Binomial and other coefficients

Mathematics for Computer Science

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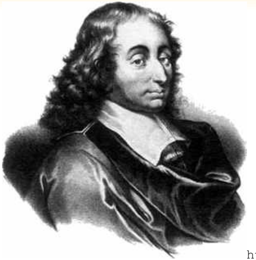
These notes are only the sketch of the lecture : the aim is to apply the basic counting techniques to the binomial coefficients and establish combinatorial equalities.

References : Concrete Mathematics : A Foundation for Computer Science *Ronald L. Graham, Donald E. Knuth and Oren Patashnik* Addison-Wesley 1989 (chapter 5)

Definition

$\binom{n}{k}$ is the number of ways to choose k elements among n elements



http://www-history.mcs.st-and.ac.uk/Biographies/Pascal.html

For all integers $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{1}$$

Hint : Prove it by a combinatorial argument *Hint : the number of sequences of k different elements among n is $n(n-1)\cdots(n-k+1)$ and the number of orderings of a set of size k is $k!$.*

Basic properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

Prove it directly from Equation 1

For all integers $0 \leq k \leq n$

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Prove it directly from 2
Prove it by a combinatorial argument *Hint : bijection between the set of subsets of size k and* ???*.*

Exercise

Give a combinatorial argument to prove that for all integers $0 \leq k \leq n$:

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{4}$$

Pascal's triangle

Recurrence equation

The binomial coefficients satisfy

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (5)$$

Prove it directly from Equation 1

Prove it by a combinatorial argument *Hint : partition in two parts the set of subsets of size k ; those containing a given element and those not.*

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Thanks to Elisha

The binomial theorem

For all integer n and a formal parameter X

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k \text{ (Newton 1666)} \tag{6}$$

Prove it by a combinatorial argument *Hint : write* $(1+X)^n = \underbrace{(1+X)(1+X)\cdots(1+X)}_{n \text{ terms}}$ *in each term chose* 1 *or* X*, what is the coefficient of* X^k *in the result (think "vector of n bits").*

Exercises

Use a combinatorial argument to prove :

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Use the binomial theorem to prove (give also a combinatorial argument)

$$\sum_{k=0 \;\; k \text{ odd}}^{n} \binom{n}{k} = \sum_{k=0 \;\; k \text{ even}}^{n} \binom{n}{k} = 2^{n-1}$$

The Vandermonde Convolution

For all integers m, n, k

$$\sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k} \quad (7)$$

Prove it by a combinatorial argument *Hint : choose k elements in two sets one of size m and the other n*.

Exercise

Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \quad (8)$$

Hint : Specify Equation 7

Summations and Decompositions (2)

Upper summation

For all integers $p \leq n$

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1} \tag{9}$$

Exercises

Establish the so classical result

$$\sum_{k=1}^{n} \binom{k}{1}$$

Compute

$$\sum_{k=2}^{n} \binom{k}{2}$$

and deduce the value of $\sum_{k=1}^{n} k^2$

The main rules in combinatorics (I)

Bijection rule

Let A and B be two finite sets if there exists a bijection between A and B then

$$|A| = |B| .$$

Summation rule

Let A and B be two **disjoint** finite sets then

$$|A \cup B| = |A| + |B| .$$

Moreover if $\{A_1, \cdots A_n\}$ is a partition of A (for all $i \neq j$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=0}^{n} A_i = A$)

$$|A| = \sum_{i=0}^{n} |A_i| .$$

The main rules in combinatorics (II)

Product rule

Let A and B be two finite sets then

$$|A \times B| = |A| \,.\, |B| \,.$$

Inclusion/Exclusion principle

Let $A_1, A_2, \cdots A_n$ be sets

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^k \sum_{S \subset \{1,\cdots,n\},\ |S|=k} \left| \bigcap_{i \in S} A_i \right| .$$

Exercises

Illustrate these rules by the previous examples, giving the sets on which the rule apply.

Derangement

Definition

A derangement of a set S is a bijection on S without fixed point.

Number of derangements $!n \stackrel{def}{=} d_n$

Inclusion/Exclusion principle

$$
\begin{aligned}
!n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)! \\
&= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \stackrel{n\to\infty}{\sim} n! \frac{1}{e}
\end{aligned}
$$

Recurrence relation

Show that

$$d_n = (n-1)(d_{n-1} + d_{n-2}) = n d_{n-1} + (-1)^n$$

Pigeons and holes

Principle

If you have more pigeons than pigeonholes
Then some hole must have at least **two** pigeons

Generalization

If there are n pigeons and t holes, then there will be at least one hole with at least

$$\left\lceil \frac{n}{t} \right\rceil \text{ pigeons}$$

History

Johann Peter Gustav Lejeune Dirichlet (1805-1859)
Principle of socks and drawers

http://www-history.mcs.st-and.ac.uk/Biographies/Dirichlet.html

Irrational approximation

Friends

Let α be a non-rational number and N a positive integer, then there is a rational $\frac{p}{q}$ satisfying

$$1 \leq q \leq N \text{ and } \left| \alpha \frac{p}{q} \right| \leq \frac{1}{qN}$$

Hint : divide $[0, 1[$ in N intervals, and decimal part of $0, \alpha, 2\alpha, \cdots, N\alpha$

Sums and others

1. Choose 10 numbers between 1 and 100 then there exist two disjoint subsets with the same sum.
2. For an integer N, there is a multiple of N which is written with only figures 0 and 1

Geometry

1. In a convex polyhedra there are two faces with the same number of edges
2. Put 5 points inside a equilateral triangle with sides 1. At least two of them are at a distance less than 1
3. For 5 point chosen on a square lattice, there are two point such that the middle is

Graphs

Friends

Six people
Every two are either friends or strangers
Then there must be a set of 3 mutual friends or 3 mutual strangers

Guess the number

Player 1 : pick a number 1 to 1 Million
Player 2 Can ask Yes/No questions
How many questions do I need to be guaranteed to correctly identify the number ?

Sorting