# Performance Evaluation : Probabilistic simulation

## Stochastic Modeling of Computer Systems
## MOSIG Master 2

Jean-Marc Vincent

Laboratoire LIG, projet Inria-Mescal
UniversitéJoseph Fourier
Jean-Marc.Vincent@imag.fr

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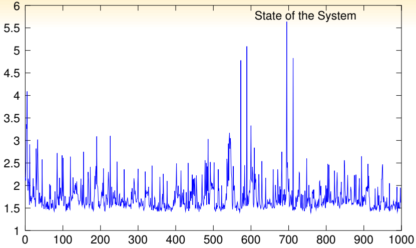
# Outline

1. **Motivation**
   - Convergence
   - Solving
   - Simulation
2. Discrete generation
3. Perfect sampling
4. Case Studies

# Long Run Evolution and Time Scaling



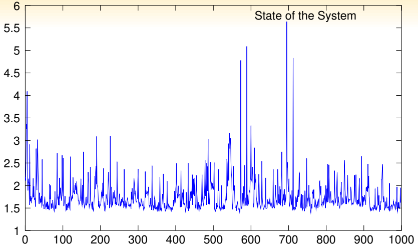

**Typical trajectory**

Performance of the system ⇒ analysis of the steady-state

Computation of the steady-state

**Main contribution**
Efficient computation in finite time of stationary samples

# Long Run Evolution and Time Scaling

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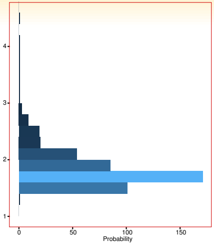


Statistical Synthesis

**Steady-State**

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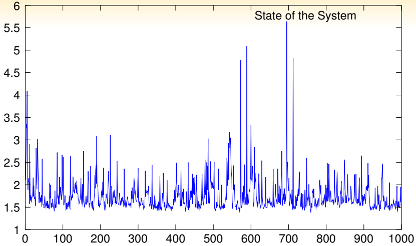


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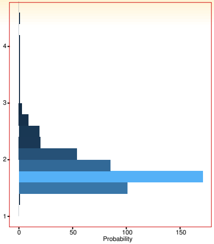


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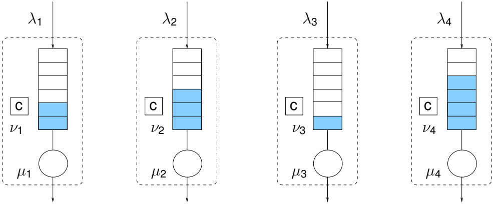
# Load sharing model

Parallel independent queues




**State space**: number of tasks in each queue; $\mathcal{X}_1 \times \cdots \times \mathcal{X}_K$
**Dynamics**: events driven by Poisson process

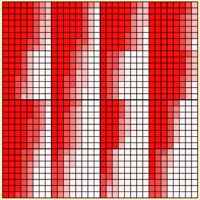
- Generation of a new task in a queue, with rate $\lambda$
- Task completion, with rate $\mu$
- Control, with rate $\nu$

**Uniformization** $\Rightarrow$ Stochastic Recurence Equation $X_{n+1} = \Phi(X_n, E_{n+1})$

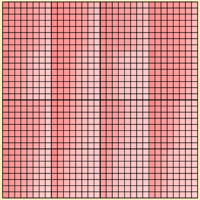
# Application

## Push on arrival

Input Load

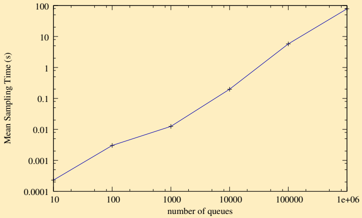


Hierarchical Load Sharing



## Scaling Toward million of nodes

Policy: Threshold Push on Arrival with priority list of 8 nodes

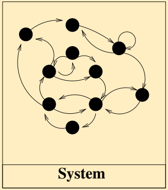



The time to simulate such system is linear with the number of nodes

[ASMTA 2010]

# Modeling and Analysis of Computer Systems

## Complex system




## Basic model assumptions

System :
- automaton (discrete state space)
- **discrete** or continuous time

Environment : non deterministic
- time homogeneous
- stochastically regular

## Problem

**Understand "typical" states**
- steady-state estimation
- ergodic simulation
- state space exploring techniques

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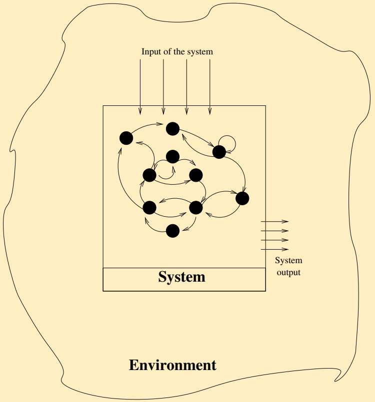


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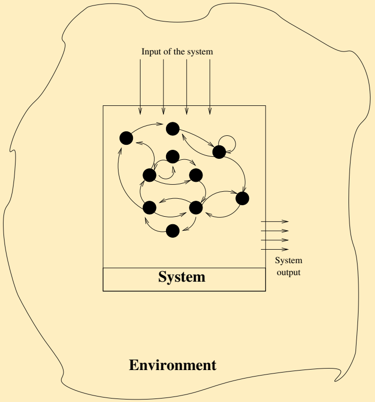


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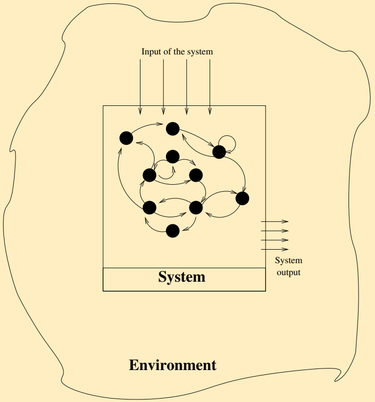


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# Convergence In Law

Let $\{X_n\}_{n\in\mathbb{N}}$ a homogeneous, irreducible and aperiodic Markov chain taking values in a discrete state $\mathcal{X}$ then

