Different Monotonicity Definitions in stochastic modelling

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Plan

Introduction

Concept of monotonicity

- Lower and Upper bounding
- Coupling of trajectories (perfect Sampling) $\longrightarrow$ Reduce the complexity.

Different notions of monotonicity

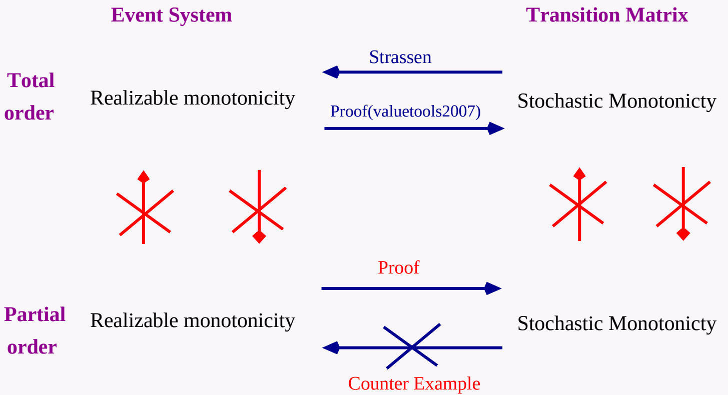
- Order on trajectories(Event monotonicity).
- Order on distribution (Stochastic monotonicity).

Monotonicity concepts depends on the relation order considerd on the state space

- Partial order and total order

Main results

Relations between monotonicity concepts in Total and Partial Orders



Markovian Discrete Event Systems(MDES)

MDES

are dynamic systems evolving asynchronously and interacting at irregular instants called *event epochs*. They are defined by:

- a state space $\mathcal{X}$
- a set of events $\mathcal{E}$
- a set of probability measures $\mathcal{P}$
- transition function Φ
- $\mathbb{P}(e) \in \mathcal{P}$ denotes the occurrence probability

Event

An event e is an application defined on $\mathcal{X}$, that associates to each state $x \in \mathcal{X}$ a new state $y \in \mathcal{X}$.

Markovian Discrete Event Systems(MDES)

Transition function with events

Let X_i be the state of the system at the i^{th} event occurrence time. The transition function $\Phi : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$,

$$X_{n+1} = \Phi(X_n, e_{n+1})$$

Φ must to obey to the following property to generate **P**:

$$p_{ij} = \mathbb{P}(\phi(x_i, E) = x_j) = \sum_{e|\Phi(x_i,e)=x_j} \mathbb{P}(E = e)$$

Discrete Time Markov Chains (DTMC)

DTMC

$\{X_0, X_1, ..., X_{n+1}, ...\}$: stochastic process observed at points $\{0, 1, ..., n+1\}$.

- It constitutes a DTMC if:

$\forall n \in \mathbb{N}$ and $\forall x_i \in \mathcal{X}$:

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, ..., X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n).$$

The one-step transition probability p_{ij} are given in a non-negative, stochastic transition matrix $\mathbf{P}$:

$$\mathbf{P} = \mathbf{P}^{(1)} = [p_{ij}] \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Discrete Time Markov Chains (DTMC)

A probability transition matrix **P**, can be described by a transition function

Transition function in a DTMC

$\Phi : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, is a transition function for **P** where :
U is a random variable taking values in an arbitrary probability space $\mathcal{U}$, such that:

- $\forall x, y \in \mathcal{X} \; : \; \mathbb{P}(\Phi(x, U) = y) = p_{xy}$
- $X_{n+1} = \Phi(X_n, U_{n+1})$

Stochastic ordering

Stochastic ordering

Let T and V be two discrete random variables and Γ an increasing set defined on $\mathcal{X}$

$$T \preceq_{st} V \Leftrightarrow \sum_{x\in\Gamma} \mathbb{P}(T = x) \leq \sum_{x\in\Gamma} \mathbb{P}(V = x), \ \ \forall\Gamma$$

Definition (Increasing set)

Any subset Γ *of* $\mathcal{X}$ *is called an increasing set if* $x \preceq y$ *and* $x \in \Gamma$ *implies* $y \in \Gamma$.

