x,x'\in\mathcal{X}$. Now, from Proposition 2, F is a β-smooth function over $\mathcal{X}$ w.r.t the $\|\cdot\|_1$ norm and following the update rules of $\bar{Z}^t$ and X^t in Lines 4 and 7 of Algorithm 3, for any t, we have:

$$\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty\leq\beta\|X^t-\bar{Z}^t\|_1=\beta\frac{t}{\sum_{s=1}^t s}\|Z^{t+\frac{1}{2}}-Z^t\|_1=\frac{2\beta}{t+1}\|Z^{t+\frac{1}{2}}-Z^t\|_1. \tag{D.20}$$

Therefore,

$$\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2\geq\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\frac{(t+1)^2}{4\beta^2}\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2$$

$$\geq\sum_{t=1}^{T}\frac{t^2}{8\sigma\eta^{t+1}\beta^2}\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2. \tag{D.21}$$

Finally, from (D.19) and (D.21), we conclude that:

$$-\sum_{t=1}^{T}\frac{1}{\eta^t}\mathrm{KL}\left(Z^{t+\frac{1}{2}}\middle\|Z^t\right)\leq\frac{2M_{\mathrm{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}}-1\right)-\sum_{t=1}^{T}\frac{t^2}{8\sigma\eta^{t+1}\beta^2}\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2. \tag{D.22}$$

[3]Indeed, $\|x-x'\|_1=\sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\left|x_{i,p}-x'_{i,p}\right|\leq\sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}|x_{i,p}|+\sum_{i\in\mathcal{N}}\sum_{p\in\mathcal{P}_i}\left|x'_{i,p}\right|=2M_{\mathrm{tot}}$.

Step 5: Conclusion.

Replace (D.15), (D.18) and (D.22) into (D.12), we obtain the following inequality for any $x \in \mathcal{X}$ and any T:

$$
\begin{aligned}
&\hat{\Delta}_T(x) \\
\leq &\frac{M_{\text{tot}} \log(M_{\max}) - \min_{x' \in \mathcal{X}} h(x')}{\eta^{T+1}} + \frac{\sigma}{2} \sum_{t=1}^{T} t^2 \eta^{t+1} \big\| \nabla F(X^t) - \nabla F(\bar{Z}^t) \big\|_\infty^2 + \frac{2M_{\text{tot}}^2}{\sigma} \left(\frac{1}{\eta^{T+1}} - 1 \right) \\
&\qquad + \frac{2M_{\text{tot}}^2}{\sigma} \left(\frac{1}{\eta^{T+1}} - 1 \right) - \sum_{t=1}^{T} \frac{t^2}{8\sigma \eta^{t+1} \beta^2} \| \nabla F(X^t) - \nabla F(\bar{Z}^t) \|_\infty^2 \\
= &\frac{M_{\text{tot}} \log(M_{\max}) - \min\limits_{x' \in \mathcal{X}} h(x') + \frac{4M_{\text{tot}}^2}{\sigma}}{\eta^{T+1}} - \frac{4M_{\text{tot}}^2}{\sigma} + \sum_{t=1}^{T} t^2 \left(\frac{\sigma \eta^{t+1}}{2} - \frac{1}{8\sigma \eta^{t+1} \beta^2} \right) \| \nabla F(X^t) - \nabla F(\bar{Z}^t) \|_\infty^2 .
\end{aligned} \tag{D.23}
$$

On the other hand, by the definition of η^{T+1} (see Line 9 of Algorithm 3), we have

$$
\begin{aligned}
\frac{1}{\eta^{T+1}} &= \sqrt{1 + \psi^T} \\
&= \sqrt{1 + \sum_{t=1}^{T} t^2 \| \nabla F(X^t) - \nabla F\big(\bar{Z}^t\big) \|_\infty^2} \\
&\leq \sum_{t=1}^{T} \frac{t^2 \| \nabla F(X^t) - \nabla F\big(\bar{Z}^t\big) \|_\infty^2}{\sqrt{1 + \sum_{s=1}^{t} s^2 \| \nabla F(X^s) - \nabla F\big(\bar{Z}^s\big) \|_\infty^2}} + 1 \\
&= \sum_{t=1}^{T} \eta^{t+1} t^2 \| \nabla F\big(X^t\big) - \nabla F\big(\bar{Z}^t\big) \|_\infty^2 + 1 .
\end{aligned} \tag{D.24}
$$

Here, the inequality is obtained by applying Lemma D.1 for $u^0 = 1$ and $u^t = t^2 \| \nabla_{i,p} F(X^t) - \nabla_{i,p} F\big(\bar{Z}^t\big) \|_\infty^2, \forall t \geq 1$ while the last equality comes from the definition of η^t (see Line 9 of Algorithm 3).