- The following limits exist (and do not depend on $i$)

$$\lim_{n\to+\infty} \mathbb{P}(X_n = j|X_0 = i) = \pi_j;$$

- $\pi$ is the unique probability vector invariant by $P$

$$\pi P = \pi;$$

- The convergence is rapid (geometric); there is $C > 0$ and $0 < \alpha < 1$ such that

$$||\mathbb{P}(X_n = j|X_0 = i) - \pi_j|| \leqslant C.\alpha^n.$$

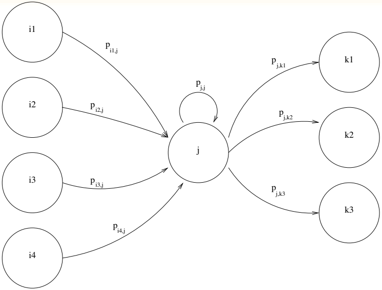
Denote

$$X_n \xrightarrow{\mathcal{L}} X_\infty;$$

with $X_\infty$ with law $\pi$
$\pi$ is the **steady-state probability** associated to the chain

# Interpretation

## Equilibrium equation




Probability to enter $j$ =probability to exit $j$
**balance equation**

$$\sum_{i \neq j} \pi_i p_{i,j} = \sum_{k \neq j} \pi_j p_{j,k} = \pi_j \sum_{k \neq j} p_{j,k} = \pi_j (1 - p_{j,j})$$

$\pi \stackrel{def}{=}$ steady-state.
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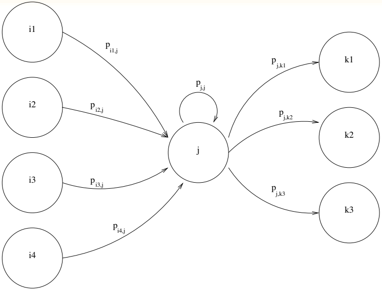


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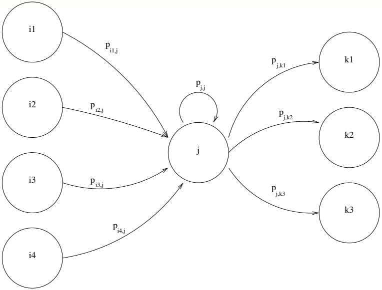


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# Ergodic Theorem

Let $\{X_n\}_{n\in\mathbb{N}}$ a homogeneous aperiodic and irreducible Markov chain on $\mathcal{X}$ with steady-state probability $\pi$ then

- for all function $f$ satisfying $\mathbb{E}_\pi|f| < +\infty$

$$\frac{1}{N}\sum_{n=1}^{N} f(X_n) \xrightarrow{P-p.s.} \mathbb{E}_\pi f.$$

generalization of the *strong law of large numbers*

- If $\mathbb{E}_\pi f = 0$ then there exist $\sigma$ such that

$$\frac{1}{\sigma\sqrt{N}}\sum_{n=1}^{N} f(X_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

generalization of the *central limit theorem*

# Fundamental question

**Given a function $f$ (cost, reward, performance,...) estimate**

$$\mathbb{E}_\pi f$$

**and give the quality of this estimation.**

# Solving methods

## Solving $\pi = \pi P$

- Analytical/approximation methods
- Formal methods $N \leqslant 50$
  Maple, Sage,...
- Direct numerical methods $N \leqslant 1000$
  Mathematica, Scilab,...
- Iterative methods with preconditioning $N \leqslant 100,000$
  Marca,...
- Adapted methods (structured Markov chains) $N \leqslant 1,000,000$
  PEPS,...
- Monte-Carlo simulation $N \geqslant 10^7$

## Postprocessing of the stationary distribution

Computation of rewards (expected stationary functions)
Utilization, response time,...

# Ergodic Sampling(1)

## Ergodic sampling algorithm

Representation : **transition fonction**

$$X_{n+1} = \Phi(X_n, e_{n+1}).$$

$x \leftarrow x_0$
{choice of the initial state at time =0}
$n = 0;$
**repeat**
  $n \leftarrow n + 1;$
  $e \leftarrow Random\_event();$
  $x \leftarrow \Phi(x, e);$
  Store $x$
  {computation of the next state $X_{n+1}$}
**until** some empirical criteria
return the trajectory

**Problem : Stopping criteria**

# Ergodic Sampling(2)

## Start-up

Convergence to stationary behavior

$$\lim_{n \to +\infty} \mathbb{P}(X_n = x) = \pi_x.$$

**Warm-up period :** Avoid initial state dependence
Estimation error :

$$||\mathbb{P}(X_n = x) - \pi_x|| \leqslant C\lambda_2^n.$$

$\lambda_2$ second greatest eigenvalue of the transition matrix
- bounds on $C$ and $\lambda_2$ (spectral gap)
- cut-off phenomena

$\lambda_2$ and $C$ non reachable in practice
(complexity equivalent to the computation of $\pi$)
some known results (Birth and Death processes)

# Ergodic Sampling(3)

## Estimation quality

Ergodic theorem :

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \mathbb{E}_\pi f.$$

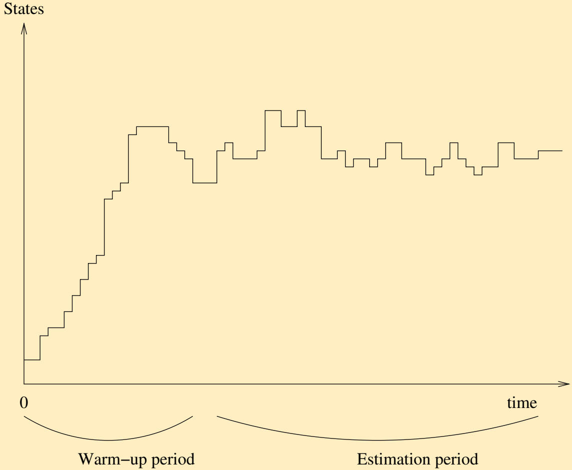
**Length of the sampling :** Error control (CLT theorem)

## Complexity

Complexity of the transition function evaluation (computation of $\Phi(x, .)$)
Related to the stabilization period + Estimation time