Stochastic ordering

Example

- Let $(\mathcal{X}, \preceq)$ be a partial ordered state space, $\mathcal{X} = \{a, b, c, d\}$.
- $a \preceq b \preceq d$, and $a \preceq c \preceq d$,
- Increasing sets:Γ_1={a,b,c,d}, Γ_2={b,c,d}, Γ_3={b,d}, Γ_4={c,d}, Γ_5={d}.
- $V1$=(0.4,0.2,0.1,0.3)
  $V2$=(0.2,0.1,0.3,0.4)

  $$V1 \preceq_{st} V2$$

  On a :

  - For Γ_1={a,b,c,d}: $0.4 + 0.2 + 0.1 + 0.3 \leq 0.2 + 0.1 + 0.3 + 0.4$
  - For Γ_2={b,c,d}: $0.2 + 0.1 + 0.3 \leq 0.1 + 0.3 + 0.4$
  - For Γ_3={b,d}: $0.2 + 0.3 \leq 0.1 + 0.4$
  - For Γ_4={c,d}: $0.1 + 0.3 \leq 0.3 + 0.4$
  - For Γ_5={d} :$0.3 \leq 0.4$

Stochastic monotonicity

Stochastic monotonicity

P a transition probability matrix of a time-homogeneous Markov chain $\{X_n, n \geq 0\}$ taking values in $\mathcal{X}$ endowed with relation order $\preceq$.
$\{X_n, n \geq 0\}$ is st-monotone if and only if,
$\forall (x, y) \mid x \preceq y$ and $\forall$ increasing set $\Gamma \in \mathcal{X}$

$$\sum_{z \in \Gamma} p_{xz} \leq \sum_{z \in \Gamma} p_{yz}$$

Realizable monotonicity

Realizable monotonicity

P a stochastic matrix defined on $\mathcal{X}$. **P** is realizable monotone, if there exists a transition function , such that Φ preserves the order relation. $\forall\ u \in \mathbf{U}$:

$$\textit{if } x \preceq y \textit{ then } \Phi(x,u) \preceq \Phi(y,u)$$

Event monotonicity

The model is event monotone, if the transition function by events preserves the order ie. $\forall\ e \in \mathcal{E}$

$$\forall (x,y) \in \mathcal{X}\ \ x \preceq y \Longrightarrow \Phi(x,e) \preceq \Phi(y,e)$$

A system is realizable monotone means that there exists a finite set of events $\mathcal{E}$ for which the system is event monotone

Monotonicity and perfect sampling

Principe

- Produce exact sampling of stationary distribution (Π) of a DTMC.
- One trajectory per state.
- The algorithm stops when all trajectories meet the same state coupling
  The evolution of the trajectories will be confused.

If the model is event monotone

- Run only trajectories from minimal and maximal states.
- All other trajectories are always between these trajectories.
- If there is coupling at time t so all the other trajectories have also coalesced.

▶ The tool PSI 2 was developed to implement this method of simulation (JM.Vincent).

Relations between monotonicity concepts

Total Order

$(\mathcal{X}, \mathcal{E})$ **: MDES** **P: Transition matrix**

$\exists \mathcal{E}: (\mathcal{X}, \mathcal{E})$ Monotone ← Strassen — **P**: Monotone

Total order

Relations between monotonicity concepts

Total Order

	$(\mathcal{X}, \mathcal{E})$: **MDES**		**P: Transition matrix**
Total order	$\exists \mathcal{E} : (\mathcal{X}, \mathcal{E})$ Monotone	← Strassen	**P**: Monotone
	$(\mathcal{X}, \mathcal{E})$ Monotone	Valuetools2007 →	$\mathbf{P}(\mathcal{E})$: Monotone

Relation between monotonicty concepts (Partial Order)

Partial Order

$(\mathcal{X}, \mathcal{E})$: MDES | **P: Transition matrix**

Total order

$\exists \mathcal{E} : (\mathcal{X}, \mathcal{E})$ Monotone ← Strassen — **P**: Monotone