From (D.24) and the fact that $0 \leq -\min_{x' \in \mathcal{X}} h(x') \leq M_{\text{tot}} \log(P/M_{\text{tot}})$ (see (C.3)), we conclude that

$$
\begin{aligned}
&\frac{M_{\text{tot}} \log(M_{\max}) - \min\limits_{x' \in \mathcal{X}} h(x') + \frac{4M_{\text{tot}}^2}{\sigma}}{\eta^{T+1}} \\
\leq &\left(M_{\text{tot}} \log\left(\frac{P M_{\max}}{M_{\text{tot}}} \right) + \frac{4M_{\text{tot}}^2}{\sigma} \right) \left[\sum_{t=1}^{T} \eta^{t+1} t^2 \| \nabla_{i,p} F\big(X^t\big) - \nabla_{i,p} F\big(\bar{Z}^t\big) \|_\infty^2 + 1 \right].
\end{aligned}
$$

Combining this with (D.23), we obtain:

$$
\begin{aligned}
&\hat{\Delta}_T(x) \\
\leq &M_{\text{tot}} \log\left(\frac{P M_{\max}}{M_{\text{tot}}} \right) \\
&+ \sum_{t=1}^{T} t^2 \left[\frac{\sigma \eta^{t+1}}{2} - \frac{1}{8\sigma \eta^{t+1} \beta^2} + \eta^{t+1} \left(M_{\text{tot}} \log\left(\frac{P M_{\max}}{M_{\text{tot}}} \right) + \frac{4M_{\text{tot}}^2}{\sigma} \right) \right] \| \nabla F(X^t) - \nabla F(\bar{Z}^t) \|_\infty^2 .
\end{aligned} \tag{D.25}
$$

Importantly, let us denote $T_0 = \max\left\{ s \in \{1,\dots,T\} : \eta^{s+1} \geq \frac{1}{2\beta} \frac{1}{\sqrt{\sigma^2 + 8M_{\text{tot}}^2 + 2\sigma M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)}} \right\}$, then for any $t \geq T_0$,[4] we have $\eta^{t+1} < \frac{1}{2\beta} \frac{1}{\sqrt{\sigma^2 + 8M_{\text{tot}}^2 + 2\sigma M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)}}$ and thus,

$$\frac{\sigma\eta^{t+1}}{2} - \frac{1}{8\sigma\eta^{t+1}\beta^2} + \eta^{t+1}\left(M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{4M_{\text{tot}}^2}{\sigma}\right) < 0. \tag{D.26}$$

As a consequence, we deduce from (D.25) that:

$$\begin{aligned}
&\hat{\Delta}_T(x) \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sum_{t=1}^{T_0} t^2 \left[\frac{\sigma\eta^{t+1}}{2} - \frac{1}{8\sigma\eta^{t+1}\beta^2} + \eta^{t+1}\left(M_{\text{tot}} \log\left(\frac{P}{M_{\text{tot}}}\right) + \frac{4M_{\text{tot}}^2}{\sigma}\right)\right] \|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|_\infty^2 \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sum_{t=1}^{T_0} t^2 \left[\frac{\sigma}{2} + M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{4M_{\text{tot}}^2}{\sigma}\right] \eta^{t+1} \|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|_\infty^2 \qquad \text{(D.27)} \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \left[\frac{\sigma}{2} + M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{4M_{\text{tot}}^2}{\sigma}\right] \sum_{t=1}^{T_0} \frac{t^2 \|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|_\infty^2}{\sqrt{1 + \sum_{s=1}^t s^2 \|\nabla F(X^s) - \nabla F(\bar{Z}^s)\|_\infty^2}} \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \left[\frac{\sigma}{2} + M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{4M_{\text{tot}}^2}{\sigma}\right] 2\sqrt{1 + \sum_{t=1}^{T_0} t^2 \|\nabla F(X^t) - \nabla F(\bar{Z}^t)\|_\infty^2} \\
& \text{(from Lemma D.1)} \\
= & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \left[\sigma + 2M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{8M_{\text{tot}}^2}{\sigma}\right] \frac{1}{\eta^{T_0+1}} \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \left[\sigma + 2M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{8M_{\text{tot}}^2}{\sigma}\right] 2\beta\sqrt{\sigma^2 + 8M_{\text{tot}}^2 + 2\sigma M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)} \\
& \text{(due to the definition of } T_0\text{)} \\
= & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{2\beta}{\sigma}\left[\sigma^2 + 8M_{\text{tot}}^2 + 2\sigma M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)\right]^{\frac{3}{2}}. \qquad \text{(D.28)}
\end{aligned}$$

Recall $\sigma = NM_{\max}$ (it is the strongly-convexity constant of h) and replace it into (D.28), we conclude that:

$$\begin{aligned}
&\hat{\Delta}_T(x) \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{2\beta}{NM_{\max}}\left[(NM_{\max})^2 + 8M_{\text{tot}}^2 + 2NM_{\max}M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)\right]^{\frac{3}{2}} \\
\leq & M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 2\beta(NM_{\max})^2\left[9 + 2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)\right]^{\frac{3}{2}} \\
& \text{(since } M_{\text{tot}} = \textstyle\sum_{i\in\mathcal{N}} M_i \leq NM_{\max}\text{)}.
\end{aligned}$$

Finally, we conclude the proof by combining (D.28) (applied for $x = x^*$ where x^* is an equilibrium normalized flow) and Lemma D.3 (note that in this static regime, we can omit the expectation in (D.3)) to obtain

$$F(X^T) - F(x^*) \leq \frac{2\,\hat{\Delta}_T(x^*)}{T^2} \leq \frac{2M_{\text{tot}} \log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 4\beta(NM_{\max})^2\left[9 + 2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)\right]^{\frac{3}{2}}}{T^2}.$$