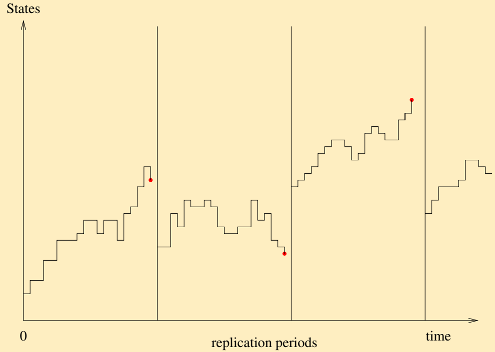
# Ergodic sampling(4)

## Typical trajectory



# Replication Method

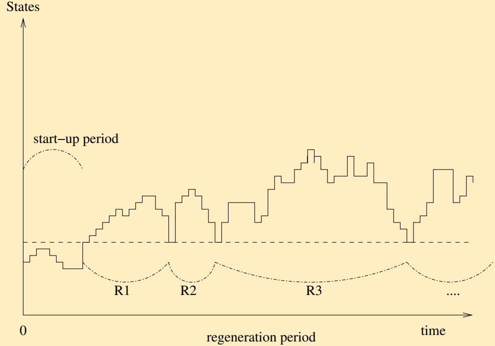
## Typical trajectory




Sample of independent states
Drawback : length of the replication period (dependence from initial state)

# Regeneration Method

## Typical trajectory




Sample of independent trajectories
Drawback : length of the regeneration period (choice of the regenerative state)

# Outline

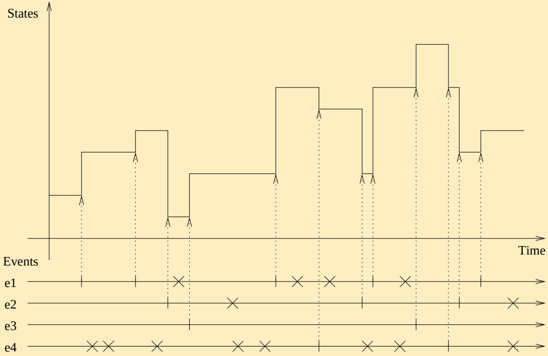
# Event Modelling

Multidimensional state space : $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K$ with $\mathcal{X}_i = \{0, \cdots, C_i\}$.

Event $e$ :

$\rightsquigarrow$ transition function $\Phi(., e)$; (skip rule)

$\rightsquigarrow$ Poisson process $\lambda_e$




1781-1840

# Event modelling

## Uniformization

$$\Lambda = \sum_e \lambda_e \text{ and } \mathbb{P}(event\ e) = \frac{\lambda_e}{\Lambda};$$

Trajectory : $\{e_n\}_{n\in\mathbb{Z}}$ i.i.d. sequence.
$\Rightarrow$ Homogeneous Discrete Time Markov Chain [Bremaud 99]
$X_{n+1} = \Phi(X_n, e_{n+1})$.

Generation among a small finite space $\mathcal{E}$ : $\mathcal{O}(1)$

# Generating random objects

Denote by $X$ the generated object ( $X$ is a random variable)
Distribution (proportion of observations, input of the load injector)

$$p_k = \mathbb{P}(X = k).$$

Remarks :

$$0 \leqslant p_i \leqslant 1; \quad \sum_k p_k = 1.$$

Expectation (average, mean)

$$\mathbb{E}X = \sum_k k.\mathbb{P}(X = k) = \sum_k kp_k.$$

Variance and standard deviation

$$\mathbb{V}arX = \sum_k (k - \mathbb{E}X)^2 \mathbb{P}(X = k) = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

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# The random function

Random bit generator (see previous lecture)

**drand48 manpage**

double drand48(void) (48 bits encoded in 8 bytes)
The rand48() family of functions generates pseudo-random numbers using a linear congruential algorithm working on integers 48 bits in size. The particular formula employed is r(n+1) = (a * r(n) + c) mod m where the default values are for the multiplicand a = 0xfdeece66d = 25214903917 and the addend c = 0xb = 11. The modulo is always fixed at m = 2 ** 48. r(0) is called the seed of the random number generator.

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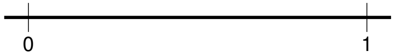


**Problem**

All the difficulty is to find a function (an algorithm) that transforms the $[0, 1[$ in a set with a good probability conserving.

**Example : flip a coin**

u= r andom()
**if** u $\leqslant \frac{1}{2}$ **then**
    return Head
**else**
    return Tail
**end if**

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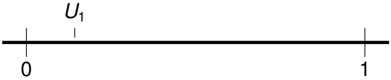


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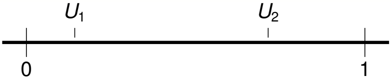


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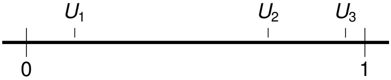


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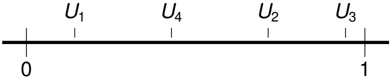


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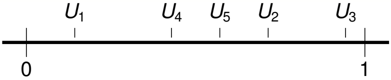


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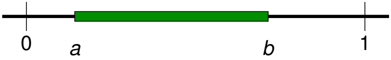


$$\mathbb{P}(U \in [a, b[) = (b - a)$$

### Problem

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### Example : flip a coin

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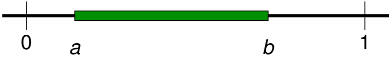
# The random function

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# Practical example : Web server

## Types of request

1. Professional customer, consult
2. Professional customer, purchase
3. Non professional customer, consult
4. Non professional customer, purchase
5. Adminstration

Build an algorithm that provides a set of requests according the observed distribution.

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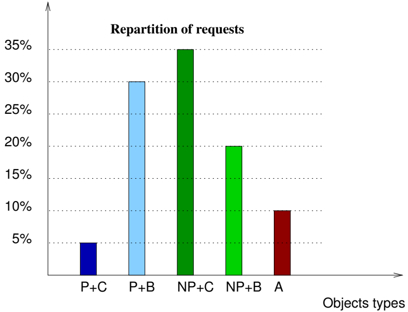
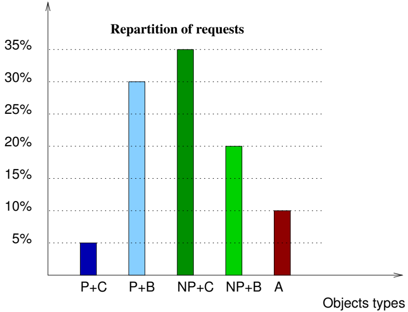



Build an algorithm that provides a set of requests according the observed distribution.

