Event flows modeling

Crimes are random processes

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Outline

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Event flow model

Continuous time modeling: occurrence of events

- traffic on a road, arrivals at a taxi station,
- birth and death in demography
- hit on web servers, messages on a link, phone calls
- crimes, delinquency,...
- ...

Basic model of a 2 time scale system

- Randomness due to complexity of the environment
- Superposition of many behaviors

$\{N_t\}_{t\in\mathbb{R}}$, where N_t = number of events in $[0, t[$

or equivalently

$\{T_n\}_{n\in\mathbb{N}}$, where T_n is the date at which event $\#n$ occurs

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Flow characteristics

Communication model: 2 counting processes

- emission/reception process

Throughput

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} N_t .$$

Volume, Streaming
Link capacity...

Latency

$$\mathbb{E}(T_{n+1} - T_n)$$

Response time
Time constraints

Jitter

$$\mathbb{V}ar(T_{n+1} - T_n)$$

Variability of inter-arrivals
Periodic behavior

Loss rates

Communication reliability
Perturbed events

$$\lambda_{emission} - \lambda_{reception}$$

Outline

Justice management

RECHERCHES

SUR LA

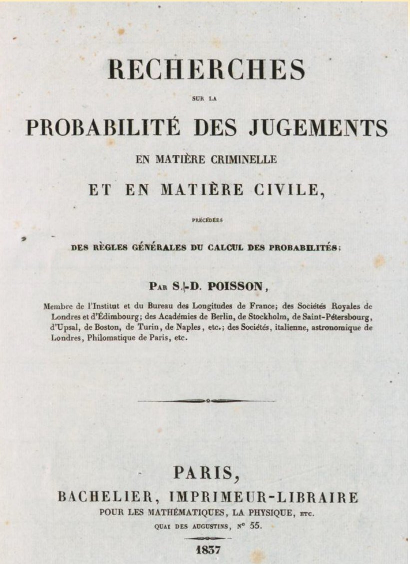
PROBABILITÉ DES JUGEMENTS

EN MATIÈRE CRIMINELLE

ET EN MATIÈRE CIVILE,

PRÉCÉDÉES

Justice management(2)



RECHERCHES

SUR LA

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EN MATIÈRE CRIMINELLE

ET EN MATIÈRE CIVILE,

PRÉCÉDÉES

DES RÈGLES GÉNÉRALES DU CALCUL DES PROBABILITÉS;

PAR S.-D. POISSON,

Membre de l'Institut et du Bureau des Longitudes de France; des Sociétés Royales de Londres et d'Édimbourg; des Académies de Berlin, de Stockholm, de Saint-Pétersbourg, d'Upsal, de Boston, de Turin, de Naples, etc.; des Sociétés, italienne, astronomique de Londres, Philomatique de Paris, etc.

PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

POUR LES MATHÉMATIQUES, LA PHYSIQUE, ETC.

QUAI DES AUGUSTINS, N° 55.

1837

Justice management

celui des accusés, a aussi augmenté d'une manière progressive (1). Voici des résultats extraits de ces documents, et que l'on pourra comparer à ce qui a lieu dans notre pays. Les nombres suivants se rapportent seulement à l'Angleterre et au pays de Galles. Ils répondent à trois périodes de chacune sept années, finissant en 1818, 1825, 1832.

	NOMBRE des accusés.	NOMBRE des condamnés.	RAPPORT du second nombre au 1er.	CONDAMNÉS à mort.	EXÉCUTÉS.	CONDAMNÉS à un emprisonnement de deux ans ou au-dessous.
1re période,	64538	41054	0,636...	5802	636	27168
2e	93718	63418	0,677...	7770	579	42713
3e	127910	90240	0,705...	9729	414	58757

Justice management

4755, 5081, 5018, 5552, 5582, 5296;

les nombres correspondants des condamnés, sous l'empire d'une même législation criminelle, se sont élevés à

882, 967, 948, 871, 834, 766,

pour les crimes de la première espèce, et à

3155, 3381, 3288, 3680, 3641, 3364,

pour ceux de la seconde. De là, on déduit

0,4649, 0,5071, 0,4961, 0,4723, 0,4657, 0,4598,

pour les rapports des nombres de condamnés à ceux des accusés de crimes contre les personnes, et

0,6635, 0,6654, 0,6552, 0,6628, 0,6523, 0,6352,

pour les rapports des nombres de condamnés à ceux des accusés de crimes contre les propriétés; où l'on voit que les uns et les autres n'ont pas beaucoup varié d'une année à une autre, mais que les derniers excèdent notablement les premiers.