To simplify the notation, we denote $A := 2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 13 > 2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 9$ (this term will appear again in the result of the stochastic case (cf. Appendix D.3)), then rewrite the above

[4] Note also that $(\eta^t)_{t\in\mathbb{N}}$ is a non-increasing sequence.

inequality as follows:

$$F(X^T) - F(x^*) < \frac{2M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 4\beta(NM_{\max})^2 A^{\frac{3}{2}}}{T^2}.$$

This concludes the proof of (6b). □

D.3. Proof of (6a): Theorem 3 in stochastic regime. In this section, we focus on the stochastic setting. The proof of (6a) is very similar to that of (6b); however, a major difference is that unlike the previous section, we now work with $c(\cdot, \omega)$ instead of $\nabla F(\cdot)$. Due to this reason, we focus on the difference between these terms. First, to simplify the notation, let us define the following random variables: $\bar{V}^t := c(\bar{Z}^t, \omega^t)$ and $V^t := c(X^t, \omega^t)$ for any $t = 1, 2, \ldots$. Since ω^t are randomly generated from i.i.d., we can rewrite:

$$\bar{V}^t = \nabla F(\bar{Z}^t) + \bar{U}^t$$
$$\text{and } V^t = \nabla F(X^t) + U^t,$$

where $\bar{U}^t$ and U^t are random variables satisfying:

$$\mathbb{E}\left[\bar{U}^t \middle| X^{t-1}, \bar{Z}^{t-1}, \omega^{t-1}, \ldots, X^1, \bar{Z}^1, \omega^1\right] = \mathbb{E}\left[U^t \middle| X^{t-1}, \bar{Z}^{t-1}, \omega^{t-1}, \ldots, X^1, \bar{Z}^1, \omega^1\right] = \mathbf{0}, \tag{D.29}$$

$$\mathbb{E}\left[\|\bar{U}^t\|_\infty^2\right] \leq \theta^2 \text{ and } \mathbb{E}\left[\|U^t\|_\infty^2\right] \leq \theta^2 \text{ where } \theta = 2KH. \tag{D.30}$$

Here, (D.29) comes from (A.1) and (D.30) comes from the fact that $\|\bar{U}^t\|_\infty \leq \|\nabla F(\bar{Z}^t)\|_\infty + \|c(\bar{Z}^t, \omega^t)\|_\infty = \max_{p\in\mathcal{P}}\left|C_p(\bar{Z}^t)\right| + \max_{p\in\mathcal{P}}\left|c_p(\bar{Z}^t, \omega^t)\right| \leq 2KH$ due to Assumption 1; the inequality corresponding to U^t is proven similarly.

Similar to the proof of (6b) in the previous section, to prove (6a), we focus on Lemma D.3 and look for an upper-bound of $\hat{\Delta}_T(x^*)$. We proceed the proof in the 4 steps as follows.

Step 1: Upper-bounding the term $\sum_{t=1}^T t\left\langle V^t, Z^{t+\frac{1}{2}} - x\right\rangle$.

First, following the steps leading to (D.12) and replacing $\nabla F(\bar{Z}^t)$ by $\bar{V}^t$ and $\nabla F(X^t)$ by V^t, we obtain the following inequality for any $x \in \mathcal{X}$:

$$\begin{aligned}
&\sum_{t=1}^T t\left\langle V^t, Z^{t+\frac{1}{2}} - x\right\rangle \\
\leq &\sum_{t=1}^T \left[\frac{1}{\eta^t}\mathcal{F}(x, \eta^t Y^t) - \frac{1}{\eta^t}\mathcal{F}(x, \eta^t Y^{t+1})\right] \\
&\quad + \left[\sum_{t=1}^T t\left\langle V^t - \bar{V}^t, \mathcal{Q}^t - Z^{t+\frac{1}{2}}\right\rangle - \sum_{t=1}^T \frac{1}{\eta^t}\mathrm{KL}\left(\mathcal{Q}^t \middle\| Z^{t+\frac{1}{2}}\right)\right] - \sum_{t=1}^T \frac{1}{\eta^t}\mathrm{KL}\left(Z^{t+\frac{1}{2}} \middle\| Z^t\right).
\end{aligned} \tag{D.31}$$

Similarly, by following Step 3 of Appendix D.2 (leading to (D.18)) and replacing $\nabla F(\bar{Z}^t)$ by $\bar{V}^t$ and $\nabla F(X^t)$ by V^t, we can also prove that:

$$\begin{aligned}
&\sum_{t=1}^T t\left\langle V^t - \bar{V}^t, \mathcal{Q}^t - Z^{t+\frac{1}{2}}\right\rangle - \sum_{t=1}^T \frac{1}{\eta^t}\mathrm{KL}\left(\mathcal{Q}^t \middle\| Z^{t+\frac{1}{2}}\right) \\
\leq &\frac{\sigma}{2}\sum_{t=1}^T t^2\eta^{t+1}\|V^t - \bar{V}^t\|_\infty^2 + \frac{2M_{\text{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}} - 1\right).
\end{aligned} \tag{D.32}$$