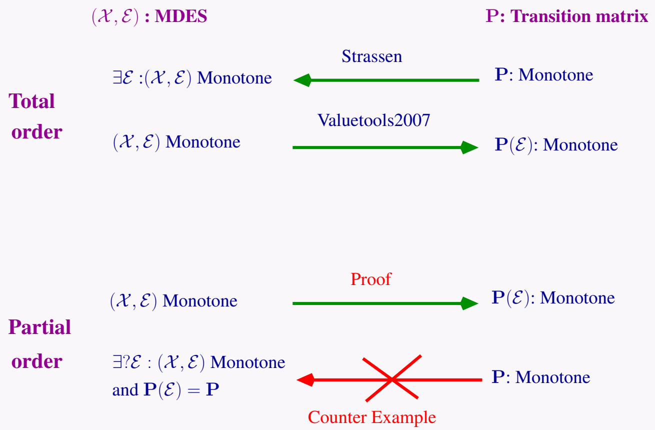
$(\mathcal{X}, \mathcal{E})$ Monotone — Valuetools2007 → **P**$(\mathcal{E})$: Monotone

Partial order

$(\mathcal{X}, \mathcal{E})$ Monotone — Proof → **P**$(\mathcal{E})$: Monotone

Relation between monotonicty concepts (Partial Order)

The reciprocal is not true



Relation between monotonicty concepts (Partial Order)

Counter Example

- $\mathcal{X} = \{a,\ b,\ c,\ d\}$, $a \preceq b \preceq d$ and $a \preceq c \preceq d$.
- **P** transition matrix in $\mathcal{X}$

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/6 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/2 & 0 & 1/6 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

	1/6	1/6	1/6	1/6	1/6	1/6
a	a			b	c	
b	a		b		d	
c	a			c	d	
d	b		c		d	

P is not realizable monotone. We have for $u \in [3/6, 4/6]$ $\Phi(a,u) = b$ is incomparable with $\Phi(c,u) = c$.

Relation between monotonicty concepts (Partial Order)

Proof

- $b \preceq d$ and $c \preceq d$
  - Transitions from states b, c, d to state d with probability 1/3 must be associated to the same interval u
- $a \preceq b$ and $a \preceq c$:
  - Transitions from a, c to a must be associated to the same interval, $e_u = 1/2$.
  - Transitions from a, b to a must be associated to the same interval, $e_u = 1/3$.
- For states b, and c it remains only an interval of $u_e = 1/3$ to assign .

	1/3	1/6	1/6	1/3
a	a	a	b	c
b	a	b	b	d
c	a	a	c	d
d	b	c		d

It is not possible to build a realizable monotone transition function for this matrix.

Relation between monotonicty concepts (Partial Order)

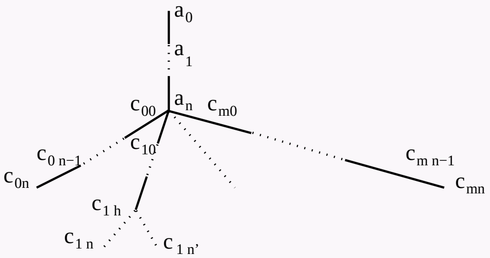
In partial orders

Define conditions on the matrix $\mathbf{P}$, that allows us to knew whether the corresponding system is realizable monotone.

Relation between monotonicty concepts (Partial Order)

Theorem

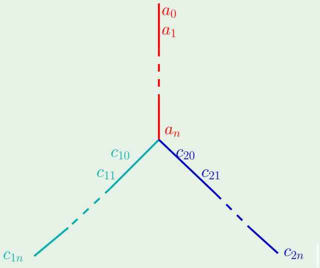
When the state space is partially ordered in a tree, if the system is stochastic monotone, then there exists a finite set of events $e_1, e_2, ..., e_n$, for which the system is event-monotone.



Define an algorithm that construct the monotone transition function Φ

Relation between monotonicty concepts (Partial Order)

- $A = \{a_1 \leq a_2 \leq ...a_n\}$: States comparable with all others.
- We consider two branches:
  - $C_1 = \{c_{10} \leq c_{11} \leq ..., c_{1n}\}$.
  - $C_2 = \{c_{20} \leq c_{21} \leq ..., c_{2n}\}$.



- For each branch C_i we find events which trigger transition to a state of C_i.
- Then we find events which trigger transition to a state of A.