En prenant pour μ et a_5 les sommes des nombres d'accusés et de condamnés dans le cas des crimes contre les personnes, et pour μ' et a'_5 leurs sommes dans le cas des crimes contre les propriétés, nous aurons

$$\mu = 11016, \quad a_5 = 5268, \quad \mu' = 31284, \quad a'_5 = 20509;$$

d'où il résulte ces deux rapports :

$$\frac{a_5}{\mu} = 0{,}4782, \qquad \frac{a'_5}{\mu'} = 0{,}6556,$$

dont le second surpasse le premier d'un peu plus du tiers de celui-ci. Au moyen de ces nombres, on trouve

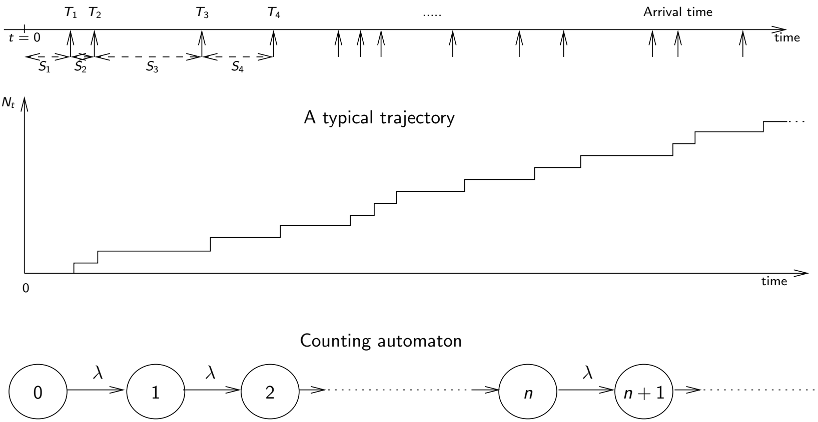
$$0{,}4782 \mp \alpha(0{,}00675)$$

pour les limites (a) de l'inconnue R_5, relative aux crimes contre les personnes, et

$$0{,}6556 \mp \alpha(0{,}00380),$$

Outline

Counting process



Macroscopic modeling

Definition (Macroscopic definition)

A continuous time stochastic process $\{N_t\}_{t\in\mathbb{R}^+}$ is a counting Poisson process with intensity λ iff

1. $N_0 = 0$
2. $\{N_t\}_{t\in\mathbb{R}^+}$ *have independent increments*

   (e.g., for $a < b < c < d$, $N_b - N_a$ is independant of $N_d - N_c$)
3. *The number of events occurring in a time interval $]a, b]$ is Poisson distributed with parameter $\lambda(b-a)$;*

$$\mathbb{P}(N_b - N_a = k) = e^{-\lambda(b-a)}\frac{(\lambda(b-a))^k}{k!}.$$

Properties

- Increments are stationary: homogeneous in time

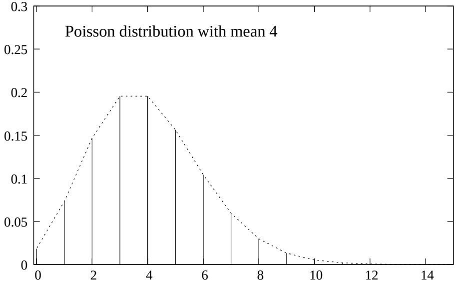
  $N_t - N_{t+\Delta}$ does not depend on t
- Linearity: $\mathbb{E}(N_b - N_a) = \lambda(b-a)$
- λ = intensity or throughput of the process (number of events per time unit)

Poisson distribution $\mathcal{P}(\lambda)$

X random variable Poisson distributed with parameter λ

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

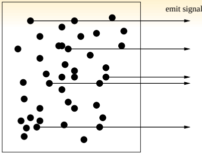
$$\mathbb{E}(X) = \lambda; \quad \mathbb{V}ar(X) = \lambda.$$



If X and Y are independent random variable Poisson distributed with mean λ and μ then

$$X + Y \sim \mathcal{P}(\lambda + \mu).$$

Interpretation



N elements, each of them $p =$ probability of signal emission
X total number of emissions: binomial distribution $\mathcal{B}(N, p)$.
$\mathbb{E}(X) = Np \stackrel{def}{=} \lambda$ mean number of emissions.