Combing (D.15), (D.19) and (D.32) into (D.31), and note again that $-\min_{x'\in\mathcal{X}} h(x') \leq M_{\text{tot}}\log(P/M_{\text{tot}})$, we have:

$$\sum_{t=1}^T t\left\langle V^t, Z^{t+\frac{1}{2}} - x\right\rangle \leq \frac{M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)}{\eta^{T+1}} + \frac{\sigma}{2}\sum_{t=1}^T t^2\eta^{t+1}\|V^t - \bar{V}^t\|_\infty^2 + \frac{2M_{\text{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}} - 1\right)$$

$$+\frac{2M_{\text{tot}}^2}{\sigma}\left(\frac{1}{\eta^{T+1}}-1\right)-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2. \tag{D.33}$$

Now, similar to the way we derived (D.24), from the definition of η^T (see Line 9-Algorithm 3) and apply Lemma D.1, we have

$$\frac{1}{\eta^{T+1}}=\sqrt{1+\psi^T}\leq\sum_{t=1}^{T}t^2\eta^{t+1}\|V^t-\bar{V}^t\|_\infty^2+1. \tag{D.34}$$

Therefore, we can rewrite (D.33) as follows:

$$\begin{aligned}&\sum_{t=1}^{T}t\left\langle V^t,Z^{t+\frac{1}{2}}-x\right\rangle\\&\leq M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{4M_{\text{tot}}^2}{\sigma}\right]\sum_{t=1}^{T}t^2\eta^{t+1}\|V^t-\bar{V}^t\|_\infty^2\\&\qquad-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2.\end{aligned} \tag{D.35}$$

Now, we set:

$H^t=\min\{\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2,\|V^t-\bar{V}^t\|_\infty^2\}$ and $\xi^t=\left[V^t-\bar{V}^t\right]-\left[\nabla F(X^t)-\nabla F(\bar{Z}^t)\right]$.

In the next steps of this proof, we aim to find upper-bounds of the second and third term in the right-hand-side of (D.35) in terms of H^t and ξ^t.

Step 2: Upper-bounding of the second term in the right-hand-size of (D.35).

First, we introduce the respective auxiliary learning rate:

$$\tilde{\eta}^t=\frac{1}{\sqrt{1+2\sum_{s=1}^{t-1}s^2H^s}}. \tag{D.36}$$

By the definition of $\tilde{\eta}^t$, we have $\frac{1}{\tilde{\eta}^t}\leq\frac{1}{\eta^t},\forall t$ and hence:

$$-\frac{1}{\eta^{t+1}}\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\leq-\frac{1}{\tilde{\eta}^{t+1}}\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\leq-\frac{1}{\tilde{\eta}^{t+1}}H^t. \tag{D.37}$$

Moreover, from the definition of ξ^t, we have:

$$\begin{aligned}&\|V^t-\bar{V}^t\|_\infty^2\leq2\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2+2\|\xi^t\|_\infty^2\\\Rightarrow&\|V^t-\bar{V}^t\|_\infty^2-\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\leq\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2+2\|\xi^t\|_\infty^2.\end{aligned} \tag{D.38}$$

which implies that

$$\begin{aligned}\|V^t-\bar{V}^t\|_\infty^2&\leq H^t+\left[\|V^t-\bar{V}^t\|_\infty^2-\min\{\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2,\|V^t-\bar{V}^t\|_\infty^2\}\right]\\&\leq H^t+\max\{0,\|V^t-\bar{V}^t\|_\infty^2-\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\}\\&\leq H^t+H^t+2\|\xi^t\|_\infty^2 \qquad (\text{D.39})\\&=2H^t+2\|\xi^t\|_\infty^2. \qquad (\text{D.40})\end{aligned}$$

Here, Inequality (D.39) comes from the fact that:

- if $\|V^t-\bar{V}^t\|_\infty^2\leq\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2$ then the result is trivially deduced from the fact that $\max\{0,\|V^t-\bar{V}^t\|_\infty^2-\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\}=0$,
- if $\|V^t-\bar{V}^t\|_\infty^2>\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2$ then $\max\{0,\|V^t-\bar{V}^t\|_\infty^2-\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2\}=\|V^t-\bar{V}^t\|_\infty^2-\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2$ and we can apply (D.38) then replace $H^t=\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\|_\infty^2$.

Now, we have

$$\begin{aligned}
&\sum_{t=1}^{T} t^2\eta^{t+1}\|V^t-\bar{V}^t\|_\infty^2\\
=&\sum_{t=1}^{T}\frac{t^2\|V^t-\bar{V}^t\|_\infty^2}{\sqrt{1+\sum_{s=1}^{t}\|V^s-\bar{V}^s\|_\infty^2}} && \text{(due to the definition of } \eta^{t+1})\\
\le&2\sqrt{1+\sum_{t=1}^{T}t^2\|V^t-\bar{V}^t\|_\infty^2} && \text{(by applying Lemma D.1)}\\
\le&2\sqrt{1+2\sum_{t=1}^{T}t^2H^t+2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2} && \text{(due to (D.40))}\\
\le&2\sqrt{1+2\sum_{t=1}^{T}t^2H^t}+2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}\\
\le&4\sum_{t=1}^{T}\frac{t^2H^t}{\sqrt{1+2\sum_{i=1}^{t}\alpha_i^2H^t}}+2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2} && \text{(by applying Lemma D.1)}\\
\le&4\sum_{t=1}^{T}t^2\tilde{\eta}^{t+1}H^t+2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}. && \text{(D.41)}
\end{aligned}$$

Here, the last inequality comes from the definition of $\tilde{\eta}$ in (D.36).