<table>
<tr><th></th><th>U0</th><th colspan="2">U1</th><th>U2</th><th>U2</th><th>U2</th></tr>
<tr><td>A</td><td>A</td><td>A</td><td>C1</td><td colspan="2">A</td><td>C2</td></tr>
<tr><td>C1</td><td>A</td><td>A</td><td>C1</td><td>A</td><td colspan="2">C2</td></tr>
<tr><td>C2</td><td>A</td><td colspan="2">C1</td><td colspan="3">C2</td></tr>
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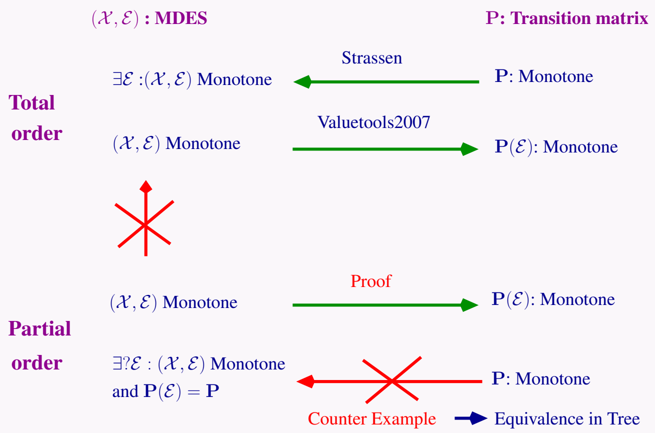
Realizable monotonicity in Partial Orders



Realizable monotonicity and Partial Orders

- Another way to reduce the complexicity $\Longrightarrow$ reduce the number of maximal and minimal states.
  - If the system is monotone according to a partial order $\preceq$, can we find a total order $\leqslant$ for which the system is monotone??.

Not possible with all partial orders.

Realizable monotonicity and Partial Orders

Counter example

- $\mathcal{X} = \{a,\ b,\ c,\ d\}$, $a \preceq b \preceq c$ and $a \preceq b \preceq d$.
- **P** transition matrix in $\mathcal{X}$

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/3 & 0 & 1/6 \\ 1/3 & 1/3 & 1/6 & 1/6 \\ 0 & 1/2 & 1/6 & 1/3 \\ 0 & 1/2 & 1/3 & 1/6 \end{pmatrix}$$

	1/6	1/6	1/6	1/6	1/6	1/6
a	a			b		d
b	a		b		c	d
c	b			c	d	
d	b			c		d

▶ Two possible orders:

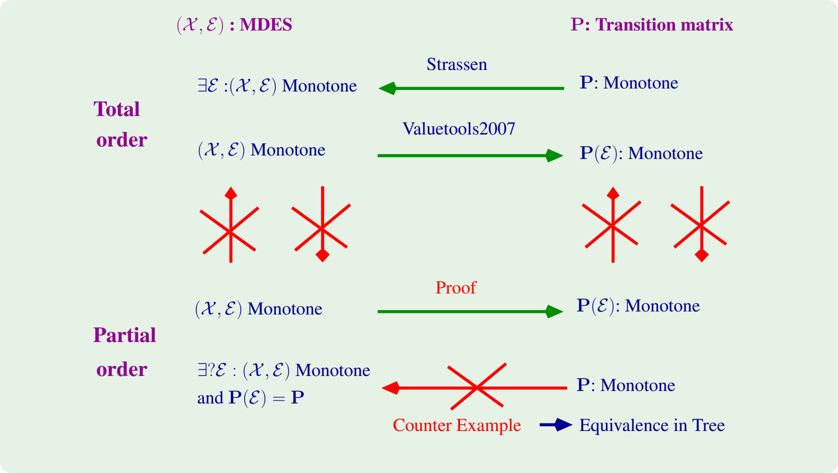
⊙ $a \leqslant b \leqslant c \leqslant d$

	1/6	1/6	1/6	1/6	1/6	1/6
a	a			b		d
b	a		b		c	d
c	b			c	d	d
d	b			c		d

⊙ $a \leqslant b \leqslant d \leqslant c$

	1/6	1/6	1/6	1/6	1/6	1/6
a	a			b		d
b	a		b		d	c
d	b			d	c	c
c	b			d		c

Monotonicity in partial and total order



- Total Order: Stochastic monotonicity $\Leftrightarrow$ Realizable monotonicity
- Partial Order:
  - Realizable monotonicity $\Rightarrow$ Stochastic monotonicity
  - Stochastic monotonicity $\nRightarrow$ Realizable monotonicity
- Monotonicity with order $\preceq$ $\nLeftrightarrow$ Monotonicity with order $\leqslant$

▶ Perspectives

- In the partial order : Find another conditions to move from the stochastic monotonicity to the realizable monotonicity
- implements