# Practical example : Web server

## Types of request

1. Professional customer, consult
2. Professional customer, purchase
3. Non professional customer, consult
4. Non professional customer, purchase
5. Adminstration

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# Tabulation method

## Pre-computation

$$p_k = \frac{m_k}{m} \text{ where } \sum_k m_k = m.$$

Create a table $T$ with size $m$.
Fill $T$ such that $m_k$ cells contains $k$.
Computation cost : $m$ steps
Memory cost : $m$

## Table construction

i=0
**for** k=1, k $\leqslant$ K, k++ **do**
  **for** j=1, j $\leqslant$ $m_k$, j++ **do**
    T[i]= k; i=i+1;
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**end for**

## Generation

Generate uniformly on the set $\{0, 1, \cdots, m-1\}$
Returns the value in the table
Computation cost : $\mathcal{O}(1)$ step
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## Generation algorithm

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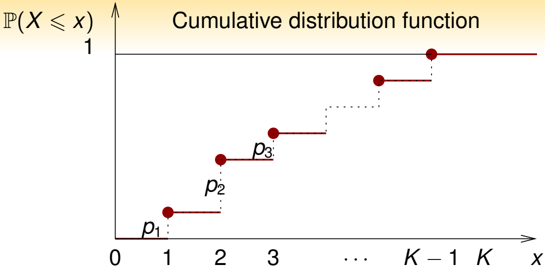


## Generation

Divide $[0,1[$ in intervals with length $p_k$
Find the interval in which *Random* falls
Returns the index of the interval
Computation cost : $\mathcal{O}(\mathbb{E}X)$ steps
Memory cost : $\mathcal{O}(1)$