Step 3: Upper-bounding of the third term in the right-hand-size of (D.35).

From the fact that $\frac{1}{\tilde{\eta}^t}\le\frac{1}{\eta^t}$, we have

$$\begin{aligned}
\sum_{t=1}^{T}\frac{1}{\tilde{\eta}^{t+1}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2\le&\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2\\
=&\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2+\sum_{t=1}^{T}\frac{1}{\eta^{t}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2-\sum_{t=1}^{T}\frac{1}{\eta^{t}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2\\
=&\sum_{t=1}^{T}\left(\frac{1}{\eta^{t+1}}-\frac{1}{\eta^t}\right)\left\|Z^{t+\frac{1}{2}}-Z^t\right\|_1^2+\sum_{t=1}^{T}\frac{1}{\eta^{t}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2\\
\le&\left(\frac{1}{\eta^{T+1}}-1\right)\cdot 4M_{\text{tot}}^2+\sum_{t=1}^{T}\frac{1}{\eta^{t}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2. \qquad \text{(D.42)}
\end{aligned}$$

Here, the last inequality comes from the fact that $\|x-x'\|_1\le 2M_{\text{tot}},\forall x,x'\in\mathcal{X}$ and the use of telescopic-sum on the first term (note that $\eta^1=1$). Now, from this, we can deduce that:

$$\begin{aligned}
&-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2\\
\le&-\frac{1}{2\sigma}\sum_{t=1}^{T}\frac{1}{\eta^t}\|Z^{t+\frac{1}{2}}-Z^t\|_1^2 && \text{(since } \eta^{t+1}\le\eta^t)\\
\le&-\frac{1}{2\sigma}\sum_{t=1}^{T}\frac{1}{\tilde{\eta}^{t+1}}\left\|Z^t-Z^{t+\frac{1}{2}}\right\|_1^2+\frac{4M_{\text{tot}}^2}{2\sigma}\left(\frac{1}{\eta^{T+1}}-1\right) && \text{(due to (D.42))}\\
\le&-\frac{1}{2\sigma}\sum_{t=1}^{T}\frac{1}{\tilde{\eta}^{t+1}}\frac{(t+1)^2\left\|\nabla F(X^t)-\nabla F(\bar{Z}^t)\right\|_\infty^2}{4\beta^2}+\frac{4M_{\text{tot}}^2}{2\sigma}\left(\frac{1}{\eta^{T+1}}-1\right) && \text{(due to (D.20))}
\end{aligned}$$

$$\leq -\frac{1}{8\beta^2\sigma}\sum_{t=1}^{T}\frac{t^2 H^t}{\tilde{\eta}^{t+1}} + \frac{4M_{\text{tot}}^2}{2\sigma}\left(\frac{1}{\eta^{T+1}} - 1\right). \tag{D.43}$$

Replacing $\nabla F(\bar{Z}^t)$ by $\bar{V}^t$ and $\nabla F(X^t)$ by V^t in the step leading to (D.24), we obtain that

$$\begin{aligned}\frac{1}{\eta^{T+1}} \leq &\sum_{t=1}^{T}\eta^{t+1}t^2\|V^t - \bar{V}^t\|_\infty^2 + 1\\ \leq &4\sum_{t=1}^{T}\tilde{\eta}^{t+1}t^2 H^t + 2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2} + 1. && \text{(due to (D.41))}\end{aligned}$$

Combining this with (D.43), we obtain:

$$-\sum_{t=1}^{T}\frac{1}{\eta^{t+1}}\frac{1}{2\sigma}\|Z^{t+\frac{1}{2}} - Z^t\|_1^2 \leq -\frac{1}{8\beta^2\sigma}\sum_{t=1}^{T}\frac{t^2H^t}{\tilde{\eta}^{t+1}} + \frac{4M_{\text{tot}}^2}{2\sigma}\left(4\sum_{t=1}^{T}t^2\tilde{\eta}^{t+1}H^t + 2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}\right). \tag{D.44}$$

<u>*Step 4:*</u> *Conclusion.*

Combine (D.41) and (D.44) into (D.35), we have

$$\begin{aligned}
&\sum_{t=1}^{T}t\left\langle V^t, Z^{t+\frac{1}{2}} - x\right\rangle\\
\leq &M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{4M_{\text{tot}}^2}{\sigma}\right]\left[4\sum_{t=1}^{T}t^2\tilde{\eta}^{t+1}H^t + 2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}\right]\\
&+\left[-\frac{1}{8\beta^2\sigma}\sum_{t=1}^{T}\frac{t^2H^t}{\tilde{\eta}^{t+1}} + \frac{4M_{\text{tot}}^2}{2\sigma}\left(4\sum_{t=1}^{T}t^2\tilde{\eta}^{t+1}H^t + 2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}\right)\right]\\
= &M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sum_{t=1}^{T}t^2H^t\left\{4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right]\tilde{\eta}^{t+1} - \frac{1}{8\beta^2\sigma}\frac{1}{\tilde{\eta}^{t+1}}\right\}\\
&+\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right]2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}\\
\leq &M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sum_{t=1}^{T}t^2H^t\left\{4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right]\tilde{\eta}^{t+1} - \frac{1}{8\beta^2\sigma}\frac{1}{\tilde{\eta}^{t+1}}\right\}\\
&+\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right]2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2}.
\end{aligned} \tag{D.45}$$