$$\begin{aligned}
\mathbb{P}(X = k) &= \binom{N}{k} p^k (1-p)^{N-k}; \\
&= \underbrace{\frac{N(N-1)\cdots(N-k+1)}{N.N\cdots N}}_{\to 1} \underbrace{\frac{1}{(1-\frac{\lambda}{N})^k}}_{\to 1} \frac{\lambda^k}{k!} \underbrace{(1-\frac{\lambda}{N})^N}_{\to e^{-\lambda}}; \\
&\simeq e^{-\lambda} \frac{\lambda^k}{k!}.
\end{aligned}$$

for very large N, X is asymptotically Poisson distributed

Flow analysis

Traffic generated by a huge amount of individuals ⇒ Poisson process

- requests arrival on a web server
- arrivals of phone calls
- routed packets in a network
- cars on a road network
- ...

How to detect non-Poisson flows

- Time dependence or correlation (burstyness, periodicity,...)
- Mean $<$ Variance: too much variability
- smoothers of the traffic (peack avoidance strategies)
- ...

Microscopic modeling

Definition (Microscopic definition)

A continuous time stochastic process $\{N_t\}_{t\in\mathbb{R}^+}$ is a counting Poisson process with intensity λ iff

1. $N_0 = 0$
2. $\{N_t\}_{t\in\mathbb{R}^+}$ *have independent and stationary increments*
3. *On a very small intervall* $]t, t+dt]$ *we have:*

$$\begin{aligned}
\mathbb{P}(N_{t+dt} - N_t = 1) &= \lambda dt + o(dt) \\
\mathbb{P}(N_{t+dt} - N_t = 0) &= 1 - \lambda dt + o(dt) \\
\mathbb{P}(N_{t+dt} - N_t \geqslant 2) &= o(dt)
\end{aligned}$$

Properties

- Increments are stationary: homogeneous in time
- Linearity: $\mathbb{E}(N_b - N_a) = \lambda(b-a)$
- $\lambda =$ intensity or throughput of the process (number of events per time unit)

Differential system

$$
\begin{aligned}
p_n(t) = &\ \mathbb{P}(N_t = n) \\
p_n(t+dt) = &\ \mathbb{P}(N_{t+dt} = n) \\
= &\ \mathbb{P}(N_{t+dt} = n | N_t = n)\mathbb{P}(N_t = n) && \text{nothing happens} \\
&+ \mathbb{P}(N_{t+dt} = n | N_t = n-1)\mathbb{P}(N_t = n-1) && \text{one arrival} \\
&+ \mathbb{P}(N_{t+dt} = n | N_t < n-1)\mathbb{P}(N_t < n-1) && \text{more than one arrival}
\end{aligned}
$$

since we have independent increments

$$
\begin{aligned}
= &\ \mathbb{P}(N_{t+dt} - N_t = 0)p_n(t) && \text{nothing happens} \\
&+ \mathbb{P}(N_{t+dt} - N_t = 1)p_{n-1}(t) && \text{one arrival} \\
&+ \mathbb{P}(N_{t+dt} - N_t \geqslant 2)\mathbb{P}(N_t < n-1) && \text{more than one arrival} \\
= &\ (1 - \lambda dt + o(dt))p_n(t) + (\lambda dt + o(dt))p_{n-1}(t) + o(dt) \\
= &\ p_n(t) + \lambda(p_{n-1}(t) - p_n(t))dt + o(dt)
\end{aligned}
$$

We end up with recurrent differential equations: $\begin{cases} p_n'(t) = \lambda(p_{n-1}(t) - p_n(t)) \\ p_0'(t) = \lambda p_0(t) \end{cases}$

, which are solved by recurrence (put $q_n(t) = e^{\lambda t} p_n(t)$) $\boxed{p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$

Differential system

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= & \quad \mathbb{P}(N_{t+dt} = n | N_t = n)\mathbb{P}(N_t = n) && \text{nothing happens} \\
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Interarrivals

Let t be a fixed time and let T_t be the time to the next arrival after time t

$$\mathbb{P}(T_t \geqslant s) = \mathbb{P}(N_{t+s} - N_t = 0) = e^{-\lambda s}.$$