## Inverse function algorithm

```
s=0; k=0;
u=random()
while u >s do
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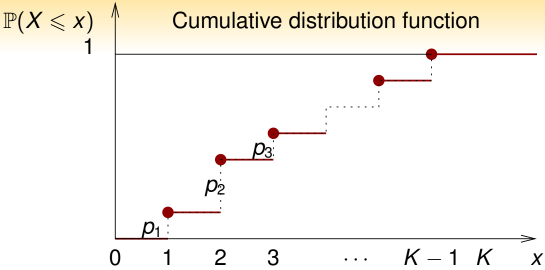


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# Searching optimization

## Optimization methods

- pre-compute the pdf in a table
- rank objects by decreasing probability
- use a dichotomy algorithm
- use a tree searching algorithm (optimality = Huffmann coding tree)

## Comments

- Depends on the usage of the generator (repeated use or not)
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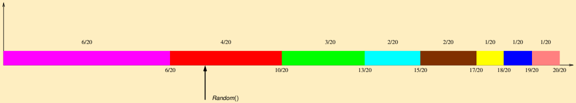


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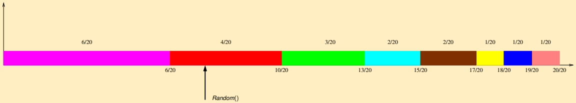


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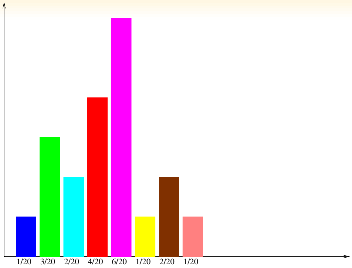


## Number of comparisons

Binary search tree structure

$$\mathbb{E}N = \sum_{k=1}^{K} p_k . l_k = 3,75, \ \text{Entropy} = \sum_{k=1}^{K} p_k(-\log_2 p_k) = 3.70$$

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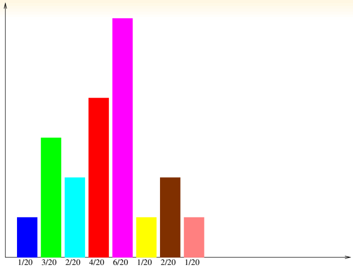


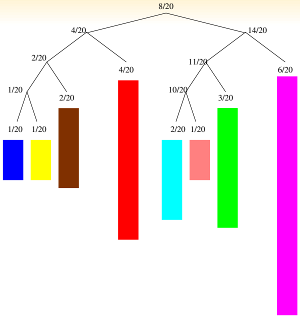


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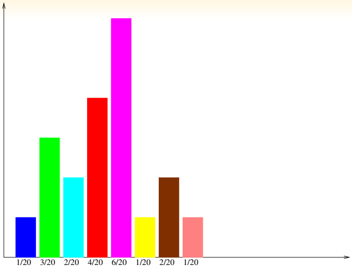


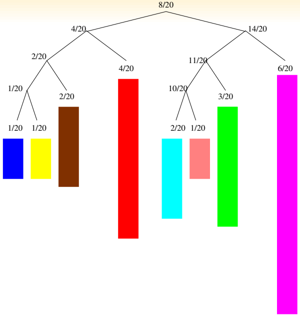


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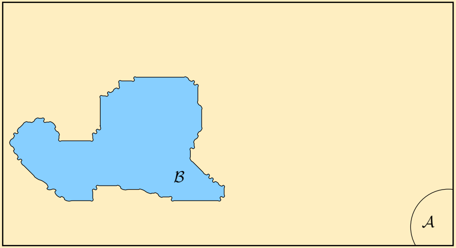
Generate uniformly on $\mathcal{A}$ accept when point is in $\mathcal{B}$.




## Rejection algorithm

**repeat**
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## Complexity

Acceptance probability

$$p_a = \frac{Size(\mathcal{B})}{Size(\mathcal{A})}$$

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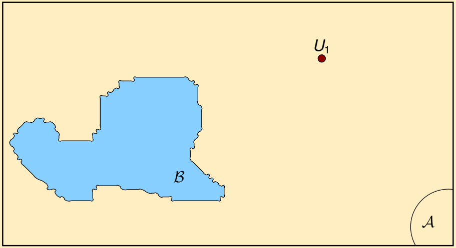


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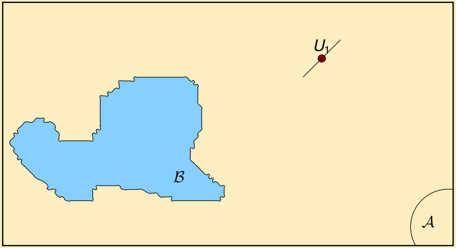


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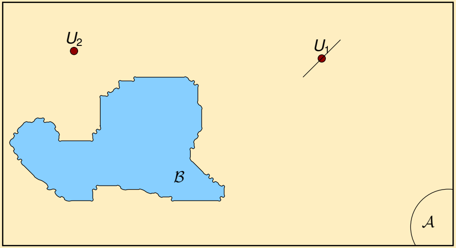


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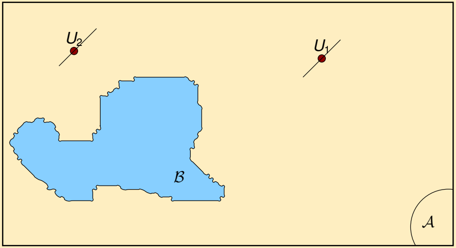