Now, let us denote $T_0 = \max\left\{1 \leq t \leq T : \tilde{\eta}^{t+1} \geq \frac{1}{4\beta\sqrt{2\sigma M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sigma^2 + 12M_{\text{tot}}^2}}\right\}$. Then for any $t \geq T_0$, we have that $\tilde{\eta}^{t+1} < \frac{1}{4\beta\sqrt{2\sigma M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \sigma^2 + 12M_{\text{tot}}^2}}$ and thus, we can trivially justify that $4\left(M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right)\tilde{\eta}^{t+1} - \frac{1}{8\beta^2\sigma}\frac{1}{\tilde{\eta}^{t+1}} < 0$. Therefore, we can upper-bound the second term in (D.45) as follows:

$$\sum_{t=1}^{T}t^2H^t\left\{4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + \frac{\sigma}{2} + \frac{6M_{\text{tot}}^2}{\sigma}\right]\tilde{\eta}^{t+1} - \frac{1}{8\beta^2\sigma}\frac{1}{\tilde{\eta}^{t+1}}\right\} \tag{D.46}$$

$$
\begin{aligned}
&\leq \sum_{t=1}^{T_0} t^2 H^t \left\{4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]\tilde{\eta}^{t+1} - \frac{1}{8\beta^2\sigma}\frac{1}{\tilde{\eta}^{t+1}}\right\}\\
&\leq \sum_{t=1}^{T_0} t^2 H^t \left\{4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]\tilde{\eta}^{t+1}\right\}\\
&=4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]\sum_{t=1}^{T_0}\frac{t^2H^t}{\sqrt{1+2\sum_{s=1}^{t-1}s^2H^s}} && \text{(by definition of } \tilde{\eta}^{t+1})\\
&\leq 4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]\sum_{t=1}^{T_0}\frac{1+t^2H^t}{\sqrt{1+2\sum_{s=1}^{t-1}s^2H^s}}\\
&\leq 4\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]2\sqrt{1+2\sum_{t=1}^{T_0}t^2H^t} && \text{(by Lemma D.1)}\\
&=\frac{4}{\tilde{\eta}^{T_0}}\left[2M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\sigma+\frac{12M_{\text{tot}}^2}{\sigma}\right] && \text{(by definition of } \tilde{\eta}^{T_0})\\
&\leq \frac{16\beta}{\sigma}\left[2\sigma M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\sigma^2+12M_{\text{tot}}^2\right]^{\frac{3}{2}}. && \text{(D.47)}
\end{aligned}
$$

Here, the last inequality comes from the definition of T_0.

Now, recall that $V^t = \nabla F(X^t) + U^t$ and combine (D.47) with (D.45), we have:

$$
\begin{aligned}
&\sum_{t=1}^{T} t\left\langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x\right\rangle\\
=&\sum_{t=1}^{T} t\left\langle V^t, Z^{t+\frac{1}{2}} - x\right\rangle - \sum_{t=1}^{T} t\left\langle U^t, Z^{t+\frac{1}{2}} - x\right\rangle\\
\leq& M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{16\beta}{\sigma}\left[2\sigma M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\sigma^2+12M_{\text{tot}}^2\right]^{\frac{3}{2}}\\
&\quad+\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]2\sqrt{2\sum_{t=1}^{T}t^2\|\xi_t\|_\infty^2} - \sum_{t=1}^{T} t\left\langle U^t, Z^{t+\frac{1}{2}} - x\right\rangle.
\end{aligned}
\tag{D.48}
$$

Now, let us define $\hat{u}^t := \mathbb{E}\big[U^t \big| X^{t-1}, \bar{Z}^{t-1}, \omega^{t-1}, \ldots, X^1, \bar{Z}^1\omega^1\big]$. From the law of total expectation, we obtain that $\mathbb{E}\Big[t\Big\langle U^t, Z^{t+\frac{1}{2}} - x\Big\rangle\Big] = \mathbb{E}\Big[t\Big\langle \hat{u}^t, Z^{t+\frac{1}{2}} - x\Big\rangle\Big]$. Combine this with (D.29), we deduce that

$$
\mathbb{E}\Big[t\Big\langle U^t, Z^{t+\frac{1}{2}} - x\Big\rangle\Big] = 0, \forall t.
$$

Moreover, from (D.30), $\mathbb{E}\big[\|\xi_t\|_\infty^2\big] = \mathbb{E}\big[\|U^t - \bar{U}^t\|_\infty^2\big] \leq 2\,\mathbb{E}\big[\|U^t\|_\infty^2\big] + 2\,\mathbb{E}\big[\|\bar{U}^t\|_\infty^2\big] \leq 4\theta^2$. Therefore,

$$
\sum_{t=1}^{T} t^2\|\xi_t\|_\infty^2 \leq 4\theta^2 \sum_{t=1}^{T} T^2 = 4\theta^2 T^3.
$$