T_t is exponentially distributed with rate λ
The inter-arrival process $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of independent exponentially distributed random variable with rate λ

Exponential distribution

Density, rate λ:

$$f(x) = \lambda e^{-\lambda x}$$

Cumulative distribution function

$$F(x) = 1 - e^{-\lambda x}$$

Mean, Variance

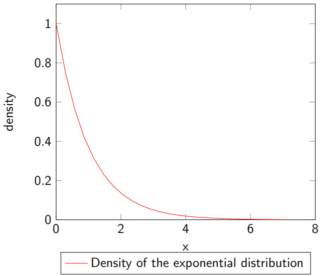
$$\mathbb{E}X = \frac{1}{\lambda}, \quad \mathrm{Var}X = \frac{1}{\lambda^2}$$

Hazard rate

$$h(x) = \lambda$$

Laplace transform

$$\mathcal{L}(t) = \mathbb{E}e^{-tX} = \frac{\lambda}{t + \lambda}$$



Memoryless property

$$\mathbb{P}(X > t + s | X > t) = \mathbb{P}(X > s)$$

Equivalence of definitions

Theorem (Global vision)

Macroscopic, **microscopic** *and* **independent exponentially distributed inter-arrivals** *are equivalent definitions of a Poisson process*

Proof: classical books

Outline

Maximum Entropy Process

Spread of Points

Let $[a, b]$ an interval, knowing $N_b - N_a = n$ the n points are distributed as the rearrangement of n points independents and uniformly distributed points on $[a, b]$

Theorem (Information Approach)

The **Poisson process** *is the model of process with a fixed intensity and* **minimal "a priori" information**

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Scale Invariance

Theorem (Superposition)

Let $\{N_t^1\}$ and $\{N_t^2\}$ be two **independent** *Poisson processes then $\{(N^1 + N^2)_t\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$*

Theorem (Extraction)

Probabilistic thinning of a Poisson process is a Poisson process.

Scale Invariance

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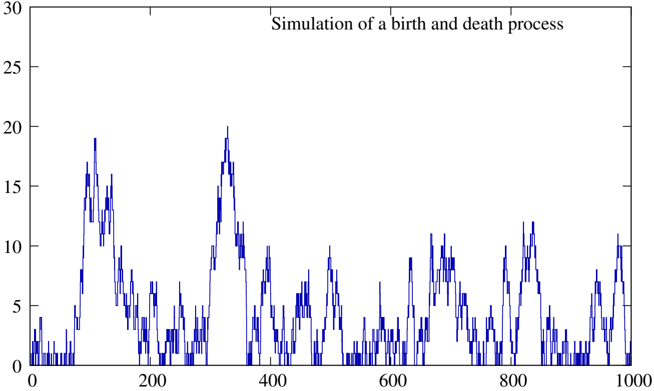
Poisson Clumping heuristic



Outline

Non-homogeneity

Definition (Macroscopic definition)

A continuous time stochastic process $\{N_t\}_{t\in\mathbb{R}^+}$ is a non-homogeneous counting Poisson process with intensity $\lambda(t)$ iff

1. $N_0 = 0$
2. $\{N_t\}_{t\in\mathbb{R}^+}$ *have independent increments*
3. *The number of events occurring in a time interval $]a, b]$ is Poisson distributed with parameter $\int_a^b \lambda(t)dt = \Lambda(b) - \Lambda(a)$;*

$$\mathbb{P}(N_b - N_a = k) = e^{-(\Lambda(b)-\Lambda(a))}\frac{(\Lambda(b)-\Lambda(a))^k}{k!}.$$

- embedded periodicity
- exceptional period
- ...

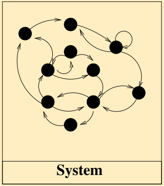
Doubly Stochastic

Randomness on the intensity

$$\{\lambda_t\}_{t \in real^+}$$

stationary process. Conditioned by λ_t, $\{N_t\}_{t\in\mathbb{R}^+}$ is a Poisson process.

Markov-modulated Poisson process



- several time scales
- algebra by composition of automata
- ON/OFF systems
- ...