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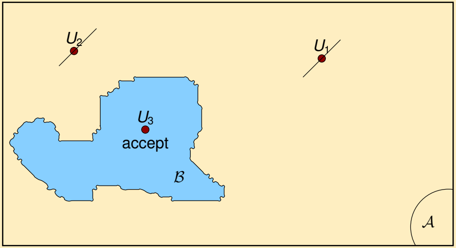


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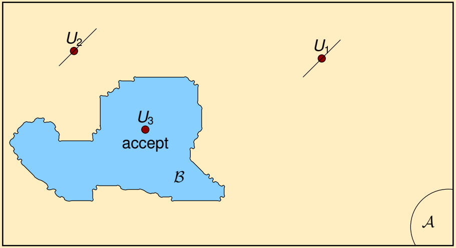


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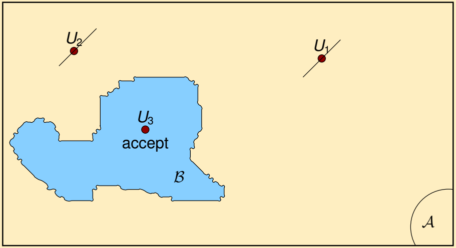


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## Rejection adaptation

$K$ objects

$$h \geqslant \max_k p_k$$

Generate uniformly on the surface $K \times h$
Accept if the point is under the distribution

## Rejection algorithm

**repeat**
  k= alea(K)
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alea(K) generate uniformly a number in $\{1, \cdots, K\}$

## Complexity

Acceptance probability $p_a = \frac{1}{hK}$
$N$ number of iterations $\mathbb{E}N = \frac{1}{p_a} = hK$.
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Uniform distribution $\Rightarrow$ no rejection
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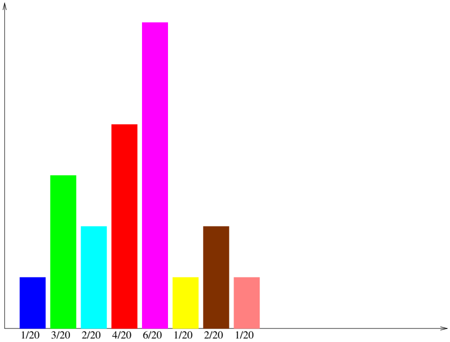

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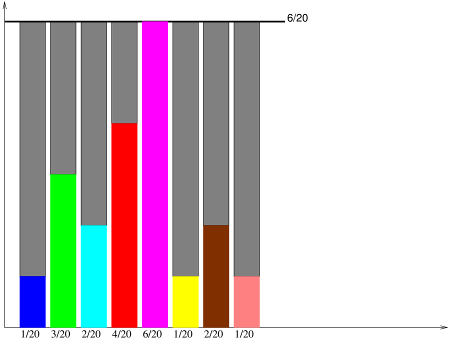

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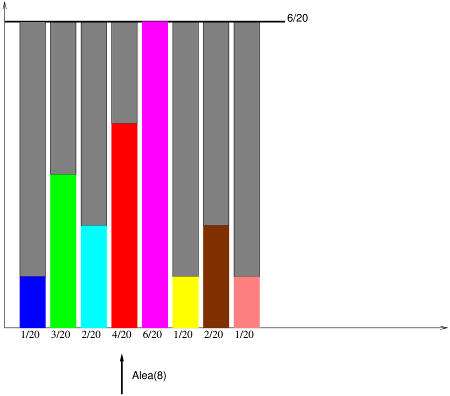

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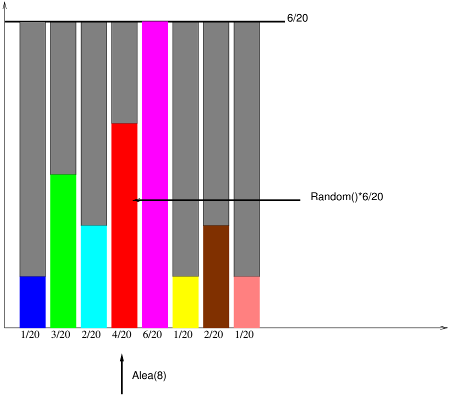

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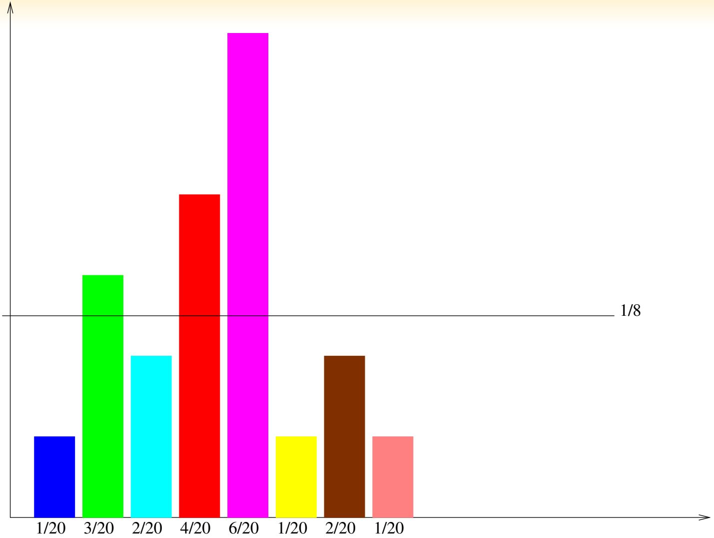

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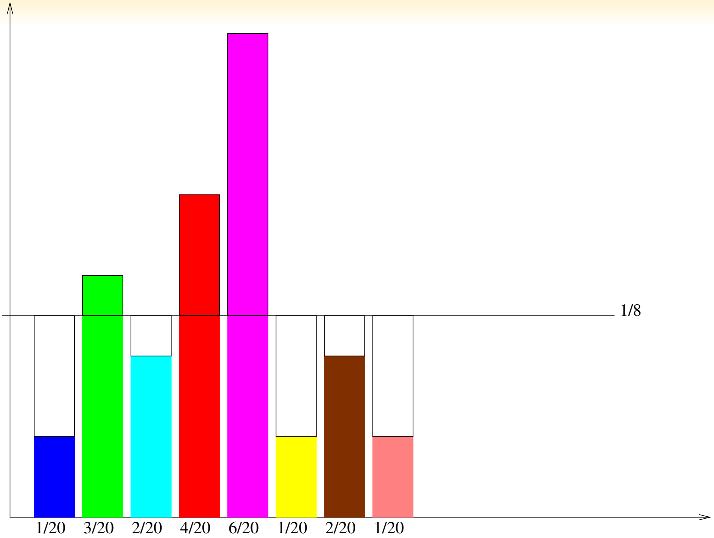

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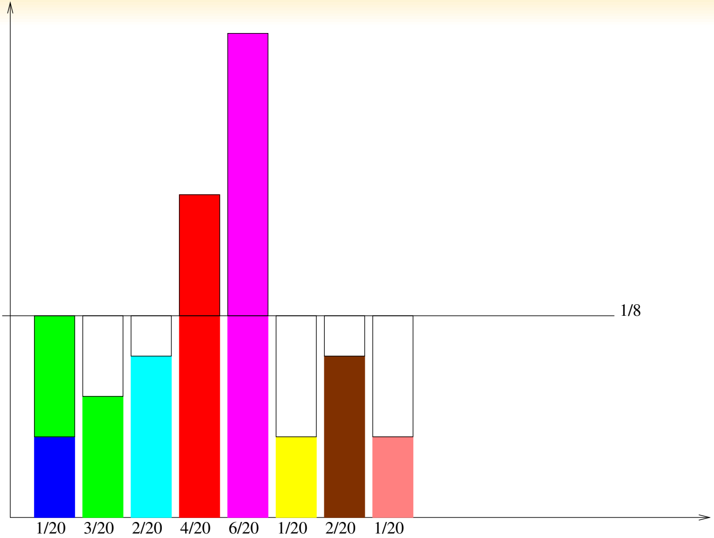

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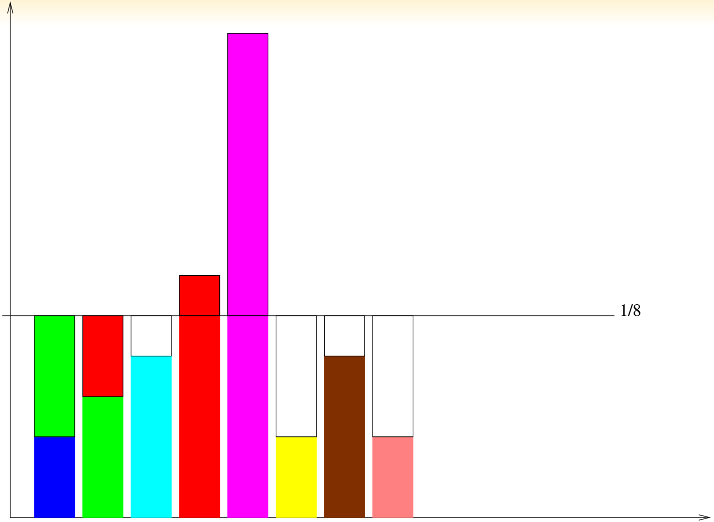

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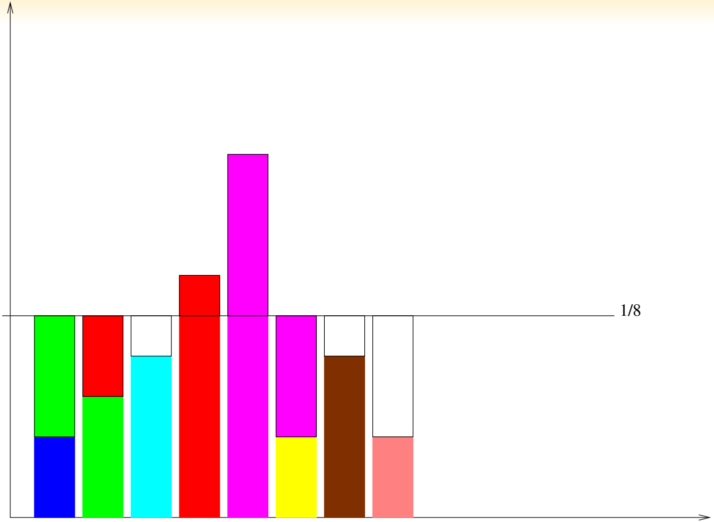

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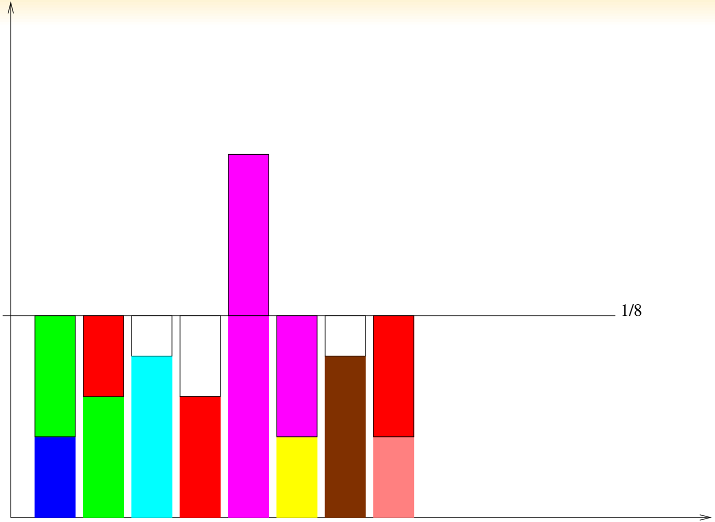

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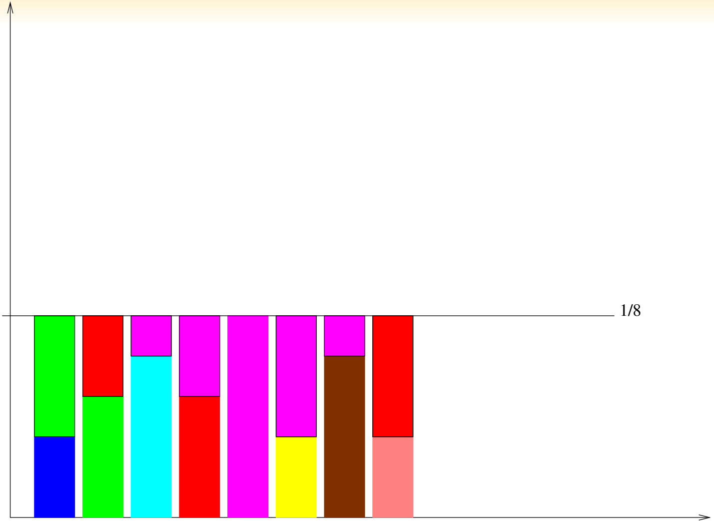

# Aliasing technique

Combine uniform and alias value when rejection