Due to these reasons, when we take expectation on both sides of (D.48), we have,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T} t\left\langle \nabla F(X^t), Z^{t+\frac{1}{2}} - x\right\rangle\right] && \text{(D.49)}\\
\leq& M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{16\beta}{\sigma}\left[2\sigma M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\sigma^2+12M_{\text{tot}}^2\right]^{\frac{3}{2}}
\end{aligned}
$$

$$+\left[M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+\frac{\sigma}{2}+\frac{6M_{\text{tot}}^2}{\sigma}\right]2\sqrt{8T^3\theta^2}. \tag{D.50}$$

Finally, recall the notations $\sigma = NM_{\max}$ and $\theta = 2KH$; in addition, recall the fact that $M_{\text{tot}} = \sum_{i\in\mathcal{N}} M_i \leq NM_{\max}$, we apply (D.50) with $x = x^*$ then use Lemma D.3 to obtain:

$$\begin{aligned}
&\mathbb{E}[F(X^t) - F(x^*)] \\
\leq &\frac{2\,\mathbb{E}\left[\hat{\Delta}_T(x^*)\right]}{T^2} \\
= &\frac{2M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+16\beta(NM_{\max})^2\left(2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+13\right)^{\frac{3}{2}}}{T^2}+\frac{2\sqrt{2}NM_{\max}\left(2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+13\right)}{\sqrt{T}}.
\end{aligned}$$

By denoting $A := 2\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right) + 13$ we simplify the notation in the above inequality as follows:

$$\mathbb{E}[F(X^t) - F(x^*)] \leq \frac{2M_{\text{tot}}\log\left(\frac{PM_{\max}}{M_{\text{tot}}}\right)+16\beta(NM_{\max})^2A^{\frac{3}{2}}}{T^2}+\frac{2\sqrt{2}NM_{\max}A}{\sqrt{T}}.$$

This concludes the proof. $\square$

E Supplementary numerical experiments

The code of our numerical experiments are available at `https://github.com/Anonymous-GT11/NumExperiments_AdaptiveRouting`.

For illustrating the convergence properties of the ExpWeight, AcceleWeight and AdaWeight algorithms, we present below more experimental results, in addition to the ones discussed in Section 5. Particularly, in Appendix E.1, we report the experiments conducted in a stochastic setting and in Appendix E.2, we report the experiments done in a static environment.

E.1. Performance in the stochastic regime. In this section, we work with the stochastic regime as described in Section 5, i.e., in a game with stochastically perturbed observations. Theorem 3 guarantees that AdaWeight converges at an $\mathcal{O}(1/\sqrt{T})$ rate. Since Theorem 3 is our core result, we re-justify it by plotting out the evolution of the term $\sqrt{T}\cdot\Delta_{\text{AdaWeight}}(T)$ (defined in Section 5). We also plot out $\sqrt{T}\cdot\Delta_{\text{ExpWeight}}(T)$, $\sqrt{T}\cdot\Delta_{\text{AcceleWeight}}(T)$ for the sake of comparison and report all of these results in Figure 2.

Figure 2 shows that $\sqrt{T}\cdot\Delta_{\text{AdaWeight}}(T)$ approaches a horizontal line as T increases. This re-confirms that the convergence rate of AdaWeight is $\mathcal{O}(1/\sqrt{T})$. On the other hand, the divergence of $\sqrt{T}\cdot\Delta_{\text{AcceleWeight}}(T)$ confirms once again that AcceleWeight does not converge in this setting.

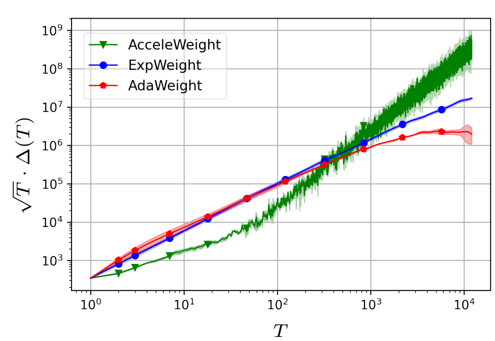
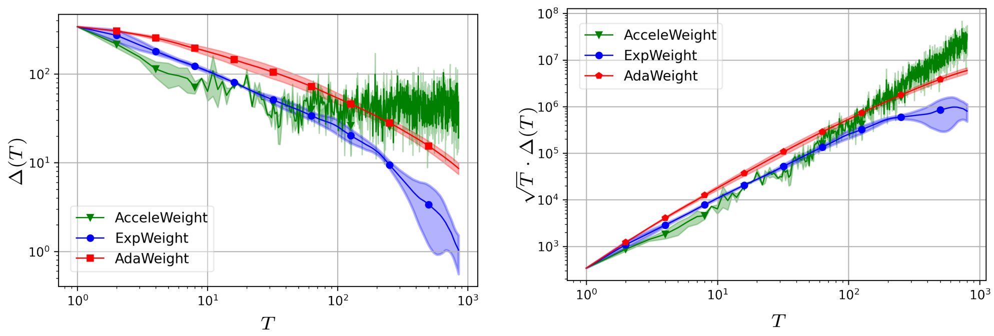


Figure 2: The order of the convergence rates in a stochastic environment.