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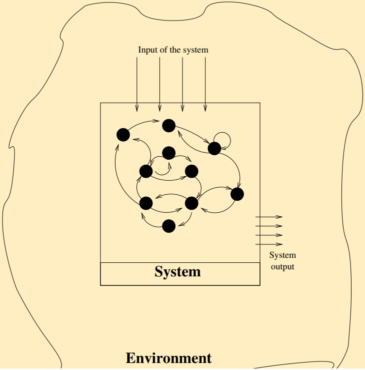

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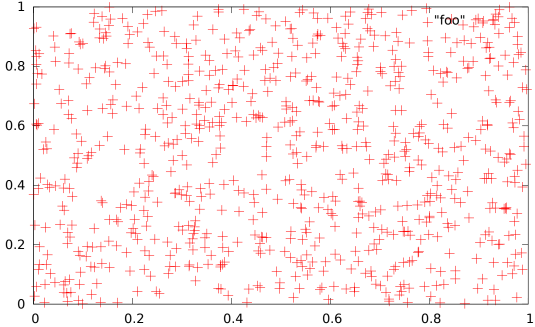
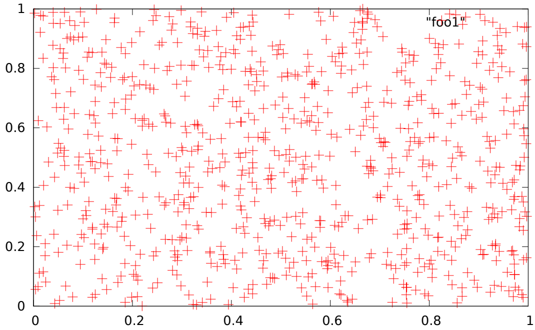
Spatial Poisson Process



$$\mathbb{P}(N_A = k) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}.$$

Spatial Poisson Process



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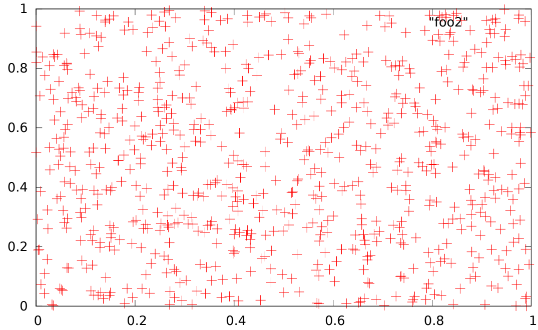

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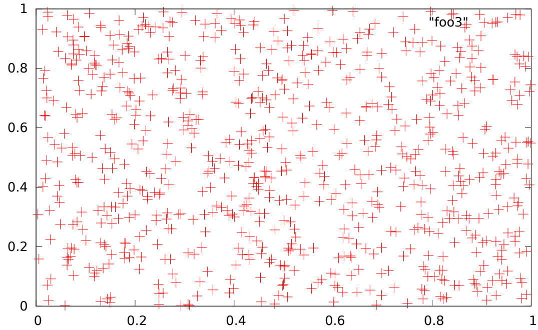

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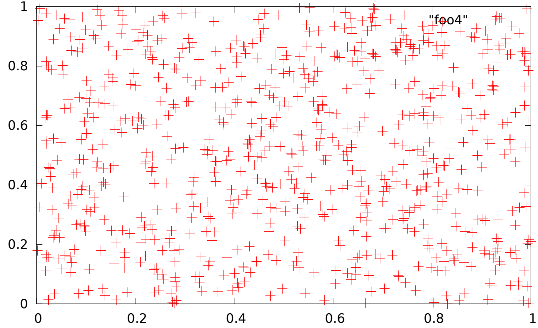

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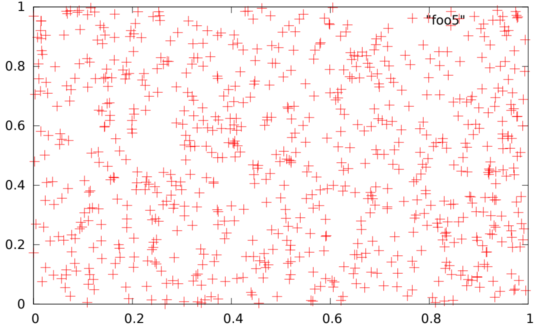

$$\mathbb{P}(N_A = k) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}.$$

Outline

Synthesis

Base model

1. reference model $\Rightarrow$ deviation
2. refinement $\Rightarrow$ model extension
3. multi-scale analysis (algebra for superposition, composition,...)
4. statistical methods $\Rightarrow$ Poisson regression