**Initialization**

$K$ objects

list L=$\emptyset$,U=$\emptyset$;
**for** k=1; k$\leqslant$ K; k++ **do**
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**Alias and threshold tables**

**while** L $\neq \emptyset$ **do**
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# Outline

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- Call centers
- Grid/cluster scheduling
- Kitting systems
- Rare event estimation
- Statistical verification of programs

## Modeling

- Poisson systems [Brémaud 1999]
- Discrete vector state-space $\mathcal{X}$
- Event based models

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  Stochastic recurrence equation
- Independent events (iid)

Provide **independent** samples of **stationary** states.

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# Perfect Sampling of Complex Markov Chains

## Applications

- Finite queuing networks
- Call centers
- Grid/cluster scheduling
- Kitting systems
- Rare event estimation
- Statistical verification of programs

## Modeling

- Poisson systems [Brémaud 1999]
- Discrete vector state-space $\mathcal{X}$
- Event based models

$$X_{n+1} = \Phi(X_n, e_{n+1}) \ , e_n \in \mathcal{E}$$

  Stochastic recurrence equation
- Independent events (iid)

Provide **independent** samples of **stationary** states.

## PSI2 : a Perfect Sampler

- Library of **monotone** events
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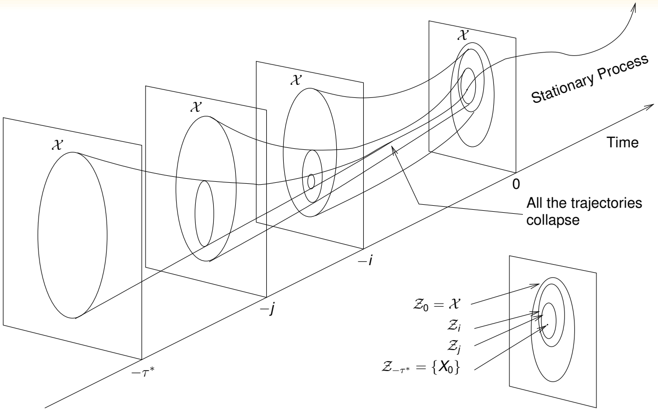
# Perfect Sampling Principle




Synchronizing pattern $\Longrightarrow$ finite backward scheme $\tau^* < \infty$

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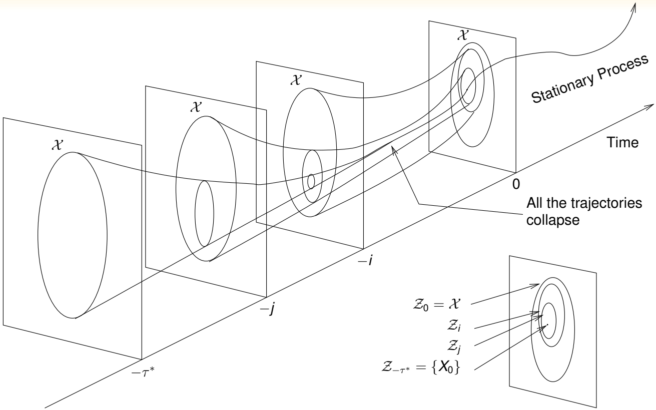


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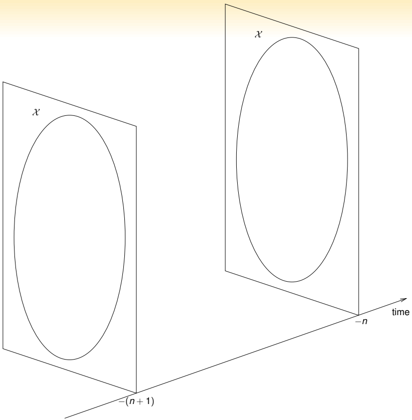
# Monotone Perfect Sampling




same convergence condition

**complexity in $\mathcal{O}(\mathbb{E}\tau^*)$** $\Rightarrow$ polynomial in model dimension

[QEST 2008]