(a) Evolution of $\Delta(T)$ **(b)** Evolution of $\sqrt{T} \cdot \Delta(T)$

Figure 3: The convergence speed of EXPWEIGHT, ACCELEWEIGHT and ADAWEIGHT in the stochastic regime *with high-level of noises*.

In the experiment reported in Figure 1 and Figure 2, the induced costs on each edge e at each time epoch t is perturbed by a noise ω_e^T that is drawn independently from the normal distribution $\mathcal{N}(0, 10)$. Now, to study the effect of noises on the performance of the algorithms, we also re-run this experiment in a "noisier" setting: the noise ω_e^T is generated from $\mathcal{N}(0, 50)$ for any T and any e. We report the results in Figure 3. At a high-level, we see that ADAWEIGHT is more stable in the presence of noise with high-variance than EXPWEIGHT (or ACCELEWEIGHT).

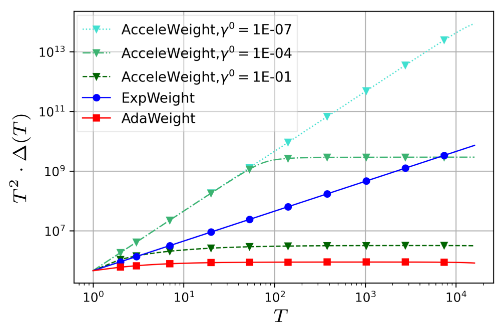
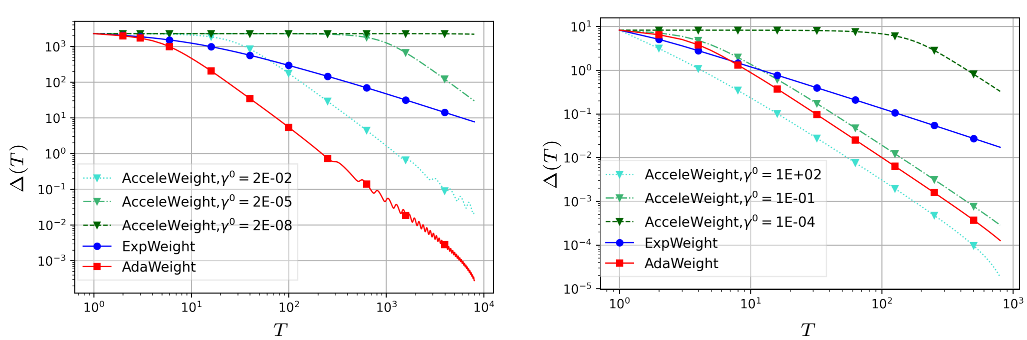


Figure 4: The order of the convergence rates in the static regime.



(a) Eastern-Massachusetts network: 74 vertices, 258 edges; we choose $N = 50$ O/D pairs, each of which is assigned with 10 shortest paths; the total paths in used is $P = 500$.

(b) Austin network: 7388 vertices, 18961 edges; we choose $N = 5$ O/D pairs which is assigned with 10 shortest paths; the total paths in used is $P = 50$.

Figure 5: The convergence speed of EXPWEIGHT, ACCELEWEIGHT and ADAWEIGHT algorithms in the static regime in Eastern-Massachusetts and Austin networks.

E.2. Performance in the static regime. In this section, we work with the static setting. SiouxFalls is a famous network that is often used as a benchmark in the field and we reported, in Section 5, the performance of the ExpWeight, AcceleWeight and AdaWeight algorithms on this network with data extracted from the real data set [14]. To support further this numerical experiments and to re-justify the theoretical convergence rates guaranteed in Theorem 1 and Theorem 3, we plot out $T^2 \cdot \Delta_{\textsc{ExpWeight}}(T)$, $T^2 \cdot \Delta_{\textsc{AcceleWeight}}(T)$ and $T^2 \cdot \Delta_{\textsc{AdaWeight}}(T)$. These results are reported in Figure 4. It shows that $T^2 \cdot \Delta_{\textsc{AcceleWeight}}(T)$ and $T^2 \cdot \Delta_{\textsc{AdaWeight}}(T)$ approach horizontal lines as T increases; this confirms that AcceleWeight and AdaWeight converge at an $\mathcal{O}(1/T^2)$ rate which are in consistent with Theorem 2 and Theorem 3. On the other hand, $T^2 \cdot \Delta_{\textsc{ExpWeight}}(T)$ increases (almost) linearly; this demonstrates that the ExpWeight algorithm fails to achieve an $\mathcal{O}(1/T^2)$-convergence rate in this setting. The slow-convergence of AcceleWeight with bad-tuned step-sizes (e.g., when $\gamma^0 = 1\mathrm{e}{-07}$) is also re-confirmed in Figure 4

Finally, we show in Figure 5 the experiments results with respected to several other networks in the Transportation Networks data set (provided by [14]) that are larger than SiouxFalls. We observe that, at a high level, the performance of ExpWeight, AcceleWeight and AdaWeight in these networks is similar to that in the SiouxFalls network.