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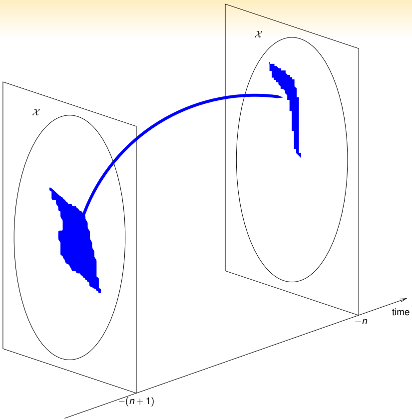


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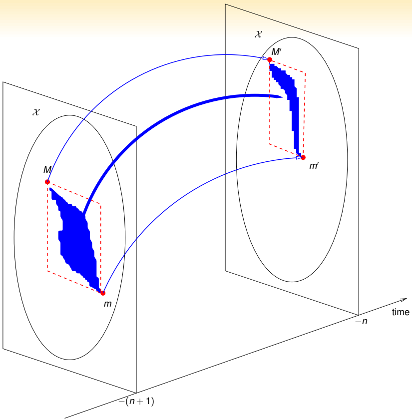


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[QEST 2008]

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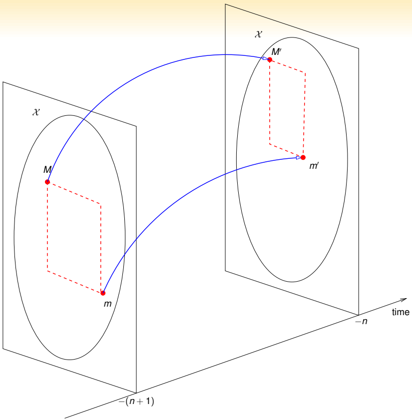



same convergence condition
**complexity in $\mathcal{O}(\mathbb{E}\tau^*)$** $\Rightarrow$ polynomial in model dimension

[QEST 2008]

# Panorama : Markov models

## Finite Monotone Systems

- large class of models : index based routing finite queueing networks
- time complexity : **polynomial** in the dimension of the system

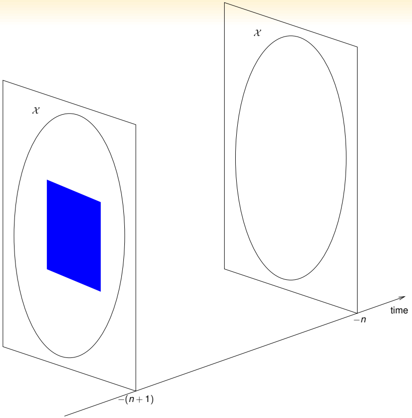
## Finite non-monotone system

- Transition function
  - almost monotone systems : bounding process
  - exhaustive : splitting
  - piecewise linear transitions
- State space extension

## Infinite systems

- Monotone transition function
- Non-monotone transitions

# Envelopes Perfect Sampling




Synchronizing pattern for envelopes
complexity unknown but practically efficient

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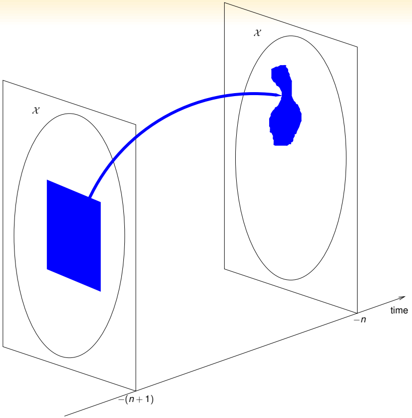


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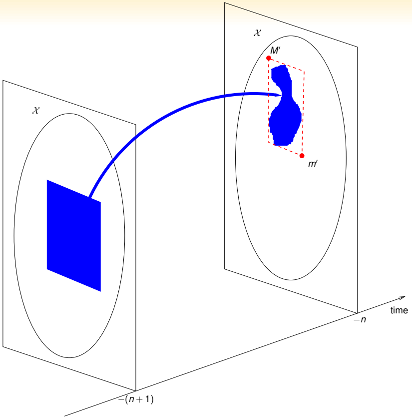


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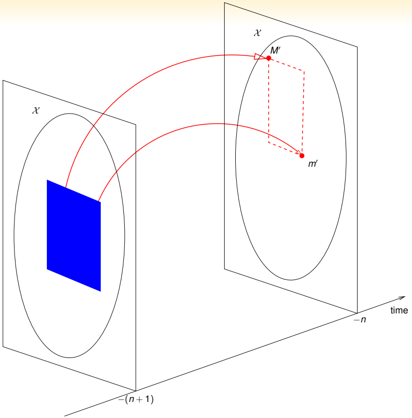


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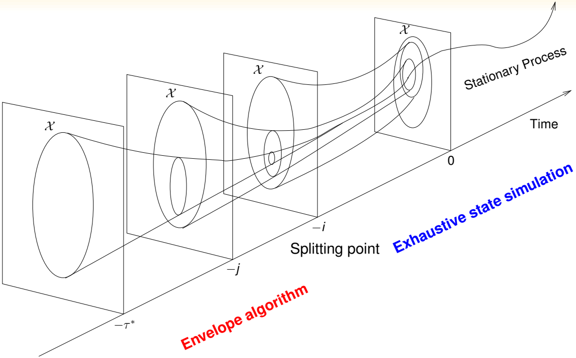
# Envelopes and Splitting Perfect Sampling

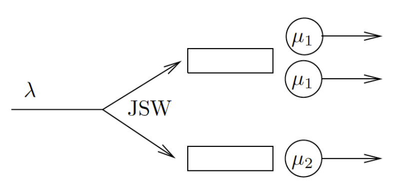



**Guarantees the convergence**
**complexity unknown** but practically more efficient

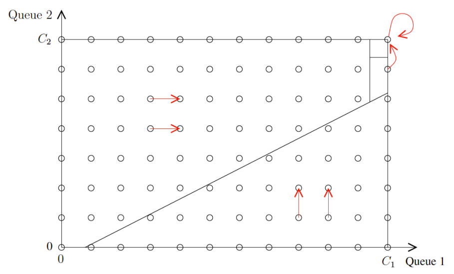
[VALUETOOLS 2008, QEST 2010]

# Computation of Envelopes

## Join the Shortest Weighted Queue




## State space




negative customers, fork and join, batch routing
**general complexity polynomial** (linear programs) but practically $\Rightarrow$ computable less tight bounds

[Performance Evaluation, 2012]

# Outline

# Case Studies